MA 261 - Lesson 6

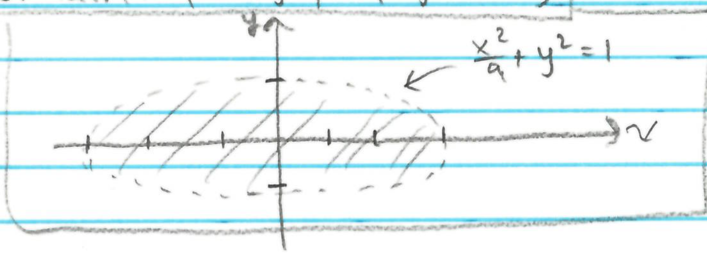
Functions of Several Variables (14.1)

A function of two variables is a rule that assigns to every pair of real numbers (x,y) in its domain a unique real number denoted $f(x,y)$. The range of f is the set of all real values that the output takes on.

Ex 1. Find the domain and range of $f(x,y) = \ln(9 - x^2 - 9y^2)$. Sketch the domain of f in the xy-plane.

Need $9 - x^2 - 9y^2 > 0 \iff x^2 + 9y^2 < 9 \iff \frac{x^2}{9} + y^2 < 1$

$\boxed{\text{Domain: } \{(x,y) \mid \frac{x^2}{9} + y^2 < 1\}}$



For the range, notice that the smallest x^2 and y^2 can be is 0, so the largest $9 - x^2 - 9y^2$ can be is $9 - 0 - 0 = 9$

$9 - x^2 - 9y^2$ can go to any number larger than 0.

So $0 < 9 - x^2 - 9y^2 \leq 9$ (subject to the domain),

so $\lim_{t \to 0^+} \ln(t) < \ln(9 - x^2 - 9y^2) \leq \ln(9)$

Hence, the range is $\boxed{(-\infty, \ln 9]}$

You can also have functions of more than two variables, which work much the same way.

Ex. 2. Find and sketch the domain of

$$f(x,y,z) = \frac{\sqrt{y-x^2}}{1-x^2} + \sqrt{9-z^2}$$

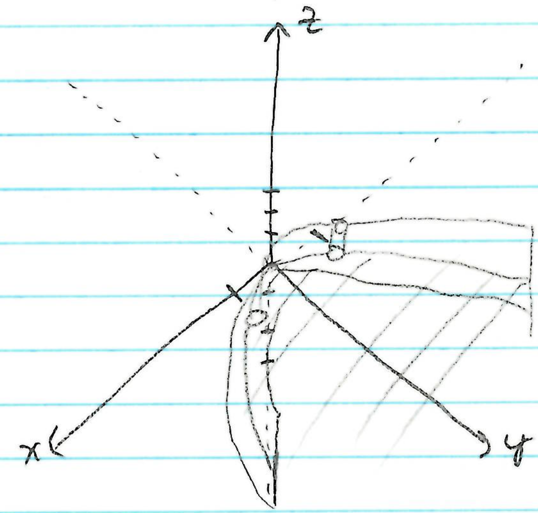
Need: $y - x^2 \geq 0 \iff y \geq x^2$

and $1 - x^2 \neq 0 \iff x \neq 1$ and $x \neq -1$

and $9 - z^2 \geq 0 \iff (3-z)(3+z) \geq 0$

$-3 \leq z \leq 3$

Domain: $\{(x,y) \mid x \neq \pm 1$ and $y \geq x^2$ and $-3 \leq z \leq 3\}$



To evaluate the function $f(x,y)$ at (a,b), you plug in a for x and b for y.
You can also create single-variable functions by evaluating $f(x,y)$ at one of the variables.

Ex 3. The temperature-humidity index I is the perceived air temperature when the actual temperature is T and the relative humidity is h, so we write $I = f(T, h)$.

Relative humidity (%)

T \ h (Actual Temp (°F))	20	30	40	50	60	70
80	77	78	79	81	82	83
85	82	84	86	88	90	93
90	87	90	93	96	100	106
95	93	96	101	107	114	124
100	99	104	110	120	132	144

(a) What is the value $f(95, 70)$? What is its meaning?

When $T = 95$ and $h = 70$, $I = 124$, so $f(95,70) = 124$.
This means that when the actual temperature is 95° F and the relative humidity is 70%, it "feels like" it is 124°F.

(b) Consider the function $g(T) = f(T, 60)$. What does this function mean and what is its behavior?

$g(T)$ tells the apparent temperature at actual temperature T when the relative humidity is fixed at 60%. As actual temperature increases, apparent temperature appears to increase at an increasing rate.

(c) What does $j(h) = f(90, h)$ mean and what is its behavior?

$j(h)$ gives the apparent temperature at relative humidity h% when actual temperature is fixed at 90° F. As h increases, j appears to have a constant increase until $h = 60$, where it speeds up.

Level Curves

One way to visualize a surface $z = f(x,y)$ is to plot the level curves of z on the xy-plane. A level curve is the projection of the intersection of the surface $z = f(x,y)$ with a plane $z = k$ for some constant k.

