As we saw in an earlier lesson, a vector function \( \mathbf{r}(t) \) can represent the position of a particle at time \( t \), traveling along the space curve.

Suppose the particle moves over a small time interval \( h \). Then the average velocity of that interval is the displacement divided by the time interval \( h \). Displacement is a vector going from the initial position to the starting position, so

\[
\mathbf{v}_{\text{ave}} = \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h} = \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t}
\]

As \( h = \Delta t \) gets smaller and smaller, the average velocity tends toward the instantaneous velocity. In other words,

\[
\mathbf{v}(t) = \lim_{{h \to 0}} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h} = \mathbf{r}'(t)
\]

While velocity is a vector quantity, speed is a scalar quantity since it has no direction.

Speed at time \( t = |\mathbf{v}(t)|\).
Ex. 1. The table gives coordinates of a particle moving through space along a smooth curve.

<table>
<thead>
<tr>
<th>t</th>
<th>x</th>
<th>y</th>
<th>z</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2</td>
<td>3</td>
<td>-1</td>
</tr>
<tr>
<td>0.5</td>
<td>2.2</td>
<td>2.9</td>
<td>-1.1</td>
</tr>
<tr>
<td>1</td>
<td>2.3</td>
<td>2.4</td>
<td>-1.3</td>
</tr>
<tr>
<td>1.5</td>
<td>2.4</td>
<td>2.5</td>
<td>-1.6</td>
</tr>
<tr>
<td>2</td>
<td>2.3</td>
<td>2.5</td>
<td>-1.4</td>
</tr>
</tbody>
</table>

(a) Find the average velocities of the particle over the time intervals [0.5, 1] and [1, 1.5]

[0.5, 1]: \( \overrightarrow{v}_{\text{ave}} = \frac{\overrightarrow{r}(1) - \overrightarrow{r}(0.5)}{1 - 0.5} = \frac{<2.3, 2.4, -1.3> - <2.2, 2.9, -1.1>}{0.5} \)

= \( \frac{0.1, -0.2, -0.2}{0.5} \) = \( <0.2, -0.4, -0.4> \) units/time unit

[1, 1.5]: \( \overrightarrow{v}_{\text{ave}} = \frac{\overrightarrow{r}(1.5) - \overrightarrow{r}(1)}{1.5 - 1} = \frac{<2.4, 2.5, -1.6> - <2.3, 2.7, -1.3>}{0.5} \)

= \( \frac{0.1, -0.2, -0.3}{0.5} \) = \( <0.2, -0.4, -0.6> \) units/time unit

(b) Estimate the instantaneous velocity \( \overrightarrow{v}(1) \).

The best we can do here is average the average velocities close to 1.

\( \overrightarrow{v}(1) \approx \frac{<0.2, -0.4, -0.4> + <0.2, -0.4, -0.6>}{2} = \frac{<0.2, -0.4, -0.5>}{2} \) units/time unit

(c) Use your answer from (b) to estimate the speed of the particle at \( t = 1 \).

\( \text{Speed} = | \overrightarrow{v}(1) |, \text{so} \quad |0.2, -0.4, -0.5| = \sqrt{0.2^2 + 0.4^2 + 0.5^2} \approx 0.67 \text{ units/time unit} \)
Acceleration is the rate of change of velocity. Going through a similar argument as before, we get

$$\ddot{\mathbf{r}}(t) = \ddot{\mathbf{v}}(t) = \dddot{\mathbf{r}}(t)$$

**Ex 2.** Find the velocity, acceleration, and speed of a particle with position

$$\mathbf{r}(t) = e^t (\cos t \mathbf{i} + \sin t \mathbf{j} + t \mathbf{k})$$

$$\mathbf{v}(t) = \mathbf{r}'(t) = e^t ((-\sin t) \mathbf{i} + (\cos t) \mathbf{j} + \mathbf{k}) + e^t ((\cos t + \sin t) \mathbf{j} + (1 + t) \mathbf{k})$$

$$= e^t ((\cos t - \sin t) \mathbf{i} + (\cos t + \sin t) \mathbf{j} + (1 + t) \mathbf{k})$$

Speed = $|\mathbf{v}(t)| = \sqrt{e^{2t} (\cos^2 t - 2 \cos t \sin t + \sin^2 t + \cos^2 t + 2 \cos t \sin t + \sin^2 t + 1 + 2t + t^2)}$

$$= \sqrt{e^{2t} (3 + 2t + t^2)} = e^t \sqrt{3 + 2t + t^2}$$

$$\mathbf{a}(t) = \mathbf{v}'(t) = e^t ((-\sin t - \cos t) \mathbf{i} + (-\sin t + \cos t) \mathbf{j} + \mathbf{k}) + e^t ((\cos t - \sin t) \mathbf{i} + (\cos t + \sin t) \mathbf{j} + (1 + t) \mathbf{k})$$

$$= e^t (-2 \sin t \mathbf{i} + 2 \cos t \mathbf{j} + (2 + t) \mathbf{k})$$

Since differentiation can be done component-wise, it follows that anti-differentiation can also be done component-wise. However, just like with real-valued functions, anti-derivatives can differ by a constant vector, since the derivative of a constant vector is 0. But given initial conditions, we can solve for an explicit solution.
Ex 3. Find the position vector of a particle that has the given acceleration and the given initial velocity and position.
\[ \vec{a}(t) = \langle 2t, \sin(2t) \rangle, \, \vec{v}(0) = \langle 1, 0, 0 \rangle, \, \vec{r}(0) = \langle 0, 1, 0 \rangle \]
\[ \vec{v}(t) = \int \vec{a}(t) \, dt = \langle t^2, -\cos t, \frac{1}{2} \sin(2t) \rangle + \vec{C} \]
\[ \langle 1, 0, 0 \rangle = \vec{v}(0) = \langle 0, -1, 0 \rangle + \vec{C}, \quad \vec{C} = \langle 1, 0, 0 \rangle \]
\[ \vec{r}(t) = \langle t^2, 1 - \cos t, \frac{1}{2} \sin(2t) \rangle \]
\[ \vec{r}(t) = \int \vec{v}(t) \, dt = \langle \frac{1}{3} t^3 + t, t - \sin t, -\frac{1}{4} \cos(2t) \rangle + \vec{D} \]
\[ \langle 0, 1, 0 \rangle = \vec{r}(0) = \langle 0, 0, -\frac{1}{4} \rangle + \vec{D}, \quad \vec{D} = \langle 0, 1, 0 \rangle - \langle 0, 0, -\frac{1}{4} \rangle = \langle 0, 1, \frac{1}{4} \rangle \]
\[ \vec{r}(t) = \langle \frac{1}{3} t^3 + t, t - \sin t + 1, -\frac{1}{4} - \frac{1}{4} \cos(2t) \rangle \]

Ex 4. A particle has position \( \vec{r}(t) = \langle t^3, \cos t, \sin t \rangle \). When is its speed minimum?
\[ \vec{v}(t) = \frac{d}{dt} \vec{r}(t) = \langle 2t, -\sin t, \cos t \rangle \]
Speed = \| \vec{v}(t) \| = \sqrt{4t^2 + \sin^2 t + \cos^2 t} = \sqrt{4t^2 + 1}
Want to minimize speed. Notice \( \sqrt{4t^2 + 1} \geq 0 \), so its minimum value will occur at the same place as the minimum value of \( (\sqrt{4t^2 + 1})^2 = 4t^2 + 1 \).
\[ \frac{d}{dt} (4t^2 + 1) = 8t \quad \Rightarrow \quad t = 0 \]
Use First or Second Derivative Test to check this is a minimum.
\[ t = 0 \]