MA 261 - Lesson 9

Directional Derivatives and Local Extrema

14.6 Directional Derivatives and the Gradient

As we mentioned in previous lessons, a surface can have infinitely many tangent lines at a point, and the "derivative" depends on the direction you're traveling away from that point in the xy-plane.

Last time, we noted that all tangent lines to a surface at a point lived on the tangent plane to the surface at that point. So if we move one unit in a direction given by unit vector $\vec{u} = \langle u_1, u_2 \rangle$ in the xy-plane, the slope of the tangent line in the direction of $\vec{u}$ should be the change in the z-value of the tangent plane. But this is precisely what the differential dz tells us!

$$dz = f_x(x,y)\,dx + f_y(x,y)\,dy$$

Since we move u_1 units in the x-direction and u_2 units in the y-direction, we get the slope of the tangent line in the direction of $\vec{u}$ is

$$f_x(x,y)\,u_1 + f_y(x,y)\,u_2$$

Notice that this can be written as

$$\langle f_x(x,y), f_y(x,y) \rangle \cdot \vec{u}$$

We call the vector $\langle f_x(x,y), f_y(x,y) \rangle$ the <u>gradient vector</u> of $f(x,y)$, and we denote it $\nabla f(x,y)$

The <u>directional derivative</u> of a two-variable function $f(x,y)$ at a point (a,b) in the direction of the unit vector $\vec{u}$ is $\nabla f(a,b) \cdot \vec{u}$ and is denoted $D_{\vec{u}} f(a,b)$.
The directional derivative in the direction of $\vec{u}$ in general is $D_{\vec{u}} f(x,y) = \nabla f(x,y) \cdot \vec{u}$

<u>Ex 1.</u> Find the gradient vector of $f(x,y) = y\cos(xy)$, then find the directional derivative of f at $(0,1)$ in the direction given by the angle $\theta = \frac{\pi}{4}$.

$f_x = -y^2\sin(xy)$, $f_y = y(-x\sin(xy)) + \cos(xy)$

$$\boxed{\nabla f(x,y) = \langle -y^2\sin(xy), \cos(xy) - xy\sin(xy) \rangle}$$

$\nabla f(0,1) = \langle -1\sin(0), \cos(0) - 0 \rangle = \langle 0, 1 \rangle$

The unit vector in the direction given by $\theta = \frac{\pi}{4}$ is $\vec{u} = \langle \cos\theta, \sin\theta \rangle = \langle \cos(\frac{\pi}{4}), \sin(\frac{\pi}{4}) \rangle = \langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle$

So $D_{\vec{u}} f(0,1) = \nabla f(0,1) \cdot \vec{u} = (0)(\frac{\sqrt{2}}{2}) + (1)(\frac{\sqrt{2}}{2}) = \boxed{\frac{\sqrt{2}}{2}}$

Ex 2. Find the directional derivative of $f(x,y) = \frac{x}{x^2+y^2}$ at $(1,2)$ in the direction of $\vec{v} = \langle 3, 5 \rangle$

First, find the gradient:

$$f_x = \frac{(x^2+y^2)(1) - (x)(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$f_y = -x(x^2+y^2)^{-2}(2y) = \frac{-2xy}{(x^2+y^2)^2}$$

$$\nabla f(x,y) = \left\langle \frac{y^2-x^2}{(x^2+y^2)^2}, \frac{-2xy}{(x^2+y^2)^2} \right\rangle$$

$$\nabla f(1,2) = \left\langle \frac{4-1}{(1+4)^2}, \frac{-4}{(1+4)^2} \right\rangle = \left\langle \frac{3}{25}, \frac{-4}{25} \right\rangle$$

Now, $\vec{v}$ is not a unit vector $\left(|\vec{v}| = \sqrt{9+25} = \sqrt{34} \right)$

So the unit vector in the direction of $\vec{v}$ is

$$\vec{u} = \left\langle \frac{3}{\sqrt{34}}, \frac{5}{\sqrt{34}} \right\rangle$$

$$D_{\vec{v}} f(1,2) = \nabla f(1,2) \cdot \vec{u} = \left(\frac{3}{25}\right)\left(\frac{3}{\sqrt{34}}\right) + \left(-\frac{4}{25}\right)\left(\frac{5}{\sqrt{34}}\right) = \boxed{-\frac{11}{25\sqrt{34}}}$$

By similar arguments, these same ideas can be extended to functions of any finite number of variables.

