We will prove that \( f = 0 \) a.e.

\[ E_k = \{ x \mid f(x) > \frac{1}{k} \} \]

- We prove that \( \forall n \geq 1, \quad \mu(E_1 \cup \cdots \cup E_n) < \pi \)

(a little bit different from the hint)

Suppose \( A = E_1 \cup \cdots \cup E_n \) has measure \( \mu(A) \geq \pi \)

By Ex. 3 (Aug. 04) \( \exists B \subseteq A \) such that \( B \subseteq A \) and \( \mu(B) = \pi \)

\[ \int_B f \leq \int_B 1 = \frac{1}{n} \mu(B) = \frac{\pi}{n} \]

(since \( B \subseteq A \) and \( \forall x \in A : f(x) > \frac{1}{n} \))

Contradiction to the hypothesis (note that \( \mu(B) = \pi \))

So \( \mu(E_1 \cup \cdots \cup E_n) < \pi \) \( \forall n \).

- By Ex. 1 (p.257) (previous HW)

\[ \{ E_n \} \text{ a countable collection of measurable sets} \]

Then

\[ \mu(\bigcup_{k=1}^{\infty} E_k) = \lim_{n \to \infty} \mu(\bigcup_{k=1}^{n} E_k) \]

By the above part \( \mu(\bigcup_{k=1}^{\infty} E_k) \leq \pi \) \( \forall n \)

\[ \Rightarrow \mu(\bigcup_{k=1}^{\infty} E_k) \leq \pi \]

So \( E = \{ x \mid f(x) > 0 \} = \bigcup_{k=1}^{\infty} E_k \) has measure \( \mu(E) \leq \pi \)

Similarly, we have \( F = \{ x \mid f(x) < 0 \} \) has measure \( \mu(F) \leq \pi \)

(in fact \( m = m \) lebesgue measure)

- Now we prove that \( \int f = 0 \), \( (\int f = 0) \)

Since \( \mu(E), \mu(F) \leq \pi, \mu(R) = \infty \), \( E \cap F = \emptyset \) so that

\[ \mu(E + G) = \mu(E) + \mu(G) = \pi \]

(\( G \) can be taken as an interval with length=\( \pi \))
By hypothesis, \( \int f = 0 \quad \text{on} \quad E \cup G \)

\[ \Rightarrow \int f + \int f = 0 \]

but \( f = 0 \) on \( G \) \( \Rightarrow \int f = 0 \quad \checkmark \)

* Now we prove the following fact:
\[ f : E \rightarrow [0, +\infty] \quad \text{measurable} \]
\[ \int f = 0 \]
Then \( f = 0 \) a.e. on \( E \)

Proof: \( \forall t > 0, \quad t \inf_{E} f \leq f \quad (\text{by def. of characteristic function}) \)

\[ \Rightarrow t \inf_{E} f \leq \int f \]

\[ \Rightarrow \inf_{E} f \leq \frac{1}{t} \int f \quad (\text{Chebyshev's inequality}) \]

Now \( \left\{ f > 0 \right\} = \bigcup_{k=1}^{\infty} \left\{ f > \frac{1}{k} \right\} \)

\[ \Rightarrow \inf_{E} f \leq \sum_{k=1}^{\infty} \frac{1}{k} \int f \leq \sum_{k=1}^{\infty} \frac{1}{k} \int f = 0 \quad \checkmark \]

Since \( \int f = 0 \)

\[ \Rightarrow \inf_{E} f = 0 \quad \Rightarrow \quad f = 0 \quad \text{a.e.} \]

*Now comeback to the problem:

We have \( E^c = \{ x \mid f(x) > 0 \} \).
\[ f : E \rightarrow (0, +\infty) \quad \text{measurable} \]
\[ \int f = 0 \]

By the above observation, \( f = 0 \) a.e. on \( E \) \( \Rightarrow \mu(E) = 0 \).

