MA 261 - Lesson 7

Limits, Continuity, and Partial Derivatives

14.2 Limits and Continuity

A limit tells you the behavior of a function nearby a particular point.

For single variable functions,

$$\lim_{x \to c} f(x)$$

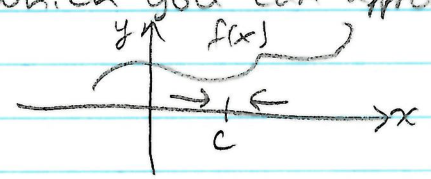
tells you if $f(x)$ tends toward a particular value if you take x-values near c and let them get closer to c.

For two-variable functions, the same applies.

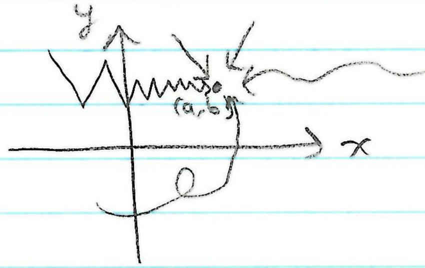
$$\lim_{(x,y) \to (a,b)} f(x,y)$$

tells you if $f(x,y)$ tends toward a particular value if you take points (x,y) near (a,b) and let them get closer to (a,b).

For single variable functions, there are two directions from which you can approach c, left and right:



For two-variable functions, there are infinitely many ways to do this!



If you can find even two paths in the xy-plane crossing through (a,b) along which you get different values for the limit, then $\lim_{(x,y)\to(a,b)} f(x,y)$ does not exist.

Ex 1. Show that $\lim_{(x,y)\to(0,0)} \frac{x^2+2y}{3x^2+4y}$ does not exist.

Two paths crossing through $(0,0)$ are the x-axis and y-axis. So let's try those.

The x-axis is the line $y=0$. If we restrict our function to $y=0$, we have $\lim_{x\to 0} \frac{x^2+2(0)}{3x^2+4(0)} = \lim_{x\to 0} \frac{x^2}{3x^2} = \frac{1}{3}$

The y-axis is the line $x=0$. If we restrict our function to $x=0$, we have $\lim_{y\to 0} \frac{(0)^2+2y}{3(0)^2+4y} = \lim_{y\to 0} \frac{2y}{4y} = \frac{1}{2}$

Since we get different values along different paths, $\lim_{(x,y)\to(0,0)} \frac{x^2+2y}{3x^2+4y}$ does not exist.

Ex 2. Show that $\lim_{(x,y)\to(0,0)} \frac{x^2+xy+3y^2}{2x^2+xy+6y^2}$ does not exist

$y=0$: $\lim_{x\to 0} \frac{x^2}{2x^2} = \frac{1}{2}$

$x=0$: $\lim_{y\to 0} \frac{3y^2}{6y^2} = \frac{1}{2}$

Limits agree along the axes, but let's try along $y=x$

$y=x$: $\lim_{x\to 0} \frac{x^2+x^2+3x^2}{2x^2+x^2+6x^2} = \lim_{x\to 0} \frac{5x^2}{9x^2} = \frac{5}{9}$

Since we get a different value along $y=x$ as we do along $x=0$, $\lim_{(x,y)\to(0,0)} \frac{x^2+xy+3y^2}{2x^2+xy+6y^2}$ does not exist.

Ex 3. Show that $\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$ does not exist.

Checking the axes, we get 0.

An arbitrary line passing through the origin is of the form $y=mx$. Trying along these lines, we get

$$\lim_{x\to 0} \frac{x^2(mx)}{x^4+(mx)^2} = \lim_{x\to 0} \frac{mx^3}{x^4+mx^2}\left(\frac{\frac{1}{x^2}}{\frac{1}{x^2}}\right) = \lim_{x\to 0} \frac{mx}{x^2+m} = 0$$

So we get the same value along any line!

But the curve $y=x^2$ passes through the origin.

$$\lim_{x\to 0} \frac{x^2(x^2)}{x^4+(x^2)^2} = \lim_{x\to 0} \frac{x^4}{2x^4} = \frac{1}{2}$$

Since we get different values along different curves passing through $(0,0)$, the limit does not exist.

Continuity

We define continuity like we do for single variable calculus.

$f(x,y)$ is continuous at (a,b) if

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

$f(x,y)$ is said to be continuous on a set D if it is continuous at every point in D.

The definition of continuity is a little subtle.

$\lim_{(x,y)\to(a,b)} f(x,y)$ must exist

$f(a,b)$ must be defined

and their values have to agree.

Continuity Facts:

- Polynomials (e.g., x^2+y^2+xy+3) are continuous on all of $\mathbb{R}^2$
- Sums, differences, products, and compositions of continuous functions are continuous.
- Quotients of continuous functions $f(x,y)$ and $g(x,y)$, given by $\frac{f(x,y)}{g(x,y)}$ are continuous wherever $g(x,y) \neq 0$.

- e^z, $\sin(z)$, $\cos(z)$, odd roots of z are continuous
- $\ln(z)$, even roots of z are continuous on their domains.

