High score: 20; (nonzero) Low score: 5; Average score: 18.77

Problem 1 (20 Points). Evaluate \( \iiint_{R} (x^2 + y^2) \, dV \), where \( R \) is the region between the spheres of radius 1 and radius 2 centered at the origin.

\[ \text{Hint: } \int_{0}^{\pi} \sin^3 u \, du = \frac{4}{3} \]

Solution. In spherical coordinates, a sphere of radius \( a \) centered at the origin has the equation \( \rho = a \). There are no restrictions on \( \theta \) or on \( \phi \), so they can take on their full ranges: \( 0 \leq \theta \leq 2\pi \), \( 0 \leq \phi \leq \pi \).

\[
\int_{0}^{\pi} \int_{0}^{2\pi} \int_{1}^{2} \left( \rho^2 \cos^2 \theta \sin^2 \phi + \rho^2 \sin^2 \theta \sin^2 \phi \right) \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi
\]

Notice that the integrand contains \( (\cos^2 \theta + \sin^2 \theta) \rho^2 \sin^2 \phi \cdot \rho^2 \sin \phi \), so we get

\[
\int_{0}^{\pi} \int_{0}^{2\pi} \int_{1}^{2} \rho^4 \sin^3 \phi \, d\rho \, d\theta \, d\phi
\]

\[
= \int_{0}^{\pi} \int_{0}^{2\pi} \frac{1}{5} \rho^5 \sin^3 \phi \bigg|_{\rho=1}^{\rho=2} \, d\theta \, d\phi
\]

\[
= \int_{0}^{\pi} \int_{0}^{2\pi} \frac{32}{5} \sin^3 \phi \, d\theta \, d\phi
\]

\[
= \frac{31}{5} \int_{0}^{\pi} \sin^3 \phi \int_{0}^{2\pi} \, d\theta \, d\phi
\]

\[
= \frac{31}{5} \cdot 2\pi \int_{0}^{\pi} \sin^3 \phi \, d\phi
\]

By the hint, we get

\[
= \frac{31}{5} \cdot 2\pi \cdot \frac{4}{3} = \frac{248\pi}{15}
\]