\[ \lim_{n \to \infty} \int_{\frac{1}{n}}^{1} \frac{\cos(x+h)-\cos(x)}{x^{n}} \, dx \quad \text{exists.} \]

**Proof)**

Claim 1: \( \frac{2}{n} - \sin(\frac{\pi}{n}) \geq 0 \) for all \( n \geq 1 \).

**Proof**

Look at \( f'(x) = x - \sin(x) \) \( \Rightarrow f(x) = 1 - \cos(x) \).

Then \( f(0) = 0 < f'(x) \geq 0 \quad \forall x \), so \( f \) is increasing, \( f(\frac{n}{n}) = \frac{2}{n} - \sin(\frac{\pi}{n}) \geq 0 \) by Claim 1.

Hence \( \forall \frac{n}{n} \leq x \), \( 2x - \sin(x+\frac{\pi}{n}) \geq 0 \).

So look at \( \int_{\frac{1}{n}}^{1} \frac{\cos(x+h)-\cos(x)}{x^{n}} \, dx = g_{n}(x) \).

By the M.V.T., \( \exists \xi \in (x, x+\frac{\pi}{n}) \), \( \sin(x) = \frac{\cos(x+h)-\cos(x)}{g_{n}(\xi)} \).

Now \( |\int_{\frac{1}{n}}^{1} \frac{\cos(x+h)-\cos(x)}{x^{n}} \, dx| \leq \int_{\frac{1}{n}}^{1} \frac{1}{x^{n}} \, dx \leq \frac{1}{1-x} \leq \frac{1}{\frac{2}{n}} \).

Now \( \int_{0}^{1} \frac{2}{x} \, dx = 4 \quad \checkmark \)

Claim 2: \( \lim_{n \to \infty} \int_{\frac{1}{n}}^{1} \frac{\cos(x+h)-\cos(x)}{x^{n}} \, dx = -\sin(\frac{\pi}{2}) \).

So by the L.C.T. (Prop 15, p. 26),

\[ \lim_{n \to \infty} \int_{\frac{1}{n}}^{1} \frac{\cos(x+h)-\cos(x)}{x^{n}} \, dx = \lim_{n \to \infty} \int_{0}^{1} g_{n}(x) \, dx \]

\[ = \int_{0}^{1} -\sin(x) \, dx \quad \text{which is finite} \]

(claiming since \( \int_{0}^{1} |g_{n}(x)| \, dx \leq \int_{0}^{1} \frac{2}{x} \, dx = 4 \) in L.C.T.)

\[ \Rightarrow \int_{0}^{1} \frac{x^{n}}{2n} \, dx < 4 \quad \checkmark \]
4) Let \( f : \mathbb{R} \to \mathbb{R} \) be a Lebesgue integrable function so that \( \int_E f(x) \, dx = 0 \) \( \forall \) set \( E \) with \( m(E) = \pi \). Then \( f = 0 \) a.e.

**Proof:** Claim: If \( E \subseteq \mathbb{R} \) is measurable then \( \forall \, b \in [0, \infty] \), there is a set \( F \subseteq E \) with \( m(F) = b \) \( \text{(F in)} \).

**Proof:** In class, we saw that \( g(x) = m(E \cap (-x,x)) \) is a continuous function \( g : \mathbb{R} \to \mathbb{R} \).

If \( b = 0 \), \( m(E) \) then clearly we are done.

So assume \( 0 < b < m(E) \).

Assume \( m(E) < \infty \), then look at \( A_n = E \cap (-n,n) \).

Clearly, \( E = \bigcup A_n \) \( \forall \) each \( A_n \) is measurable.

\( A_n \subseteq A_{n+1} \subseteq \ldots \) \( A_1 \subseteq \ldots \).

Applying Prop 14 to \((E \setminus A_0) \), we find

\[
\lim_{n \to \infty} m(E \setminus A_n) = m(E) \setminus (\bigcup_{n=1}^{\infty} A_n) = m(E \setminus \bigcup_{n=1}^{\infty} A_n).
\]

\[
\exists \, n \text{ s.t. } m(E \setminus A_n) = m(E) - m(A_n) = m(E) - b
\]

\[
\Rightarrow g(n) = m(A_n) = m(E \setminus (-n,n)) > b
\]

Yet \( g(0) = m(E) = b \).