Ex 4. Find $\lim\limits_{(x,y)\to(2,1)} \cos\left(x^2y+2y+\sqrt{3y-x}\right)$

Since cosine is continuous, we get

$$\cos\left(\lim_{(x,y)\to(2,1)}\left(x^2y+2y+\sqrt{3y-x}\right)\right)$$

x^2y+2y is a polynomial, so it is continuous
$\sqrt{3y-x}$ has domain $3y-x \geq 0$, which $(2,1)$ satisfies,
and sums of continuous functions are continuous,
so we get

$$\cos\left((2)^2(1)+2(1)+\sqrt{3(1)-(2)}\right)$$

$$= \cos(4+1)$$

$$= \boxed{\cos(5)}$$

The real power of limits come from where functions are discontinuous. Often, we can use algebra to help us find the values of these limits. Since $\lim_{(x,y)\to(a,b)} f(x,y)$ only cares about points (x,y) near (a,b) but not (a,b) itself, we can often use cancellation.

Ex 5. Find $\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2-xy+y^2}$

Since the denominator is 0 at $(0,0)$, the function is not continuous at $(0,0)$. But, notice

$$x^3+y^3 = (x+y)(x^2-xy+y^2)$$

so

$$\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2-xy+y^2} = \lim_{(x,y)\to(0,0)} \frac{(x+y)\cancel{(x^2-xy+y^2)}}{\cancel{x^2-xy+y^2}}$$

(can cancel since $(x,y) \neq (0,0)$)

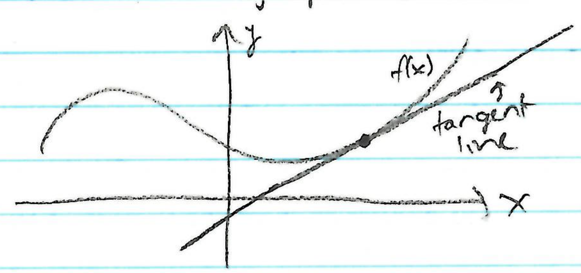
$$= \lim_{(x,y)\to(0,0)} x+y$$

$$= (0)+(0) = \boxed{0}$$

since polynomials are continuous.

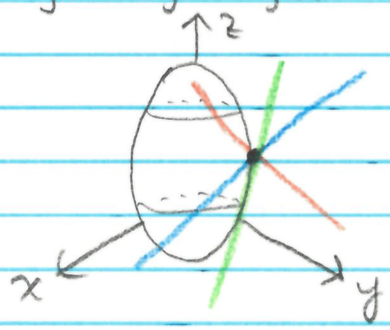
14.3 Partial Derivatives

As we know from single variable calculus, the derivative of a function $f(x)$ gives the slope of the tangent line to the graph of $f(x)$.



This is fine because there are only two directions in which the domain can change, so there's only one tangent line.

For two variable functions, there are infinitely many directions you can go from a point, so there can be infinitely many tangent lines



So in order to talk about derivatives of multivariable functions, we need to specify a direction.

It's not easy to think about this at first, but maybe you could come up with the idea to turn a multivariable function into a single variable function by fixing all but one variable and then taking the derivative. This would give us a derivative in that variable's direction.

We call these derivatives partial derivatives, and denote them in many different ways, but frequently as $f_x(x,y) = \frac{\partial f}{\partial x}$ or $f_y(x,y) = \frac{\partial f}{\partial y}$.

Formally,

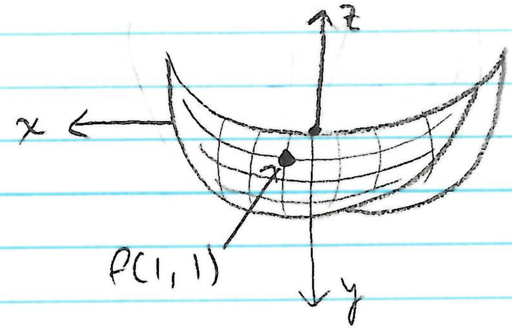
$$f_x(x,y) = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$f_y(x,y) = \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

Since in f_x, we are fixing y and varying x

in f_y, we are fixing x and varying y.

Ex 6. Determine the signs of $f_x(1,1)$ and $f_y(1,1)$ for the surface graphed below.



The tangent line in the positive x-direction has positive slope (since z increases) so $f_x(1,1) > 0$
The tangent line in the positive y-direction has negative slope (since z decreases) so $f_y(1,1) < 0$.

Notations for partial derivatives of $z = f(x,y)$

$$f_x(x,y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x,y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$$
$$f_y(x,y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x,y) = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f$$

Rule for finding partial derivatives of $z = f(x,y)$

To find f_x, treat y as if it is a constant and differentiate with respect to x.
To find f_y, treat x as if it is a constant and differentiate with respect to y.

This makes sense, since we were holding other variables fixed.

Ex 7. Let $z = \frac{x}{y}$. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

Treating y like a constant, $z = \underbrace{\frac{1}{y}}_{\text{const.}} \cdot x$, so $\boxed{\frac{\partial z}{\partial x} = \frac{1}{y}}$

Treating x like a constant, $z = \underbrace{x}_{\text{const,}} \cdot y^{-1}$, so $\boxed{\frac{\partial z}{\partial y} = -xy^{-2} = -\frac{x}{y^2}}$

Partial derivatives also work the same way for more than two variables. Treat all variables except one as constants.