But \( f(x) > 0 \quad \forall x \in E \)

Apply the same method for \( -f \) on \( F \) we have \( \mu(F) = 0 \). So \( f = 0 \) a.e. on \( F \)
\[ f_n = n \left( \frac{\cos(x + \frac{1}{n}) - \cos x}{\sqrt{x}} \right) \bigg|_{\frac{1}{n}}^{1} \]

We need to compute \( \lim_{n \to \infty} \int_{0}^{1} f_n(x) \, dx \).

\[
\cos(x + \frac{1}{n}) - \cos x = -2 \sin(x + \frac{1}{2n}) \sin \frac{1}{2n}
\]

\[
\Rightarrow |f_n| = \left| \frac{\sin(x + \frac{1}{2n})}{x} \right| \left| \frac{\sin \frac{1}{2n}}{\frac{1}{2n}} \right| \cdot \frac{1}{\sqrt{x}} \bigg|_{\frac{1}{n}}^{1} \bigg| \frac{1}{n}, 1 \bigg|
\]

Note that

\[
\lim_{x \to 0} \frac{\sin x}{x} = 1
\]

and \( 0 < \sin x < x \) when \( 0 < x < \frac{\pi}{2} \)

since \( \frac{1}{2n} \to 0 \) when \( n \to \infty \),

when \( n \) large enough (\( n \gg 0 \)), \( 0 < x + \frac{1}{2n} < \frac{\pi}{2} \) \( \forall x \in [0, 1] \)

and \( \left| \frac{\sin \frac{1}{2n}}{\frac{1}{2n}} \right| < 2 \)

\Rightarrow when \( n \) large enough \( \left| \frac{\sin(x + \frac{1}{2n})}{x} \right| < 1 \)

\[
\left| \frac{\sin \frac{1}{2n}}{\frac{1}{2n}} \right| < 2
\]

\Rightarrow \( n \gg 0 \Rightarrow |f_n| < 2 \left| \frac{1}{\sqrt{x}} \right| \)

but \( \int_{0}^{1} \frac{1}{\sqrt{x}} \, dx = 2 \times \frac{1}{2} \bigg|_{0}^{1} = 2 \times 2 = 4 \)

So we have \( |f_n| < g(x) = \frac{2}{\sqrt{x}} \) when \( n \gg 0 \)

and \( \int_{0}^{1} g(x) < +\infty \).
Apply Lebesgue Convergence Thm (91) we have
\[
\lim_{n \to \infty} \int_0^1 f_n(x) \, dx = \int_0^1 \lim_{n \to \infty} f_n(x) \, dx
\]
\[
f_n(x) = -\sin \left(x + \frac{1}{2n}\right) \frac{\sin \frac{1}{2n}}{x^{3/2}} + \frac{1}{x^{3/2}} \quad \text{if} \quad \left[\frac{1}{n}, 1\right]
\]
\[
\lim_{n \to \infty} f_n(x) = -\frac{\sin x}{x^{3/2}}
\]
\[
0 \leq \int_0^1 \frac{\sin x}{x^{3/2}} \, dx \leq \int_0^1 \frac{1}{x^{3/2}} \, dx < \infty
\]
\[
\Rightarrow \lim_{n \to \infty} \int_0^1 f_n(x) \, dx = -\int_0^1 \frac{\sin x}{x^{3/2}} \, dx < \infty
\]

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Ex5 - Aug 04

It is clear that $T_o^x(f)$ is monotone increasing (and $\geq 0$).

Since $x < y$

\[
0 = a_0 < a_1 < \cdots < a_n < x < a_{n+1} < \cdots < a_m = y
\]

\[
= \frac{1}{x} \left| f(a_1) - f(a_0) \right| + \cdots + \frac{1}{x} \left| f(a_n) - f(a_{n-1}) \right| \leq \frac{1}{x} \left| f(a_1) - f(a_0) \right| + \cdots + \frac{1}{x} \left| f(a_m) - f(a_{m-1}) \right|
\]

\[
= T_o^x(f) \leq T_o^y(f)
\]

\[
T_o^x(f) = \sup \text{LHS} \leq T_o^y(f)
\]

\[
\Rightarrow T_o^x(f) = 0 \quad \text{and} \quad T_o^x(f) \neq\ \\
\Rightarrow \quad \frac{1}{T_o^x(f)} \quad \text{upward}
\]

\[
\text{But} \quad f(x) + \left(\frac{1}{T_o^x(f)}\right)^\frac{1}{2} = \text{const} = 1 \quad \text{downward}
\]

\[
\Rightarrow T_o^x(f) = f(0) - f(x)
\]