So by the intermediate value Thm, \( \exists \, 0 < x < n \) s.t. \( g(x) = m(E \cap (-x,x)) = b \).

Now if \( m(E) = \infty \), \( m \left( \bigcup_{n=1}^{\infty} A_n \right) = \sum \infty \wedge \forall K \text{ yet } m \left( \bigcup_{n=1}^{K} A_n \right) = m(E) = \infty \text{ free above).

So for some \( K \), \( m \left( \bigcup_{n=1}^{K} A_n \right) = b \), now apply the result for the finite case.

Write \( f = f^+ - f^- \). If \( m(\text{supp} f^+) = 0 \) then

by the claim \( \exists \) a set \( E \subseteq \text{supp} f^+ \) s.t.

\( m(E) = 0 \Rightarrow \sum_{E} f^+ = \sum_{E} f^- = \sum_{E} f^+ = 0 \) \( (\text{supp} f^- \cap \text{supp} f^+ = \emptyset) \).

Then by Prop 13, \( f^+ = 0 \) a.e. on \( E \), but this is a contradiction since \( f \neq 0 \) on \( E \) \( E \subseteq \text{supp} f^+ \) \( m(E) \neq 0 \).

This gives \( m(\text{supp} f^+) \neq \pi \), a similar argument yields \( m(\text{supp} f^-) < \pi \).

So \( m(E \cap \text{supp} f^+ \cup \text{supp} f^-) = \infty \Rightarrow \exists \) a set \( E \subseteq \mathbb{R} \setminus (\text{supp} f^+ \cup \text{supp} f^-) \) with \( m(E) = \pi - m(\text{supp} f^+) \Rightarrow m(E \cup \text{supp} f^+) = \pi \) \( \left( \text{E and supp} f^+ \text{ disjoint}\right) \).

\[
\Rightarrow \sum_{E \cup \text{supp} f^+} f^+ = \sum_{E \cup \text{supp} f^+} f^- = \sum_{E \cup \text{supp} f^+} f^+ = 0 \Rightarrow f^+ = 0 \text{ a.e. on } E \cup \text{supp} f^+
\]

\[
\Rightarrow m(\text{supp} f^+) = 0 \text{ a sym. arg. shows } m(\text{supp} f^-) = 0 \Rightarrow m(\text{supp} f) = m(\text{supp} f^+ \cup \text{supp} f^-) = 0 \]
Find all \( f : [0,1] \to \mathbb{R} \) with B.V. \( s.t. \) \( f(x) + (T_0^x f)^{1/2} = 1 \) \( \forall x \in [0,1] \) \( \int_0^1 f(x) \, dx = \frac{1}{3} \).

**Proof:** First we note \( f \) is non-increasing since if \( x \in [0,1] \), clearly \( T_0^x f \leq T_0 f \implies 1 - T_0^x f \leq 1 - T_0 f \implies f(y) \leq f(x) \).

So \( T_0^x f = f(0) - f(x) \) since \( f \) is non-increasing.

Also by hypothesis \( T_0^x f = (1 - f(x))^2 \implies f(0) - f(x) = (1 - f(x))^2 \).

Hence \( 1 - f(x) = (1 - f(x))^2 \implies (1 - f(x))(1 - f(x)) = (1 - f(x))(f(x)) \implies f(x) = 1 \) or \( f(x) = 0 \).

Now \( f(x) = \begin{cases} 1 & x \in [0,\frac{1}{3}] \\ 0 & x \in (\frac{1}{3},1] \end{cases} \) or \( f(x) = \begin{cases} 0 & x \in [0,\frac{1}{3}] \\ 1 & x \in (\frac{1}{3},1] \end{cases} \).

To find \( a \), consider \( \int_0^1 f(x) \, dx = c = \frac{1}{3} \).

