# MA 261 - Lesson 8

## Linear Approximations and the Chain Rule

### 14.4 Tangent Planes and Linear Approximation

One key concept from Calculus 1 is that if you zoom into a function whose derivative exists and is continuous, then the function looks pretty close to a straight line. As such, the tangent line is a good approximation of the function near the point they share.

For a two variable function, if the first partial derivatives exist and are continuous at $(x_0, y_0, f(x_0,y_0))$ on the surface, then zooming in on that point, the surface looks pretty close to a plane. Indeed, we can find a tangent plane to the surface at that point containing all tangent lines at that point. And because of this, being near to the point, the tangent plane gives a good approximation of the surface.

### How to find Tangent planes

Suppose you have a surface $f(x,y)$ having a tangent plane at the point $(a,b)$. The partial derivatives $f_x(a,b)$ and $f_y(a,b)$ tell us how quickly $z = f(x,y)$ changes as we move in the x- or y-direction, respectively. Hence, the tangent line to f in the x-direction has direction vector $\langle 1, 0, f_x(a,b) \rangle$. The tangent line to f in the y-direction has direction vector $\langle 0, 1, f_y(a,b) \rangle$. Since both of these vectors lie on the tangent plane, we can find the normal vector to the tangent plane by taking the cross product.

$\vec{n} = \langle 0, 1, f_y(a,b)\rangle \times \langle 1, 0, f_x(a,b)\rangle = \langle f_x(a,b), f_y(a,b), -1\rangle$

Since the tangent plane passes through $(a,b,f(a,b))$, we know from lesson 2 that it has equation

$$f_x(a,b)(x-a) + f_y(a,b)(y-a) - (z - f(a,b)) = 0$$

or $$z - f(a,b) = f_x(a,b)(x-a) + f_y(a,b)(y-a)$$

> If $f$ has continuous first partial derivatives, then the tangent plane to $f$ at the point $(a,b,c)$ is given by the equation
> $$z - c = f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

Ex 1. Find the equation of the tangent plane to the surface $z = 2x^2 + y^2 - 5y$ at the point $(1,2,-4)$.

$\frac{\partial z}{\partial x} = 4x$ so $\left.\frac{\partial z}{\partial x}\right|_{(1,2,-4)} = 4$

$\frac{\partial z}{\partial y} = 2y - 5$ so $\left.\frac{\partial z}{\partial y}\right|_{(1,2,-4)} = -1$

Hence, the tangent plane has equation

$$z + 4 = 4(x-1) - 1(y-2)$$
$$z + 4 = 4x - 4 - y + 2$$
$$\boxed{z = 4x - y - 6}$$

As we already mentioned, the tangent plane is a good approximation of the surface near the point of tangency. So we get the linear approximation by replacing $z$ with $L(x,y)$ in the equation of the tangent plane.

**Linear Approximation**

The linear approximation to a surface $z = f(x,y)$ at the point $(a,b)$ is given by

$$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

Ex 2. Use Linear Approximation to estimate the value of $\sqrt{3.1^2 + 3.9^2}$

Let $z = f(x,y) = \sqrt{x^2+y^2}$. Want to look at $(3,4)$.

$$f_x = \tfrac{1}{2}(x^2+y^2)^{-1/2} \cdot 2x = \frac{x}{\sqrt{x^2+y^2}} \quad \text{so } f_x(3,4) = \frac{3}{5}$$

$$f_y = \tfrac{1}{2}(x^2+y^2)^{-1/2} \cdot 2y = \frac{y}{\sqrt{x^2+y^2}} \quad \text{so } f_y(3,4) = \frac{4}{5}$$

$$f(3,4) = \sqrt{3^2+4^2} = 5$$

So

$$L(x,y) = 5 + \tfrac{3}{5}(x-3) + \tfrac{4}{5}(y-4) = \tfrac{3}{5}x + \tfrac{4}{5}y$$

Therefore,

$$f(3.1, 3.9) \approx L(3.1, 3.9) = \tfrac{3}{5}(3.1) + \tfrac{4}{5}(3.9) = \boxed{4.98}$$

(Actual value is $\approx 4.981967483$)

Sometimes, you may want to estimate the error in a computation, and linear approximations can give you the approximate error. Or if you want to estimate how a function changes as you change the inputs, linear approximation can help with that.

