MA 261 - Lesson 5

Motion in Space: Velocity & Acceleration (13.4)

As we saw in an earlier lesson, a vector function $\vec{r}(t)$ can represent the position of a particle at time t, traveling along the space curve.

Suppose the particle moves over a small time interval h. Then the average velocity of that interval is the displacement divided by the time interval h. Displacement is a vector going from the initial position to the starting position, so

$$\vec{v}_{ave} = \frac{\vec{r}(t+h) - \vec{r}(t)}{h} = \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t}$$

As $h = \Delta t$ gets smaller and smaller, the average velocity tends toward the instantaneous velocity. In other words,

$$\vec{v}(t) = \lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h} = \vec{r}\,'(t)$$

While velocity is a vector quantity, speed is a scalar quantity since it has no direction.

speed at time $t = |\vec{v}(t)|$

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Ex 1. The table gives coordinates of a particle moving through space along a smooth curve.

t	x	y	z
0	2	3	-1
0.5	2.2	2.9	-1.1
1	2.3	2.7	-1.3
1.5	2.4	2.5	-1.6
2	2.3	2.5	-1.4

(a) Find the average velocities of the particle over the time intervals [0.5, 1] and [1, 1.5]

$[0.5, 1]$: $\vec{v}_{ave} = \dfrac{\vec{r}(1) - \vec{r}(0.5)}{1-0.5} = \dfrac{\langle 2.3, 2.7, -1.3\rangle - \langle 2.2, 2.9, -1.1\rangle}{0.5}$

$= \dfrac{\langle 0.1, -0.2, -0.2\rangle}{(\frac{1}{2})} = \boxed{\langle 0.2, -0.4, -0.4\rangle}$ units/time unit

$[1, 1.5]$: $\vec{v}_{ave} = \dfrac{\vec{r}(1.5) - \vec{r}(1)}{1.5-1} = \dfrac{\langle 2.4, 2.5, -1.6\rangle - \langle 2.3, 2.7, -1.3\rangle}{0.5}$

$= \dfrac{\langle 0.1, -0.2, -0.3\rangle}{(\frac{1}{2})} = \boxed{\langle 0.2, -0.4, -0.6\rangle}$ unit/time unit

(b) Estimate the instantaneous velocity $\vec{v}(1)$.

The best we can do here is average the average velocities close to 1.

$\dfrac{\langle 0.2, -0.4, -0.4\rangle + \langle 0.2, -0.4, -0.6\rangle}{2} = \boxed{\langle 0.2, -0.4, -0.5\rangle}$ units/time unit

(c) Use your answer from (b) to estimate the speed of the particle at $t=1$.

Speed $= |\vec{v}(t)|$, so $|\langle 0.2, -0.4, -0.5\rangle| = \sqrt{0.2^2 + 0.4^2 + 0.5^2}$

$= \boxed{\sqrt{0.45} \approx 0.67 \text{ units/time unit}}$

Acceleration is the rate of change of velocity. Going through a similar argument as before, we get

$$\boxed{\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t)}$$

Ex 2. Find the velocity, acceleration, and speed of a particle with position

$$\vec{r}(t) = e^t(\cos t\,\hat{\imath} + \sin t\,\hat{\jmath} + t\,\hat{k})$$

$$\vec{v}(t) = \vec{r}\,'(t) = e^t(-\sin t\,\hat{\imath} + \cos t\,\hat{\jmath} + \hat{k}) + e^t(\cos t\,\hat{\imath} + \sin t\,\hat{\jmath} + t\,\hat{k})$$
$$= \boxed{e^t((\cos t - \sin t)\hat{\imath} + (\cos t + \sin t)\hat{\jmath} + (1+t)\hat{k})}$$

$$\text{speed} = |\vec{v}(t)| = \sqrt{e^{2t}(\underbrace{\cos^2 t - 2\cos t\sin t + \sin^2 t}_{1} + \underbrace{\cos^2 t + 2\cos t \sin t + \sin^2 t}_{1} + 1 + 2t + t^2)}$$
$$= \sqrt{e^{2t}(3+2t+t^2)} = \boxed{e^t\sqrt{3+2t+t^2}}$$

$$\vec{a}(t) = \vec{v}\,'(t) = e^t((-\sin t - \cos t)\hat{\imath} + (-\sin t + \cos t)\hat{\jmath} + \hat{k})$$
$$+ e^t((\cos t - \sin t)\hat{\imath} + (\cos t + \sin t)\hat{\jmath} + (1+t)\hat{k})$$
$$= \boxed{e^t(-2\sin t\,\hat{\imath} + 2\cos t\,\hat{\jmath} + (2+t)\hat{k})}$$

Since differentiation can be done component-wise, it follows that anti-differentiation can also be done component-wise. However, just like with real-valued functions, anti-derivatives can differ by a constant vector, since the derivative of a constant vector is $\vec{0}$. But given initial conditions, we can solve for an explicit solution.

Ex 3. Find the position vector of a particle that has the given acceleration and the given initial velocity and position.

$\vec{a}(t) = \langle 2t, \sin t, \cos(2t) \rangle$, $\vec{v}(0) = \langle 1,0,0 \rangle$, $\vec{r}(0) = \langle 0,1,0 \rangle$

$\vec{v}(t) = \int \vec{a}(t)\,dt = \langle t^2, -\cos t, \frac{1}{2}\sin(2t) \rangle + \vec{C}$

$\langle 1,0,0 \rangle = \vec{v}(0) = \langle 0,-1,0 \rangle + \vec{C}$

$\vec{C} = \langle 1,0,0 \rangle - \langle 0,-1,0 \rangle = \langle 1,1,0 \rangle$

so $\vec{v}(t) = \langle t^2+1, 1-\cos t, \frac{1}{2}\sin(2t) \rangle$

$\vec{r}(t) = \int \vec{v}(t)\,dt = \langle \frac{1}{3}t^3+t, t-\sin t, -\frac{1}{4}\cos(2t) \rangle + \vec{D}$

$\langle 0,1,0 \rangle = \vec{r}(0) = \langle 0,0,-\frac{1}{4} \rangle + \vec{D}$

$\vec{D} = \langle 0,1,0 \rangle - \langle 0,0,-\frac{1}{4} \rangle = \langle 0,1,\frac{1}{4} \rangle$

$$\boxed{\vec{r}(t) = \langle \frac{1}{3}t^3+t, t-\sin t+1, \frac{1}{4}-\frac{1}{4}\cos(2t) \rangle}$$

Ex 4. A particle has position $\vec{r}(t) = \langle t^2, \cos t, \sin t \rangle$. When is its speed minimum?

$\vec{v}(t) = \vec{r}\,'(t) = \langle 2t, -\sin t, \cos t \rangle$

Speed $= |\vec{v}(t)| = \sqrt{4t^2+\sin^2 t+\cos^2 t} = \sqrt{4t^2+1}$

Want to minimize speed. Notice $\sqrt{4t^2+1} \geq 0$, so its minimum value will occur at the same place as the minimum value of $(\sqrt{4t^2+1})^2 = 4t^2+1$.

$\frac{d}{dt}(4t^2+1) = 8t \overset{\text{set}}{=} 0 \iff t=0$

Use First or Second Derivative Test to check this is a minimum.

$$\boxed{t=0}$$