\( a = \begin{cases} 1 & x \in [0,\frac{1}{3}] \\ 0 & x \in (\frac{1}{3},1] \end{cases} \) or \( a = \begin{cases} 0 & x \in [0,\frac{1}{3}] \\ 1 & x \in (\frac{1}{3},1] \end{cases} \).
If \( f \) is continuous on \([a, b]\) and \( P^+ \) is everywhere non-negative on \((a, b)\), then \( f \) is non-decreasing on \([a, b]\).

**Proof.** First, we assume \( P^+ f > 0 \) then fix \( y \in (a, b) \).

\([a, y]\) is compact so \( f \) attains its max on \([a, y]\) at a point \( c \in [a, y] \).

Now if \( c < y \) then \( y - c = \delta > 0 \), so we can find \( \delta > 0 \) such that \( f(c + \delta) > f(c) \)

since \( \lim_{h \to 0^+} (f(c + \delta) - f(c)) > 0 \). Hence, \( z \in (c, y) \) but \( f(z) < f(c + \delta) \) which contradicts that \( f(c) \) is a max.

\( \therefore z = y \Rightarrow f(y) \) is the maximum value on \([a, y]\) = \( \forall x \leq y, \ f(x) \leq f(y) \).

Since this holds true for arbitrary \( y \), \( f(x) \leq f(y) \) \( \forall x \leq y, \ x, y \in [a, b] \).

Now define \( g_n(x) = f(x) + \frac{1}{n} x \). Then \( g_n(x) \) is continuous on \([a, b]\).

Moreover \( P^+ g_n(x) = \lim_{h \to 0^+} \left( \frac{g_n(x + h) - g_n(x)}{h} \right) = \lim_{h \to 0^+} \left( \frac{f(x + h) - f(x) + \frac{1}{n}(x + h - x)}{h} \right) = \lim_{h \to 0^+} \left( \frac{f(x + h) - f(x)}{h} \right) + \frac{1}{n} \geq \frac{1}{n}. \)

So, by the previous part \( \forall x \leq y, \ g_n(x) \leq g_n(y) \)

\( \therefore \lim_{n \to \infty} g_n(x) \leq \lim_{n \to \infty} g_n(y) \) \( \forall x \leq y \)

\( \Rightarrow f(x) + \lim_{n \to \infty} \left( \frac{1}{n} x \right) \leq f(y) + \lim_{n \to \infty} \left( \frac{1}{n} y \right) \)

\( \Rightarrow f(x) \leq f(y) \) \( \forall x \leq y, \ x, y \in [a, b] \).
Let \( f \) be BV, then \( A \in (a,b) \), \( \lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) \) exists.

**Proof.** Clearly, it is sufficient to show this for monotone functions since if \( f \) is BV, it is the sum of two monotone functions.

So, WLOG, suppose \( f \) is non-decreasing.

\[
S = \{ f(t) | a < c \leq t \} \quad \text{is bounded above by} \quad f(c)
\]

so \( L = \sup S \) exists as a real number.

Fix \( \varepsilon > 0 \), since \( L \) is sup \( S \) \( \exists \delta \) s.t. \( a < c < c + \delta \) \( \Rightarrow \) \( f(c) - f(y) < \varepsilon \)

\[
\forall \varepsilon > 0, \quad L - \varepsilon < f(c) \leq L \quad \text{if} \quad c \in (a,b) \quad \Rightarrow \quad f(t) - f(c) < \varepsilon
\]

A symmetric argument using \( T = \inf \{ f(t) | c \leq t < b \} \) shows \( \lim_{x \to b^-} f(x) \) exists.

A monotone function can have only a countable number of discontinuities.