If $f(x_1, \ldots, x_n)$ is a differentiable function of n variables, then the gradient vector is $\nabla f = \langle f_{x_1}, \ldots, f_{x_n} \rangle$ and the directional derivative in the direction of the n-dimensional unit vector $\vec{u}$ is

$$D_{\vec{u}} f = \nabla f \cdot \vec{u}$$

Ex 3. Find the directional derivative of $f(x,y,z) = xy^2 \tan^{-1} z$ at $(2,1,1)$ in the direction $\vec{v} = \langle 1,1,1 \rangle$

$f_x = y^2 \tan^{-1} z$, $f_y = 2xy \tan^{-1} z$, $f_z = \frac{xy^2}{1+z^2}$

$\nabla f(x,y,z) = \langle y^2 \tan^{-1} z,\ 2xy \tan^{-1} z,\ \frac{xy^2}{1+z^2} \rangle$

$\nabla f(2,1,1) = \langle 1 \cdot \tan^{-1}(1),\ 4\tan^{-1}(1),\ \frac{2}{1+1} \rangle = \langle \frac{\pi}{4}, \pi, 1 \rangle$

$|\vec{v}| = \sqrt{1+1+1} = \sqrt{3}$, so $\vec{u} = \langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \rangle$

$D_{\vec{u}} f(2,1,1) = \nabla f(2,1,1) \cdot \vec{u} = \left(\frac{\pi}{4}\right)\left(\frac{1}{\sqrt{3}}\right) + (\pi)\left(\frac{1}{\sqrt{3}}\right) + (1)\left(\frac{1}{\sqrt{3}}\right) = \boxed{\frac{4+5\pi}{4\sqrt{3}}}$

Direction of Maximal Increase

An important question that often arises is the direction that a function changes the most.
To find the direction $\vec{u}$ in which $D_{\vec{u}} f$ is greatest, notice that

$$D_{\vec{u}} f = \nabla f \cdot \vec{u} = |\nabla f||\vec{u}| \cos\theta = |\nabla f| \cos\theta$$
$$(\text{since } |\vec{u}| = 1)$$

The maximum possible value occurs when $\cos\theta = 1$, i.e., $\theta = 0$, i.e., $\vec{u}$ points in the same direction as the gradient and the maximum value of $D_{\vec{u}} f$ is $|\nabla f|$.

Similar reasoning shows that the direction of maximal decrease is exactly opposite of ∇f, which you will explain on your homework.

Ex 4. Find the maximum rate of change of $f(x,y) = ye^{xy}$ at $(0,2)$ and the direction in which it occurs.

$\nabla f = \langle y^2 e^{xy}, xye^{xy} + e^{xy} \rangle$

$\nabla f(0,2) = \langle 4, 1 \rangle$

$|\nabla f(0,2)| = \sqrt{16+1} = \sqrt{17}$

$\boxed{\text{maximum rate of change is } \sqrt{17} \text{ in the direction } \langle 4,1 \rangle}$

Ex 5. Find all points in the xy-plane at which the direction of fastest change of $f(x,y) = x^2 + y^2 - x + y$ is $\hat{i} - \hat{j}$.

$\nabla f = \langle 2x-1, 2y+1 \rangle$

Since direction of fastest change is ∇f, we want all points (x,y) so $\nabla f = c(\hat{i} - \hat{j})$ for some scalar c.