So we have
\[
f(x) + f(0) - f(x) = 1, \quad f(x) < 1 \quad \forall x
\]

\[
\Rightarrow f(0) - f(x) = 1 - 2f(x) + f^2(x) \quad \text{(*)}
\]

\[
f(0) + \left(\frac{T_o^x(f)}{2}\right)^\frac{1}{2} = 1 \Rightarrow f(0) = 1
\]
\( f^2(x) = f(x) \quad \forall x \)

\[ \Rightarrow f(x) (f(x) - 1) = 0 \quad \forall x \]

\[ \Rightarrow \begin{cases} f(x) = 0 \\ f(x) = 1 \end{cases} \checkmark \]

Let \( E \subseteq [0,1] \), \( E = \{ x \mid f(x) = 0 \} \)

\[ \int_0^1 f(x) \, dx = \int_0^{[0,1] \setminus E} f(x) \, dx + \int_E f(x) \, dx \]

\[ = \int_0^{[0,1] \setminus E} 1 \, dx = \mu([0,1] \setminus E) = \mu([0,1]) - \mu(E) = 1 - \mu(E) \]

\[ \int_0^1 f(x) = \frac{1}{3} \quad \Rightarrow \quad \mu(E) = \frac{2}{3} \checkmark \quad \overline{E} = \left[ \frac{2}{3}, 1 \right] \]

So \( f(x) = 0 \) or \( f(x) = 1 \quad \forall x \in [0,1] \)

with \( \mu(\{ x \mid f(x) = 0 \}) = \frac{2}{3} \).
Suppose \( g \) is continuous, \( D^+ g \geq \varepsilon > 0 \).

Suppose \( x_0 < y_0 \in [a, b] \) and \( g(x_0) > g(y_0) \).

Since \( D^+ g(x) = \lim \limits_{h \to 0^+} \frac{g(x+h) - g(x)}{h} \geq \varepsilon \quad \forall x \)

\( \Rightarrow \exists \delta > 0 \text{ s.t. } \forall t, 0 < t < \delta_x, \frac{g(x+t) - g(x)}{t} \geq \varepsilon \).

(Note that \( \lim \limits_{x \to y} f(x) = \inf \lim \sup \limits_{\delta > 0} f(x) \), we can see Ex 49, p50)

So \( \forall x' \in [x, x+\delta_x], \ g(x') - g(x) \geq 0 \)

\( \Rightarrow \forall x' \in [x, x+\delta_x], \ g(x') - g(x) \geq 0 \)

Since \( g \) is continuous and \( g(x_0) > g(y_0) \), \( \exists x_1 > x_0 \) and \( y_1 < y_0 \) so that

\[ g(x_1) > g(y_1) \]

\[ x_0 \ | \ x_1 \ | \ y_1 \ | \ y_0 \]

(we can choose \( x_1 < y_1 \) since \( x_0 < y_0 \))

Now since \([x_1, y_1] \) is compact, \( \exists \) finite intervals s.t.

\[ [x_1, y_1] \subset \bigcup \text{finite intervals} \bigcup \{ x, x + \frac{\delta x}{2} \} \]

\[ \frac{x_1}{x_1} \ | \ \frac{x^1}{x^2} \ | \ \frac{x^3}{x^4} \ | \ \frac{x^3 + \delta x}{2} \ | \ \frac{y_1}{y_2} \ | \ \frac{x^3 + \delta x}{2} \ | \ \frac{x^3 + \delta x}{2} \]

By definition of \( \Gamma(x, x + \frac{\delta x}{2}) \) we have

\[ f(y_1) > f(x^3) = f(x^2 + \frac{\delta x^2}{2}) \geq f(x^2) \geq f(x_1) \]

(I write in a concrete case, the general case is similar)

So we get a contradiction.