Sketching level curves of a surface produces a contour map or topographical map of the surface.

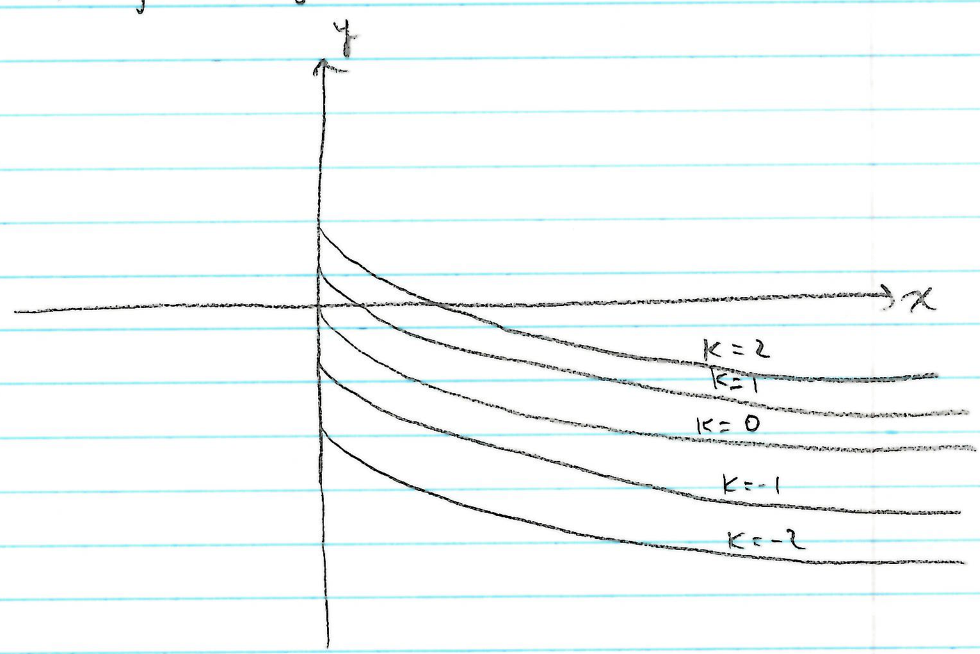
Ex 4. Draw a contour map of the function showing several level curves, where $f(x,y) = \sqrt{x} + y$

$k = 0$, get $\sqrt{x} + y = 0 \iff y = -\sqrt{x}$

$k = 1$, get $\sqrt{x} + y = 1 \iff y = -\sqrt{x} + 1$

$k = -1$, get $\sqrt{x} + y = -1 \iff y = -\sqrt{x} - 1$

etc.

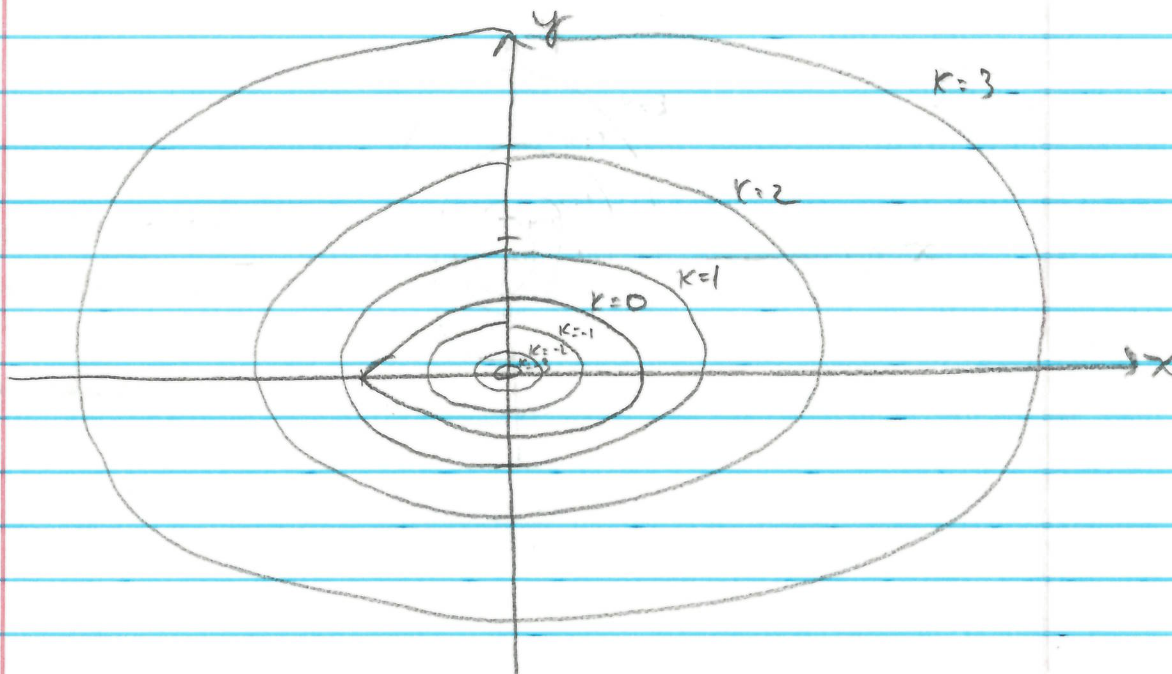


Ex 5. Sketch a contour map of $f(x,y)=\ln(x^2+4y^2)$, showing several level curves.

