Eddie Price   I have a line or surface integral. What do I do?!   Summer 2018

I have a line integral. What do I do?!

Is it the line integral of a scalar field? ($\int_C f\,ds$) If so, then you must parameterize the curve and use the arc length differential.

It's the line integral of a vector field. ($\int_C \vec{F}\cdot d\vec{r}$) *What do I do?!*

Is $\vec{F}$ conservative? (For 2-d vector fields $F = P\hat{i} + Q\hat{j}$, check if $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$; for 3-d vector fields $F = P\hat{i} + Q\hat{j} + R\hat{k}$, check if $\operatorname{curl}\vec{F} = \nabla\times\vec{F} = \vec{0}$). If yes, find a potential function f, and use the Fundamental Theorem for Line Integrals. (Note: By FTfLI, if $\vec{F}$ is conservative and C is closed, you get 0 no matter what, so you don't need to find f in this case!)

$\vec{F}$ *is not conservative. What do I do?!*

Is C closed? If not, then you must parameterize C and use the definition for a line integral.

C *is closed. What do I do?!*

Is $\vec{F}$ 2-dimensional? ($\vec{F}(x,y) = P\hat{i} + Q\hat{j}$) If yes, then determine the orientation of C and use Green's Theorem ($\oint_C \vec{F}\cdot d\vec{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\,dA$). If Green's Theorem is not useful, you could parameterize C and use the definition of a line integral.

$\vec{F}$ *is 3-dimensional.* ($\vec{F}(x,y,z) = P\hat{i} + Q\hat{j} + R\hat{k}$) *What do I do?!*

Is $\operatorname{curl}\vec{F}$ easy to work with? If yes, determine the orientation of C, find an easy-to-work-with surface S having C as its boundary ∂S (with the induced orientation being the orientation of C), and use Stokes' Theorem ($\int_{\partial S}\vec{F}\cdot d\vec{r} = \iint_S \operatorname{curl}\vec{F}\cdot d\vec{S}$). If Stokes' Theorem is too difficult to use, you may have to parameterize the curve and use the definition of a line integral.

I have a surface integral. What do I do?!

Is it the surface integral of a scalar field? ($\iint_S f\, dS$) If so, then you must parameterize S and use the surface area differential.

It's the surface integral of a vector field. ($\iint_S \vec{F} \cdot d\vec{S}$) *What do I do?!*

Are you given that your vector field is the curl of some vector field, and the surface is not closed? If so, use Stokes' Theorem to replace the surface integral with the line integral over the boundary curve with the induced orientation, or replace the surface with a much simpler surface having the same boundary curve with correct orientation and compute the surface integral by definition (useful if the boundary curve must be broken up into several different parameterizations).

It doesn't appear that $\vec{F}$ *is the curl of some vector field, and the surface is not closed. What do I do?!*

Your best bet is probably to parameterize the surface and use the definition of a surface integral.

The surface is closed. What do I do?!

Is $\operatorname{div} \vec{F}$ easy to work with and the region E enclosed by S is easy to work with? If so, determine the orientation of the surface and use the Divergence Theorem ($\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F}\, dV$, assuming S has positive orientation).

If not, then you may have to parameterize the surface and use the definition of a surface integral.