So \( g \) is monotone increasing.
- Now come back to the problem.

let \( g(x) = f(x) + \varepsilon x \), \( \varepsilon > 0 \)

\[ D^*g(x) = D^*f(x) + \varepsilon \geq \varepsilon \quad \text{since} \quad D^*f \geq 0 \]

\( \Rightarrow g \uparrow \) for every \( \varepsilon \).

Fix \( x, y \in [a, b], \ x < y \).

\( \forall \varepsilon^* > 0. \)

\[ g(x) = f(t) + \frac{\varepsilon^*}{y-x} t \quad \text{by above observation} \]

\( \Rightarrow g(y) - g(x) > 0 \)

\[ \Rightarrow f(y) - f(x) + \frac{\varepsilon^*}{y-x} (y-x) > 0 \]

\( \Rightarrow f(y) - f(x) + \varepsilon^* > 0 \quad \forall \varepsilon^* > 0 \)

\( \Rightarrow f(y) > f(x) \)

and we are done. \( \square \)
Ex 7 - p104: \[ 10/10 \]
a) Since a bounded variation function is a difference of two monotone increasing functions, it is enough to assume that \( f \) is monotone increasing.

We need to show that \( \forall x_n \to c \), \( x_n > c \), \( f(x_n) \to \alpha \).

Since \( \{f(x_n)\} \) is bounded \( \{r \leq f(x_n) \leq f(b)\} \), by Bolzano-Weierstrass theorem, \( \exists \{x_{n_k}\} \) s.t. \( f(x_{n_k}) \to \alpha \).

First, \( f(x_0) > \alpha \ \forall n \). Suppose \( f(x_{n_0}) < \alpha \)

\[
\begin{align*}
& f(x_{n_k}) < \alpha \\
& \Rightarrow f(x_{n_k}) < f(x_{n_0}) < \alpha \ \forall k \gg 0 \\
& \text{Contradiction, since } f(x_{n_k}) \to \alpha.
\end{align*}
\]

Second, we will show that \( f(x_n) \to \alpha \). \( \forall \epsilon > 0 \):

\[
\begin{align*}
f(x_{n_k}) & \to \alpha \\
& \Rightarrow \exists k_0 \text{ s.t. } k \geq k_0 , 0 \leq f(x_{n_k}) - \alpha < \epsilon \\
\end{align*}
\]

Since \( x_n \to c \), \( \exists N \) s.t. \( \forall n \geq N \)

\[
\begin{align*}
c < x_n < x_{n_{k_0}}
\end{align*}
\]

So \( \forall n \geq N \), \( 0 \leq f(x_n) - \alpha < f(x_{n_{k_0}}) - \alpha < \epsilon \)

(since \( f \) is monotone increasing)

So \( f(x_n) \to \alpha \).

* Now we need to show that \( \forall y_n \to c \), \( f(y_n) \) also tends to \( \alpha \).

By the above proof \( \exists \beta = \lim f(y_n) \). Suppose \( \beta < \alpha \)

\[
\begin{align*}
\frac{f(y_n)}{f(x_n)} &= \frac{\beta}{\alpha} \\
\beta &= \frac{\alpha f(x_n)}{f(y_n)} \\
\end{align*}
\]

\( \Rightarrow \exists \ N \text{ s.t. } \forall n \geq N : \beta \leq f(y_n) \leq \beta + \frac{\alpha - \beta}{4} < \alpha \).

But \( x_n \to c \), \( \exists N_1 \) s.t. \( c < x_n < y_n \).
but $f$ is monotone increasing, $f(x_n) \leq f(y_N) < a$

Contradiction to $f(x_n) \geq a \quad \forall \ n$.

Then we are done, i.e., $\forall \ x_n \rightarrow c, \ x_n > c, \ f(x_n) \rightarrow \text{same}$

$\Rightarrow \exists \lim_{x \rightarrow c^+} f(x)$.

And we are done.

