42. Suppose \( g \) satisfies the hypotheses of the lemma for \( p = 1 \). Then \( |g(x)| \leq M \) a.e. For suppose this were false, then there must exist some \( N > M \) for which \( |g(x)| > N \) for a set \( E \) of positive measure. (If there does not exist such an \( N \), then \( \mu \{ x : |g(x)| > N \} = 0 \) \( \forall N > M \), so \( \mu \{ x : |g(x)| > N \} = \mu \{ \cap_{N} \} = 0 \).) But then, letting \( E_1 \) be the subset of \( E \) for which \( g(x) > 0 \) and \( E_2 \) the subset for which \( g(x) < 0 \), we have by hypothesis:

\[
\mu (E) = \mu (E_1) + \mu (E_2) \geq 1 \int g(x) \, d\mu \geq N \mu (E) = M \mu (E),
\]

which is a contradiction, since \( 0 < \mu (E) \leq \mu (x) < \infty \).

Hence, the conditions of the lemma imply \( g \leq M \) a.e., which implies \( g \in L^p (\mu) \).

43. The lemma will not always be true for measure spaces for which \( \mu \) is not \( \sigma \)-finite. Consider \( X = [0,1] \), with \( \mu (x) = 0 \), \( \mu ([0,1]) = \infty \). Then for \( 1 < p < \infty \), it will be true that for simple functions \( g \) and all \( x \in X \):
This is more or less a statement about the completeness of the complex numbers. Let \( \{\Lambda_1, \Lambda_2, \ldots\} \) be a Cauchy sequence of linear functionals over \( V \). Then for each \( k \geq 0 \) there is an increasing sequence \( N_k \) such that for all \( f \in V \) and all \( n, m \geq N_k \),

\[
|\Lambda_n f - \Lambda_m f| \leq \frac{1}{2^k} \|f\|_V.
\]

(Note that \( \sup_{n \geq 0} |\Lambda_n f| \leq M \) implies \( \|\Lambda_n f\|_V \leq M \), or \( |\Lambda f| \leq M \|f\|_V \) for all \( f \in V \).) Define, for each fixed \( f \),

\[
\Lambda f = \Lambda_{N_0} f + \sum_{k=0}^{\infty} (\Lambda_{N_{k+1}} f - \Lambda_{N_k} f).
\]

Because \( |\Lambda_{N_{k+1}} f - \Lambda_{N_k} f| \leq \frac{1}{2^k} \|f\|_V \), the sum defining \( \Lambda \) is convergent, and then for \( \Lambda \) is well-defined.

Now, it is trivial to verify that \( \Lambda (\alpha f + \beta g) = \alpha \Lambda (f) + \beta \Lambda (g) \). So \( \Lambda \) is a linear functional. Further, \( \Lambda \) is bounded, since,

\[
|\Lambda f| \leq \|\Lambda_{N_0} f\| + \sum_{k=0}^{\infty} |\Lambda_{N_{k+1}} f - \Lambda_{N_k} f| \\
\leq \|\Lambda_{N_0} f\| + \sum_{k=0}^{\infty} \frac{1}{2^k} \|f\|_V \\
\leq (\|\Lambda_{N_0}\|_V + 2) \|f\|_V.
\]

Hence \( \Lambda \in \mathcal{V}^* \).

We now show that \( V^* \) is complete by proving \( \lim_{n \to \infty} \|\Lambda_n - \Lambda\|_V = 0 \). Given any \( \varepsilon > 0 \), let \( h \) be such that \( \frac{1}{2^h} < \frac{\varepsilon}{2^h} \). Then \( \forall n \geq N_h \) (because the sum for \( \Lambda \) is telescopic),

\[
|\Lambda f - \Lambda_n f| = \left| (\Lambda_{N_h} f - \Lambda_n f) + \sum_{k=h}^{\infty} (\Lambda_{N_{k+1}} f - \Lambda_{N_k} f) \right| \\
\leq \frac{1}{2^h} \|f\|_V + \frac{1}{2^h} \|f\|_V \\
< \varepsilon \|f\|_V.
\]
b) Cont.) so that \( \|A - A_n\| < \varepsilon \), hence \( \|A - A_n\| \to 0 \) as claimed, and \( y^* \) is complete.

This will in fact imply the Riesz-Fisher Theorem that \( L^2 \) is complete for \( 1 < p < \infty \), when coupled with the fact that for \( k < q < \infty \), \( (L^q)^* = L^q \). Indeed, it is easy to show that \( L^2 \) is a normal linear space, see that \( L^p = (L^q)^* \) is complete for \( k < p < \infty \). (Proof: denominator \( p' = 1 \), \( x^* \neq (L^q)^* \).

c) It is obvious that \( L^\infty (X, \mu) = \{ \alpha X + \beta X_b : \alpha, \beta \in \mathbb{R} \} \), since all these functions are contained in \( L^\infty \), and this characterizes all functions on \( X \). Also \( L^p (X, \mu) \) is just the set \( \{ \alpha X_b : \alpha \in \mathbb{R} \} \), since all these functions are in \( L^p \), and for a function to be in \( L^p \), its value at \( b \) must be \( 0 \).