$\langle 2x-1, 2y+1 \rangle = \langle c, -c \rangle$

$2x - 1 = c$ and $2y + 1 = -c \Leftrightarrow c = -2y - 1$

$2x - 1 = -2y - 1$

$2y = -2x$

$y = -x$

$\boxed{\text{All points on the line } y = -x}$

Ex 6. A heat source is placed at the origin. The temperature T of a point in space is inversely proportional to the distance from the origin. The temperature at $(4, 2, 4)$ is $100°$. Show that, at any point in space, the direction of the greatest increase in temperature is given by a vector that points toward the origin.

The distance of (x,y,z) from the origin is $\sqrt{x^2+y^2+z^2}$
So since T is inversely proportional to the distance,

$$T = \frac{k}{\sqrt{x^2+y^2+z^2}}$$ for some constant K.

$$100 = T(4,2,4) = \frac{k}{\sqrt{16+4+16}} = \frac{k}{\sqrt{36}} = \frac{K}{6} \iff K = 600$$

So $$T(x,y,z) = \frac{600}{\sqrt{x^2+y^2+z^2}}$$

$$\nabla T = \left\langle \frac{-600x}{(x^2+y^2+z^2)^{3/2}}, \frac{-600y}{(x^2+y^2+z^2)^{3/2}}, \frac{-600z}{(x^2+y^2+z^2)^{3/2}} \right\rangle$$

$$\nabla T = \underbrace{\frac{-600}{(x^2+y^2+z^2)^{3/2}}}_{\text{always negative, regardless of values of } x,y,z} \langle x, y, z \rangle$$

So ∇T points in the direction opposite of the position vector, hence toward the origin.

14.7 Local Extrema

A surface $z = f(x,y)$ has a local minimum at (a,b) if for all points (x,y) near (a,b), the value of z is larger at (x,y) than at (a,b).

$z = f(x,y)$ has a local maximum at (a,b) if for all points (x,y) near (a,b), the value of z is smaller at (x,y) than it is at (a,b).

In order for f to have a local max or min at (a,b), the surface has to change from increasing to decreasing or vice versa, regardless of which direction you are pointing. Hence,

$D_{\vec{u}} f(a,b) = 0$, regardless of the unit vector $\vec{u}$.

$0 = D_{\vec{u}} f(a,b) = \nabla f(a,b) \cdot \vec{u}$ for all unit vectors $\vec{u}$

can only be true if $\nabla f(a,b) = \vec{0}$, i.e., if $\langle f_x(a,b), f_y(a,b) \rangle = \langle 0, 0 \rangle$

A point (a,b) in the xy-plane is a critical point of $f(x,y)$ if $f_x(a,b) = 0$ and $f_y(a,b) = 0$ (or if either does not exist).

So to find critical points of $f(x,y)$ we find $f_x(x,y)$ and $f_y(x,y)$ and set them both equal to 0 simultaneously. This gives a system of (possibly nonlinear) equations to solve for all critical points.

After finding all critical points, we can use the second derivatives test to find local extrema.

Second Derivatives Test

Suppose f has continuous second partial derivatives near a critical point (a,b).

Let $D = D(a,b) = f_{xx}(a,b)f_{yy}(a,b) - [f_{xy}(a,b)]^2$.

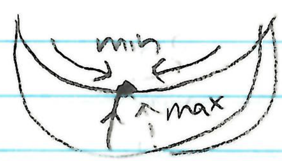
(a) If $D > 0$ and $f_{xx}(a,b) > 0$, then f has a local min at (a,b)

(b) If $D > 0$ and $f_{yy}(a,b) < 0$, then f has a local max at (a,b)

(c) If $D < 0$, then f has neither a local max nor a local min at (a,b); it has a saddle point at (a,b).

The test is inconclusive if $D = 0$.

A saddle point is a point on a surface which is a minimum in one direction, but a maximum in another direction.