Differentials

A differential is the difference in the value of the tangent plane. For a surface $z = f(x,y)$, $\Delta z$ is the actual change (measuring the exact difference in the values of $z$) and $dz$ is the differential (measuring the change in the $z$ values of the tangent plane).

We set
$$dz = L(x,y) - f(a,b)$$
$$= f_x(a,b)\underbrace{(x-a)}_{\text{the change in } x \text{ from } a} + f_y(a,b)\underbrace{(y-b)}_{\text{the change in } y \text{ from } a}$$

So if we want the differential from an arbitrary starting point, we get

$$\boxed{dz = f_x(x,y)\,dx + f_y(x,y)\,dy}$$

$dx$ and $dy$ are independent variables, which we usually set equal to $\Delta x$ and $\Delta y$, to estimate the change in $z$.

If $dx = \Delta x$ and $dy = \Delta y$, then $dz \approx \Delta z$

Ex 3. A triangle's base and height are measured as 10 cm and 15 cm, respectively, with an error in measurement of at most 0.1 cm in base and at most 0.2 cm in height. Use differentials to estimate the maximum error in the calculated area of the triangle.

$A = \frac{1}{2}bh$. want $dA = \frac{\partial A}{\partial b}db + \frac{\partial A}{\partial h}dh$

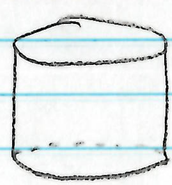
$\frac{\partial A}{\partial b} = \frac{1}{2}h$, $\frac{\partial A}{\partial h} = \frac{1}{2}b$

$dA = \frac{1}{2}h\,db + \frac{1}{2}b\,dh$

$dA = \frac{1}{2}(15)(0.1) + \frac{1}{2}(10)(0.2) = 1.75$

The maximum error is approximately $\boxed{1.75 \text{ cm}^2}$

Ex 4. Use differentials to estimate the amount of tin in a closed tin can with diameter 6 cm and height 14 cm if the tin is 0.02 cm thick.



$V = \pi r^2 h$ $\quad dh = 0.02 + 0.02 = 0.04$ (top + bottom of can)

$dr = \frac{dd}{2} = \frac{(0.02+0.02)}{2} = 0.02$

$$dV = \frac{\partial V}{\partial r}dr + \frac{\partial V}{\partial h}dh = (2\pi rh)dr + (\pi r^2)dh$$
$$= 2\pi(3)(14)(0.02) + \pi(3)^2(0.04)$$
$$= 1.68\pi + 0.36\pi$$
$$\boxed{\begin{aligned} &= 2.04\pi \text{ cm}^3 \\ &\approx 6.41 \text{ cm}^3 \end{aligned}}$$

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A two-variable function $z = f(x,y)$ is called <u>differentiable</u> if $dz \to \Delta z$ as $dx \to 0$ and $dy \to 0$.
This happens when $z$ has a unique tangent plane, which, as we mentioned earlier, occurs when $f_x$ and $f_y$ exist and are continuous.

## 14.5 The Chain Rule

Suppose $z = f(x,y)$ is a differentiable function of $x$ and $y$, and $x = x(t)$ and $y = y(t)$ are differentiable functions of $t$.
Then $z$ is a function of a single variable $t$.
So we can calculate $\frac{dz}{dt}$.

We know from differentials that

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

If we divide this by $dt = \Delta t$ and let it tend to 0, we get

<u>The Chain Rule (case 1)</u>

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

If $z = f(x,y)$ is differentiable functions of $x$ and $y$, which are differentiable functions of $t$.

Ex 5. Suppose $f(x,y) = \cos(x) + \sin(y)$,
$x = \frac{\pi}{4}t^2$, $y = \frac{3\pi}{4}t^2$. Find $\frac{df}{dt}$ when $t = 1$.

