Problem 1 (20 Points). Find the length of the curve \( \mathbf{r}(t) = \mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k} \), \( 0 \leq t \leq 1 \).

Solution.

Recall that \( L = \int_0^1 |\mathbf{r}'(t)| \, dt \), so we first find \( \mathbf{r}'(t) \):

\[
\mathbf{r}'(t) = 2t\mathbf{j} + 3t^2\mathbf{k}
\]

\[
|r'(t)| = \sqrt{(2t)^2 + (3t^2)^2} = \sqrt{4t^2 + 9t^4} = \sqrt{t^2(4 + 9t^2)} = |t|\sqrt{4 + 9t^2} = t\sqrt{4 + 9t^2}
\]

\((|t| = t \text{ since } 0 \leq t \leq 1)\)

\[
L = \int_0^1 t\sqrt{4 + 9t^2} \, dt
\]

Make the following \( u \)-substitution: \( u = 4 + 9t^2, \, du = 18t \, dt \). Also, \( u(0) = 4, \, u(1) = 13 \).

\[
\frac{1}{18} \int_4^{13} u^{1/2} \, du
\]

\[
= \frac{1}{18} \cdot \frac{2}{3} u^{3/2} \bigg|_4^{13} = \frac{1}{27} (13^{3/2} - 4^{3/2}) = \frac{1}{27} (13^{3/2} - 8)
\]