The same method for $\lim_{x \rightarrow c^-} f(x)$.

b) First we prove that: for every decomposition

\[ a < x_1 < x_2 < \ldots < x_k < b \]

We have $A = \left[ f(x_{i+}) - f(x_{i-}) \right] + \left[ f(x_{i+}) - f(x_{i-}) \right] + \ldots + \left[ f(x_{k+}) - f(x_{k-}) \right]$ \leq $f(b) - f(a)$

(f monotone increasing)

Choose $c_1, \ldots, c_{k+1}$ s.t.

\[ a < c_1 < x_1 < c_2 < x_2 < \ldots < c_k < x_k < c_{k+1} < b \]

By definition:

\[ c_1 \leq f(x_1) \leq f(x_{i+}) \leq f(c_2) \]

$\Rightarrow f(x_{i+}) - f(x_{i-}) \leq f(c_2) - f(c_1)$

Similarly:

\[ f(x_{i+}) - f(x_{i-}) \leq f(c_3) - f(c_2) \]

\[ f(x_{k+}) - f(x_{k-}) \leq f(c_{k+1}) - f(c_k) \]

$\Rightarrow A \leq f(c_{k+1}) - f(c_1) \leq f(b) - f(a)$ (f monotone increasing)

* $f$ has at most finite discontinuous points $x \sqrt{\text{s.t.} \ \left| f(x^+) - f(x^-) \right| \geq \varepsilon}$

for fixed $\varepsilon > 0$. We can see that (the number of points $\ldots$) \leq \left[ \frac{f(b) - f(a)}{\varepsilon} \right]

(By the above part, $\varepsilon \leq \frac{\left| f(x^+) - f(x^-) \right|}{\varepsilon} \leq f(b) - f(a)$).
Now let $A_k = \{ x \mid f(x^+) - f(x^-) > \frac{1}{k} \}$

We already know that $\# A_k < \infty$.

Let $A = \{ \}$ discontinuous points $\}$. It is clear by def. that $A = \bigcup_{k \geq 1} A_k$. But $\# A_k < \infty$, $\# A$ is at most countable.

Ex 10 - p.104 (Only part a)

$f$ is not bounded variation.

We can restrict to $[0,1]$. Choose $x_k = \frac{1}{\sqrt{2k\pi + \pi}} \{ 0, x_{k+1}, x_k, x_{k+1}, 1 \}$

$$f(x_{k+1}) = \left( \frac{1}{\sqrt{2k\pi + \pi}} \right)^2 \sin\left( 2 (2k\pi + \pi + \pi) \right) = \left( \frac{1}{2k\pi + \pi} \right)^k \sqrt{2}$$

$$f(x_{k+1}) = \frac{1}{\sqrt{2k\pi + \pi + \pi}} \sin\left( 2 (2k\pi + \pi + \pi) \right) = \frac{1}{2k + 3\pi} \sqrt{2}$$

\[ t \geq \sum_{i=1}^{k} \left| f(x_{i+1}) - f(x_{i}) \right| \quad \text{(we omit two extreme subintervals)}
\]

\[ \Rightarrow \ t \geq \left( \left| f(x_{i+1}) + f(x_{i}) \right| + \left| f(x_{i+1}) + f(x_{i}) \right| \right) + \cdots + \left( \left| f(x_{i+1}) + f(x_{i}) \right| \right)
\]

\[ \Rightarrow \ t \geq \sum_{i=1}^{k} \left| f(x_{i}) \right| \quad \text{(since $f(x_k)$ and $f(x_{k+1})$ have opposite signs)}
\]

\[ t \geq \sum_{i=1}^{k} \frac{1}{\pi + \pi} \quad \text{but} \quad \frac{1}{\pi + \pi} \sim \frac{1}{\pi} \quad \text{and} \quad \sum_{i=1}^{\infty} \frac{1}{\pi} > \infty \quad \text{bounded variation} \]
a) The first statement is false.

Consider the function \( f(x) = \begin{cases} x \cos \frac{1}{x} & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases} \)

\( f : [0, 1] \to \mathbb{R} \)

- Since \( |\cos \frac{1}{x}| \leq 1 \Rightarrow x \to 0 \Rightarrow f(x) \to 0 = f(0) \)

so \( f \) is continuous at 0.

