High score: 20; (nonzero) Low score: 15; Average score: 17.34

Problem 1 (20 Points). Use Green’s Theorem to evaluate $\int_C y^3 \, dx - x^3 \, dy$, where $C$ is the positively-oriented circle $x^2 + y^2 = 4$.

Solution.

Let $D$ be the region enclosed by $C$. Since $C$ is a positively-oriented closed curve, by Green’s Theorem, we have

$$\int_C y^3 \, dx - x^3 \, dy = \iint_D \left( \frac{\partial}{\partial x} [-x^3] - \frac{\partial}{\partial y} [y^3] \right) \, dA = \iint_D (-3x^2 - 3y^2) \, dA$$

Since the region $D$ is the interior of the circle $x^2 + y^2 = 4$, this double integral is best done in polar coordinates:

$$\int_0^{2\pi} \int_0^2 -3r^2 \cdot r \, dr \, d\theta$$

$$= -3 \int_0^{2\pi} \frac{1}{4}r^4 \bigg|_{r=0}^{r=2} \, d\theta$$

$$= -3 \cdot 4 \int_0^{2\pi} \, d\theta$$

$$= -12 \cdot 2\pi = -24\pi$$