$f_x = -\sin x$, $\frac{dx}{dt} = \frac{\pi}{2}t$
$f_y = \cos y$, $\frac{dy}{dt} = \frac{3\pi}{2}t$

$\frac{df}{dt} = -\frac{\pi}{2}t\sin x + \frac{3\pi}{2}t\cos y$
When $t=1$, $x = \frac{\pi}{4}(1)^2 = \frac{\pi}{4}$, $y = \frac{3\pi}{4}(1)^2 = \frac{3\pi}{4}$

So $\left.\frac{df}{dt}\right|_{t=1} = -\frac{\pi}{2}(1)\sin\left(\frac{\pi}{4}\right) + \frac{3\pi}{2}(1)\cos\left(\frac{3\pi}{4}\right)$
$= -\frac{\pi}{2}\cdot\frac{\sqrt{2}}{2} + \frac{3\pi}{2}\left(-\frac{\sqrt{2}}{2}\right)$
$= -\frac{4\pi\sqrt{2}}{4} = \boxed{-\pi\sqrt{2}}$

Now, suppose $z = f(x,y)$ is differentiable in $x$ and $y$, and $x = g(s,t)$ and $y = h(s,t)$ are differentiable functions. For the same reason as before, we get

Chain Rule (case 2)

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}, \quad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

Ex 6. Let $R(s,t) = G(u(s,t), v(s,t))$ where $G$, $u$, and $v$ are differentiable, $u(1,2) = 5$, $u_s(1,2) = 4$, $u_t(1,2) = -3$, $v(1,2) = 7$, $v_s(1,2) = 2$, $v_t(1,2) = 6$, $G_u(5,7) = 9$, $G_v(5,7) = -2$. Find $R_s(1,2)$ and $R_t(1,2)$.

$R_s(s,t) = G_u(u,v)\cdot u_s(s,t) + G_v(u,v)\cdot v_s(s,t)$
$R_s(1,2) = G_u(u(1,2), v(1,2))\cdot u_s(1,2) + G_v(u(1,2), v(1,2))\cdot v_s(1,2)$
$= G_u(5,7)\cdot 4 + G_v(5,7)\cdot 2$
$= 9\cdot 4 + (-2)\cdot 2$
$= \boxed{32}$

(Ex. 6 cont.) $R_t(s,t) = G_u(u,v)\cdot u_t(s,t) + G_v(u,v)\cdot v_t(s,t)$

$$R_t(1,2) = G_u(u(1,2), v(1,2))\cdot u_t(1,2) + G_v(u(1,2), v(1,2))\cdot v_t(1,2)$$
$$= 9 \cdot (-3) + (-2) \cdot (6)$$
$$= \boxed{-39}$$

The same argument applies to a differentiable function of any finite number of variables which are differentiable functions of any finite number of variables.

<u>The Chain Rule (General Version)</u>
If $u$ is a differentiable function of the $n$ variables $x_1, \dots, x_n$ and each $x_i$ is a differentiable function of the $m$ variables $t_1, \dots, t_m$, then:

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \dots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

for any $i$ with $1 \le i \le m$.

e.g., if $w = f(x,y,z)$, $x = g_1(s,t,u)$, $y = g_2(s,t,u)$, $z = g_3(s,t,u)$ are all differentiable, then

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t}$$

<u>Implicit Differentiation</u>.
The Chain Rule is the key to implicit differentiation. If you have a function $F(x,y)$ with $F(x,y) = 0$, then $y$ is implicitly a function of $x$ (under some conditions, which we will always satisfy in this class). Hence, $F$ is a function of $x$, and we can find $\frac{dF}{dx}$.

Taking derivatives on both sides of $F(x,y)=0$ with respect to $x$, we get

$$\frac{\partial F}{\partial x}\underbrace{\frac{dx}{dx}}_{=1}+\frac{\partial F}{\partial y}\frac{dy}{dx}=0$$

Solving for $\frac{dy}{dx}$, we get

Equation 6

$$\boxed{\frac{dy}{dx}=\frac{-\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}=\frac{-F_x}{F_y}}$$

This is an easier way to solve for $\frac{dy}{dx}$ in the situation we have $F(x,y)=0$.

Ex 7. Use Equation 6 to find $\frac{dy}{dx}$ when

$$e^y\sin x = x+xy$$

Subtracting $e^y\sin x$ from both sides, get

$$0=x+xy-e^y\sin(x)=F(x,y)$$

$$\boxed{\frac{dy}{dx}=\frac{-F_x}{F_y}=-\frac{(1+y-e^y\cos x)}{x-e^y\sin x}}$$