**Proof.** Suppose \( f \) is non-decreasing, then from the above construction of \( \lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) \), it is clear \( \lim_{x \to \pm \infty} f(x) = f(c^-) - f(c^+) = L \) \( \forall \varepsilon > 0 \)

Let \( E_n = \{ c \in (a,b) | f(c^-) - f(c^+) > \varepsilon \} \)

For each \( n \), the set \( E_n \) is finite since

\[
f(b) - f(a) = \sum_{i=1}^{k} (f(c_i) - f(c_{i-1})) \quad \text{for} \quad (c_i), i = 1, \ldots, k \text{ a finite subset of } E_n
\]

\[
2, \frac{f(b) - f(a)}{\varepsilon} \quad \text{is} \quad \frac{1}{n}
\]

so \( 1 \in E_n \leq \{ f(b) - f(a) \} \)

Now \( E = \bigcup_{n=1}^{\infty} E_n \) is a countable union of finite sets, so it is countable.

If \( c \in E \) then \( f(c^-) = f(c^+) \Rightarrow f \) is continuous.

If \( a < c \in E \) \( \Rightarrow \) \( f \) is continuous at \( c \) so \( E \) contains all points of discontinuity.

The set of discontinuities of \( f \) is finite.
Let $f$ be defined by $f(0) = 0$ and $f(x) = x^2 \sin \left( \frac{1}{x} \right)$ for $x \neq 0$.

Then, $f$ is not of bounded variation.

**Proof:** From calculus we know

$$\frac{1}{\pi} \sum_{t=1}^{\infty} \frac{1}{2n\pi + \pi/2} = \infty \text{ thus, } \sqrt{N} \sum_{t=1}^{N} \frac{1}{2nt + \pi/2}$$

for all $N \in \mathbb{N}$, \exists $M$ s.t. $N \leq M \leq \frac{2M}{2n\pi} - \frac{\sqrt{N}}{2}$

Then consider the partition of $[0, 1]$, $P = \{x_t \}_{t=0}^{2M}$

$x_0 = 1$, $x_{2n+1} = -1$, $x_t = \frac{1}{2n\pi} \sin \left( \frac{1}{2n\pi} \right)

Note: $x_1 = x_3 = x_5 = \ldots$

Then $\xi_P = \sum_{t=1}^{2M} |f(x_t) - f(x_{t-1})|

= \sum_{t=1}^{2M} \left| 0 - f(x_{t-1}) \right| + \sum_{t=2}^{2M} \left| f(x_t) - 0 \right|

= \sum_{t=1}^{2M} \left| \frac{1}{2n\pi} \sin \left( \frac{1}{2n\pi} \right) \right|

= \sum_{t=1}^{2M} \left| \frac{1}{2n\pi} \right| \sin \left( \frac{1}{2n\pi} \right)

\geq N \sqrt{N} \geq N \sqrt{N} \geq N \sqrt{N}

For every $N \in \mathbb{N}$ we can find a partition $P$ so that $\xi_P \geq N \Rightarrow \sup \xi_P = \infty$.

Hence $f$ is not of bounded variation on $[0, 1]$. Notice this proof shows in fact it is not of bounded variation on $[0, 1]$. (Just choose $x_{2n+1} = 0$.)
Let $f$ be absolutely continuous in $[a, b]$, for each $x > 0$.

Does continuity of $f$ at $0$ imply that $f$ is absolutely continuous on $[0, 1]$? No.

Proof) Define $f(0) = 0$. If $f(x) = x^2 \sin \left( \frac{1}{x} \right)$, then it was commented that $f$ is not of bounded variation on $[0, 1]$. So by the contrapositive of Lemma 14, $f$ is not absolutely continuous on $[0, 1]$. Notice $x^2 \geq x^2, \sin \left( \frac{1}{x^2} \right) = -x^2$, so by the squeeze theorem, $\lim_{x \to 0} x^2 \sin \left( \frac{1}{x^2} \right) = 0$. So $f$ is continuous at $0$.

Also, for any $x > 0$, we have $f'$ exists on $[a, b]$ (since $(x^2)' = (x^2)' \sin \left( \frac{1}{x^2} \right)$ exist there).

So if $f = \int_a^b f'(t) \, dt + f(a)$.

So by Thm 14, $f$ is absolutely continuous on $[a, b]$. A.e.

If $f$ is also of bounded variation on $[0, 1]$, then $f$ is absolutely continuous on $[0, 1]$.