(Looks like a Pringle at that point or saddle)

The proof of part of the Second Derivatives Test is in the textbook, but here is an imperfect explanation for why it makes sense.

Notice: if $D > 0$, then $f_{xx}(a,b)f_{yy}(a,b) > [f_{xy}(a,b)]^2 \geq 0$ so $f_{xx}(a,b)$ and $f_{yy}(a,b)$ must have the same sign.

If $f_{xx}(a,b)$ and $f_{yy}(a,b) > 0$, f is concave up in both directions, giving a min. If $f_{xx}(a,b)$ and $f_{yy}(a,b) < 0$, f is concave down in both directions giving a max.

If $D<0$, then $f_{xx}(a,b)$ and $f_{yy}(a,b)$ could have different signs, giving a min in one direction and a max in another.
(The other case is too complicated without formal proof)

Ex 7. Find the local minimum and maximum values and saddle points for $f(x,y) = (y-x)(3-2xy)$

Find critical points $f(x,y) = 3y - 2xy^2 - 3x + 2x^2y$

$f_x = 2y^2 - 3 + 4xy$, $f_y = 3 - 4xy + 2x^2$

Setting $f_x = 0$ and $f_y = 0$

$$2y^2 - 3 + 4xy = 0$$
$$2x^2 + 3 - 4xy = 0$$
$$\overline{\quad 2y^2 + 2x^2 = 0 \quad}$$
$$y^2 + x^2 = 0$$

So $y = \pm x$ ← two cases

If $y = x$, $0 = f_x = 2x^2 - 3 + 4x^2 = 6x^2 - 3 \iff 3 = 6x^2$

$\iff x = \pm \frac{1}{\sqrt{2}}$ so $y = \pm \frac{1}{\sqrt{2}}$ $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

If $y = -x$, $0 = f_x = 2x^2 - 3 - 4x^2 = -2x^2 - 3$ no solutions.

Find $f_{xx} = 4y$, $f_{yy} = -4x$, $f_{xy} = 4y + 4x$

So $D = f_{xx} \cdot f_{yy} - [f_{xy}]^2 = (4y)(-4x) - (4y+4x)^2$
$= -16xy - (4y+4x)^2$

$D\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) < 0$ saddle

$D\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) < 0$ saddle

$\boxed{\text{$f$ has saddle points at } \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \text{ and } \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) \text{ and no extrema}}$

Ex 8. Find the local max and min values and saddle points of $f(x,y) = 3x^2y + 48xy + 4y^2$

Find critical points:

$0 \overset{set}{=} f_x = 6xy + 48y$, $0 \overset{set}{=} f_y = 3x^2 + 48x + 8y$

$6y(x+8) = 0$

$y = 0$ or $x = -8$

Case $y=0$: $0 = f_y = 3x^2 + 48x + 0 = 3x(x+16)$

$x = 0$ or $x = -16$

$(0,0)$, $(-16,0)$

Case $x=-8$: $0 = f_y = 3(-8)^2 + 48(-8) + 8y \Leftrightarrow 192 = 8y$

$\Leftrightarrow y = 24$

$(-8, 24)$

$f_{xx} = 6y$, $f_{yy} = 8$, $f_{xy} = 6x + 48$

$D = f_{xx} \cdot f_{yy} - [f_{xy}]^2 = (6y)(8) - (6x+48)^2$

$= 48y - (6x+48)^2$

$D(0,0) = 48(0) - (0+48)^2 < 0$ saddle

$D(-16,0) = 48(0) - (-96+48)^2 < 0$ saddle

$D(-8,24) = 48(24) - (-48+48)^2 > 0$

$f_{xx}(-8,24) = 6(24) > 0$ local min

$f(-8,24) = -2304$

$\boxed{\text{f has a local min value of } -2304 \text{ at } (-8,24) \text{ and has saddle points at } (0,0) \text{ and } (-16,0)}$