Now, for any linear functional \( \Lambda \) on \( L^p \), any function \( f = \alpha X + \beta X_b \in L^p \), we have \( \Lambda f = \Lambda (\alpha X + \beta X_b) = \alpha \Lambda X + \beta \Lambda X_b \), where \( \Lambda X = \int X d\mu \). So \( (L^p)^* = \{ \Lambda : \Lambda f = \int_X f d\mu \} \), \( \Lambda \in \mathbb{R} \).

Now, if two normed linear spaces are isomorphic, it will follow easily that the cardinality of their dimension is the same, since an isomorphism will preserve spans and linear independence. Yet \( \dim_{\mathbb{R}} L^p (X, \mu) = \dim_{\mathbb{R}} (L^p)^* = 2 \), plain, with the basis \( \{ X, X_b \} \), and \( \dim_{\mathbb{R}} L^2 (X, \mu) = 1 \), with a basis \( \{ \Lambda_0 : \Lambda_0 f = \int f d\mu \} \).

Therefore it cannot be that \( L^p (X, \mu) \) and \( L^2 (X, \mu) \) are isomorphic.

This shows that the Riesz Representation Theorem cannot be extended for \( p \neq 2 \) to measures that are not or finite.

\( \ast \)

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d) Given any \( \varepsilon > 0 \), there is a decomposition of \( X \) into disjoint sets \( A, B \) such that \( \mu (B) < \infty \), and \( \int A |f|^p d\mu < E/\varepsilon \). Now, by Fatou's lemma, we know that

\[ \int_X |f|^p d\mu = \lim \inf \int_B |f|^p d\mu \leq \lim \inf \int_B |f|^p d\mu = \lim \inf \int_X |f|^p d\mu - \sum \int_B |f|^p d\mu = \int_X |f|^p d\mu - \sum \int_B |f|^p d\mu \]

This shows that \( \lim \sup \int_B |f|^p d\mu \leq \int_X |f|^p d\mu < E/\varepsilon \). Now, note that for \( \Delta(x) = 2^p \int x |f|^p d\mu \), \( \lim \Delta(x) = 0 \). (This has been proven before.) Hence this is a \( \delta > 0 \) for which \( f \in L^p \) for all \( \mu (E) < \delta \). By Egorov's Theorem, there is a decomposition of \( B \) into disjoint sets \( C \) and \( D \) for which \( \mu (C) < \delta \), and \( f \to f \) uniformly on \( C \). We have defined \( \delta \) such that \( \int_D |f|^p d\mu < E/\varepsilon \). As above, this implies \( \lim \sup \int_D |f|^p d\mu < E/\varepsilon \). (Continued.)
\[
\int_A |f - f_n|_p^p \, d\mu = \int_A |f - f_n|_p^p \, d\mu + \int_C |f - f_n|_p^p \, d\mu + \int_D |f - f_n|_p^p \, d\mu \\
\leq \left( \int_A |f|_p^p \, d\mu + \int_C |f - f_n|_p^p \, d\mu + \int_D |f - f_n|_p^p \, d\mu \right) \left( \int_A (|f|_p)^{p'} + \int_C (|f - f_n|_p)^{p'} + \int_D (|f - f_n|_p)^{p'} \right) \frac{1}{\delta^p} \\
< \frac{2^p(\delta^{p'})^p}{\delta^{p'}} + \frac{\varepsilon}{\delta^{p'}} \mu(A) + \frac{(2(\delta^{p'})^p}{\delta^{p'}} \\
= \frac{\varepsilon}{\delta} + 2^{p+1} \frac{\varepsilon}{\delta} = \frac{1}{2} \frac{\varepsilon}{\delta}.\]

As this can be made arbitrarily small through choice of \( \varepsilon \), so too can \( \|f - f_n\|_p \) (\( \forall n \geq N \)). Hence \( \|f - f_n\|_p \to 0 \).

Proof of \( \star \): For \( \sigma \)-finite \( X \), this is obvious; since \( \mu \) can write 

\( X = U \cup \mathcal{X} \) for increasing sets \( \mathcal{X} \) of finite measure, then 

\[
\int_X |f|_p \, d\mu = \lim \int_{X_n} |f|_p \, d\mu,
\]

as so we may have 

\[
\int_{X_0} |f|_p \, d\mu = \int_X |f|_p \, d\mu - \int_{X_0} |f|_p \, d\mu < \frac{\varepsilon}{2} \quad \text{for some} \ i.
\]

This is easy to extend to any \( \sigma \)-finite measure \( \mu \); simply consider any \( \sigma \)-finite collection of sets \( X \) which has \( \mu \) as their union \( X \).