Ex 8. Let $f(x,y,z) = \ln(x+2y+3z)$. Find the partial derivatives f_x, f_y, and f_z.

Treating y and z like constants, use the chain rule:

$$f_x = \frac{1}{x+2y+3z} \cdot \frac{\partial}{\partial x}(x + \underbrace{2y+3z}_{\text{const}}) = \frac{1}{x+2y+3z} \cdot 1$$

$$= \boxed{\frac{1}{x+2y+3z}}$$

Treating x and z like constants, use the chain rule:

$$f_y = \frac{1}{x+2y+3z} \cdot \frac{\partial}{\partial y}(\underbrace{x}_{\text{const.}} + 2y + \underbrace{3z}_{}) = \frac{1}{x+2y+3z}(2)$$

$$= \boxed{\frac{2}{x+2y+3z}}$$

Treating x and y like constants, use the chain rule:

$$f_z = \frac{1}{x+2y+3z} \cdot \frac{\partial}{\partial z}(\underbrace{x+2y}_{\text{const.}} + 3z) = \frac{1}{x+2y+3z}(3)$$

$$= \boxed{\frac{3}{x+2y+3z}}$$

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Ex 9 Use implicit differentiation to find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for $yz + x\ln y = z^2$

Here, we treat z as a function of x and y, implicitly. In other words, say $z = f(x,y)$. Now take partial derivatives.

$\frac{\partial}{\partial x}$: $\quad \underbrace{y}_{\text{const.}}z + x\underbrace{\ln y}_{\text{const.}} = z^2$

$$\xrightarrow{\frac{\partial}{\partial x}} \quad y\frac{\partial z}{\partial x} + \ln y = 2z\frac{\partial z}{\partial x}$$

$$(y-2z)\frac{\partial z}{\partial x} = -\ln y$$

$$\boxed{\frac{\partial z}{\partial x} = \frac{-\ln y}{y-2z}}$$

$\frac{\partial}{\partial y}$: $\quad yz + \underbrace{x}_{\text{const}}\ln y = z^2$

$$\xrightarrow{\frac{\partial}{\partial y}} \quad \left(y\frac{\partial z}{\partial y} + z\right) + \frac{x}{y} = 2z\frac{\partial z}{\partial y}$$

$$(y-2z)\frac{\partial z}{\partial y} = -z - \frac{x}{y}$$

$$\frac{\partial z}{\partial y} = \frac{-z-\frac{x}{y}}{y-2z}$$

$$\boxed{\frac{\partial z}{\partial y} = \frac{-yz-x}{y^2-2yz}}$$

Higher Order Partial Derivatives

Partial derivatives are functions, so you can take partial derivatives of those.

$$f_{xx} = (f_x)_x = \frac{\partial^2 f}{\partial x^2}$$

$$f_{yy} = (f_y)_y = \frac{\partial^2 f}{\partial y^2}$$

$$f_{xy} = (f_x)_y = \frac{\partial^2 f}{\partial y\,\partial x} \quad \left(\text{because } \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}(f)\right)\right)$$

$$f_{yx} = (f_y)_x = \frac{\partial^2 f}{\partial x\,\partial y}$$

and similarly for 3rd, 4th, etc. partial derivatives

Ex 10. Find the 2nd order partial derivatives of
$f(x,y) = x^2y - 3xy^4 + xy$

$f_x = 2xy - 3y^4 + y$, $f_y = x^2 - 12xy^3 + x$

$f_{xx} = (f_x)_x = \frac{\partial}{\partial x}(2xy - 3y^4 + y) = 2y$
$f_{yy} = (f_y)_y = \frac{\partial}{\partial y}(x^2 - 12xy^3 + x) = -36xy^2$
$f_{xy} = (f_x)_y = \frac{\partial}{\partial y}(2xy - 3y^4 + y) = 2x - 12y^3 + 1$
$f_{yx} = (f_y)_x = \frac{\partial}{\partial x}(x^2 - 12xy^3 + x) = 2x - 12y^3 + 1$

Notice in the above example that $f_{xy} = f_{yx}$.
This is no coincidence, as French mathematician Alexis Clairaut noticed.

Clairaut's Theorem. Suppose f is defined on a disk D containing the point (a,b). If the functions f_{xy} and f_{yx} are both continuous on D, then
$f_{xy}(a,b) = f_{yx}(a,b)$.

Most of the functions we discussed as continuous earlier have continuous derivatives, so usually, Clairaut's Theorem applies. It also applies to higher order partial derivatives.

Ex 11. Find f_{yx} for $f(x,y) = x^2y + y\ln(y)\cos(y) + e^{(y+2)/y}$

Technically, $f_{yx} = (f_y)_x$, but all of these functions are continuous and have continuous derivatives except when $y \leq 0$.
So by Clairaut's Theorem, $f_{yx} = f_{xy} = (f_x)_y$
$f_x = 2xy$
$f_{yx} = f_{xy} = \boxed{2y}$