Proof) First: suppose that $f$ is a.v. $\Rightarrow f'$ exists. i.e. on $[0, 1]$, $x$ is measurable. (Thm 3)

By the proof of Lemma 14, on $[a, b]$, $f(x) = \int_a^x f'(t) \, dt + f(a)$.

Fix $x \in (0, 1)$ then we have $f$ is continuous at $0$, so $\exists \varepsilon > 0$ s.t. $0 < y < \varepsilon \Rightarrow |f(y) - f(0)| < 1$.

Then since $f$ is b.v. on $[0, 1]$, $f = f'$ is b.v. on $[0, x)$.

We have by the proof of Thm 14, $\int_a^x f'(t) \, dt < 1$ (Only the B.V. of $f'$ is used to get this in the proof).

As $|f'(t)|$ is integrable on $[0, x)$ for any $x \in (0, 1)$.

Moreover $\int_0^x f'(t) \, dt = \int_0^x f'(t) \, dt + \int_0^x f'(t) \, dt = \int_0^x f'(t) \, dt$ a.e. $x \geq \frac{1}{x}$.
Hence $f'(t)$ is non-negative, i.e., $\forall t \in \mathbb{R}, t \geq 0 \Rightarrow f'(t) = f$.

So by the MCT, $\lim_{n \to \infty} \int_{\frac{1}{n}}^{x} f(t) \, dt = \int_{0}^{x} f(t) \, dt$, $\checkmark$

So, $f(x) = \lim_{n \to \infty} f(x) = \lim_{n \to \infty} \left( \int_{\frac{1}{n}}^{x} f'(t) \, dt + f\left(\frac{1}{n}\right) \right)$$ = \left[ \int \frac{f'(t)}{dt} \, dt \right] + \lim_{n \to \infty} f\left(\frac{1}{n}\right)$, $\checkmark$

$= \int_{0}^{x} f'(t) \, dt + f(0)$ (if cont at 0)

Hence, by Thm 14, $f$ is abs. continuous on $[0,1]$.
Let $F$ be absolutely continuous on $[a,b]$ then $T_a^b(F) = \int_a^b F'(t) \, dt$

Proof. Since $F$ is A.C., $F' \in L^1(a,b)$, $F(x) = \int_a^x F'(t) \, dt + F(a)$ by Thm 14.

Notice for $x \geq a$, $F(x) - F(a) = \int_a^x F'(t) \, dt = \int_a^y F'(t) \, dt + (F(x) - F(y))$.

Notice $T_a^b(F) = \int_a^b \left[ \int_a^y F'(t) \, dt \right] \, dy$

$$= \int_a^b \sum_{i=1}^{\infty} F'(x_i) \Delta x_i$$

Now $\int_a^b \int_a^y |F'(t)| \, dt = \int_a^b \sum_{i=1}^{\infty} |F'(x_i)| \Delta y$.

Write $g(x) = \int_a^x F'(t) \, dt + C$, $h(x) = \int_a^x F'(t) \, dt - F(a)$

Then $F(x) = g(x) + h(x) = \int_a^x F'(t) \, dt - F(a)$ so

$g(x) - h(x)$ are absolutely continuous by Thm 14,

$g'$ and $h$ are abs. (cont.) Also, by definition

$F'(x) = (F')' = (F')^- \geq 0 \Rightarrow g(x) - h(x)$ are increasing functions.

Notice that $\int_a^b |F'(t)| \, dt$ is actually equal to $F(b) - F(a)$.

So we have $T_a^b(g) = P_a^b g = \int_a^b F'(t) \, dt$

Also $T_a^b(h) = P_a^b h = -\int_a^b (F'(t))^- \, dt$

Finally notice $f(b) - f(a) = P_a^b f - P_a^b f$

So repeating the proof of Lemma 4, yields $T_a^b(F) = \int_a^b F'(t) \, dt$

To get the second result realize,

$f(b) - f(a) + T_k^b(f(t) = 2P_a^b f + 2P_a^b g$

$\Rightarrow P_a^b f + P_a^b g = g(b) - g(a) = \int_a^b F'(t) \, dt.$

$\Box$