Intersect $z=\ln(x^2+4y^2)$ with $z=k$ and get

$$k=\ln(x^2+4y^2)$$
$$e^k=x^2+4y^2$$
$$\frac{x^2}{e^k}+\frac{y^2}{\left(\frac{e^k}{4}\right)}=1$$

ellipses with major axis $e^{\frac{k}{2}}$ in x-direction
minor axis $\frac{e^{k/2}}{2}$ in y-direction



Surface should look kind of like a funnel, going toward $-\infty$ at the origin

Level curves can also help us identify surfaces. (see attached)

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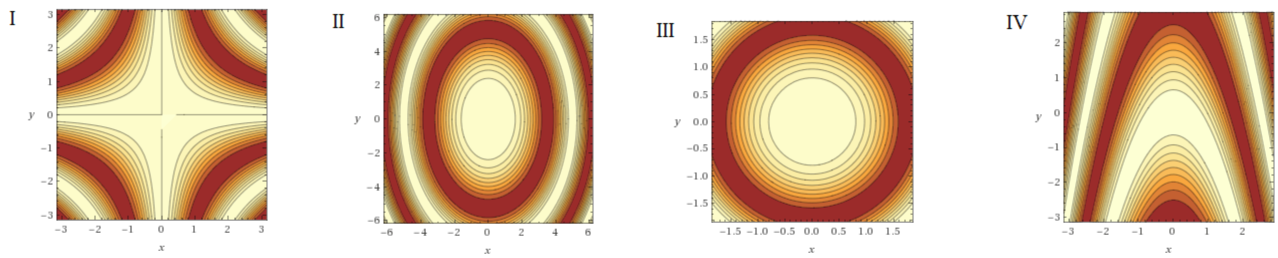
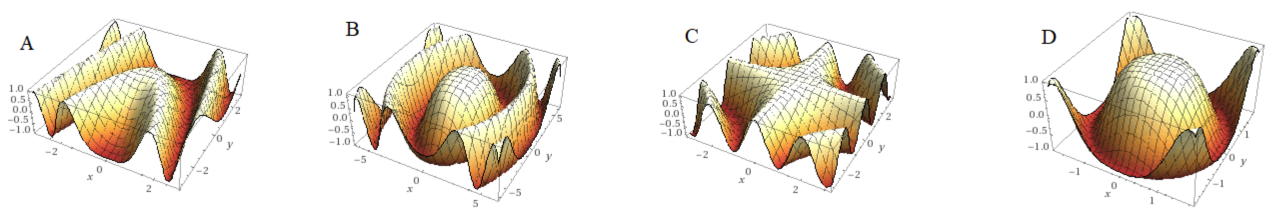
Ex 6. Match the following equations with their contour plots (I-IV) and surfaces (A-D).

$$1. \cos\left(x^2 + y^2\right) \qquad 2. \cos(xy) \qquad 3. \cos\left(x^2 + y\right) \qquad 4. \cos\left(\frac{x^2}{4} + \frac{y^2}{9}\right)$$





(Answer on next page)

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Level curves for 1 are of the form

$$k = \cos\big(x^2 + y^2\big) \Leftrightarrow \cos^{-1}(k) = x^2 + y^2$$

which are each a series of circles with different radii. Hence, the correct contour plot is III, and the matching graph with that contour plot is D.

Level curves for 2 are of the form

$$k = \cos(xy) \Leftrightarrow \cos^{-1}(k) = xy \Leftrightarrow y = \frac{\cos^{-1}(k)}{x}$$

which are a series of hyperbolas. Hence, the correct contour plot is I, and the matching graph with that contour plot is C.

Level curves for 3 are of the form

$$k = \cos\big(x^2 + y\big) \Leftrightarrow \cos^{-1}(k) = x^2 + y \Leftrightarrow y = -x^2 + \cos^{-1}(k)$$

which are a series of parabolas opening downward, shifted. Hence, the correct contour plot is IV, and the matching graph with that contour plot is A.

Level curves for 4 are of the form

$$k = \cos\left(\frac{x^2}{4} + \frac{y^2}{9}\right) \Leftrightarrow \cos^{-1}(k) = \frac{x^2}{4} + \frac{y^2}{9} \Leftrightarrow \frac{x^2}{4\cos^{-1}(k)} + \frac{y^2}{9\cos^{-1}(k)} = 1$$

which are a series of ellipses with major axis in the y-direction and minor axis in the x-direction. Hence, the correct contour plot is II, and the matching graph with that contour plot is B.