- By the same method as in Ex 10a, p. 104, \( f \) is not bounded variation. Hence \( f \) is not absolutely continuous.

(In fact, \( x_k = \frac{1}{k\pi}, \quad t > \sum_{i=1}^{k} |f(x_{i+}) - f(x_i)| \)

\( \Rightarrow t > \frac{1}{\pi} \sum_{i=1}^{k} \frac{1}{i} \quad \checkmark \)

and use the fact that \( \sum_{i=1}^{\infty} \frac{1}{i} > \infty \))

- Now we check that \( f \) is absolutely continuous on \([\varepsilon, 1] \neq \emptyset\).

It is easy to see that if \( f \) satisfies Lipschitz on \([a, b] \) then \( f \) is absolutely continuous on \([a, b] \):

\( f \in \text{Lip}([a, b]) \Rightarrow \exists L \in \mathbb{R} \ | f(y) - f(x) | \leq L |y - x| \)

\( \forall x, y \in [a, b] \).

So \( \forall \varepsilon > 0 \), can choose \( \delta = \frac{\varepsilon}{L} \)

\( \Rightarrow \forall \text{ decomposition s.t. } \sum_{i=1}^{k} |b_i - a_i| \leq \delta \text{ we have:} \)

\( \sum_{i=1}^{k} |f(b_i) - f(a_i)| \leq L \sum_{i=1}^{k} |b_i - a_i| \leq L \delta = \varepsilon \)

In particular \( f \in C^1 \Rightarrow f \) is AC

\( C^1 : \{ f \text{ if } f' \text{ and } f \text{ } \text{continuous}\} \)

Since \( \forall y, x \in [a, b] \), \(|f(y) - f(x)| = |f'(x)| |y - x| \), \( f \) is continuous.

but \( f' \) continuous on \([a, b] \Rightarrow |f'(b)| \leq L \forall \varepsilon \)

\( \Rightarrow f \in \text{Lip}([a, b]) \) and therefore \( f \) is AC.
So now we only need to check that \( f \in C^1[\varepsilon, 1] \) with:

\[
f(x) = \frac{x \cos 1}{x}, \quad x \in [\varepsilon, 1]
\]

\[
f'(x) = \frac{\cos 1}{x} + x \left( -\frac{1}{x^2} \right) \left( -\sin 1 \right) = \frac{\cos 1}{x} + \frac{1}{x} \sin 1
\]

Since \( \varepsilon > 0 \Rightarrow \) it is clear that \( f' \in C[\varepsilon, 1] \) and therefore \( f \in C^1[\varepsilon, 1] \) and we are done

(Main idea: find \( f \) s.t. near 0 \( f \) has "sin graph" \( \Rightarrow \) not \( C^1 \) \( \Rightarrow \) not \( A \)

and outside 0, \( f \) smooth (i.e. \( f \in C^4[\varepsilon, 1] \)).

* If \( f \) is bounded variation, then \( f = g - h \), with \( g, h \) are monotone increasing functions.

Fact: \( g \) monotone increasing \( \Rightarrow \) \( g \) AC.

By Ex. 7a (p.104), \( g \) can has only countable discontinuities\n
\( \Rightarrow \) can assume \( g \) is continuous (measure of discontinuities) = 0\n
\( \Rightarrow \) \( g \) is uniformly continuous (we are on the closed interval)\n
\( \Rightarrow \) \( \forall \varepsilon > 0 \exists s \text{ s.t. } |x - x'| < s \Rightarrow |f(x) - f(x')| < \varepsilon \).

So for every non-overlapping intervals \([x_i, x'_i], (x_j, x'_j), \ldots\)

\( x_i, x'_i \) s.t. \( \sum |x'_i - x_i| < s \Rightarrow \sum |f(x'_i) - f(x_i)| < \varepsilon \)

since \( f \) satisfies \( \alpha) \) and \( f \) is AC.

(?) Suppose we can prove that \( f \) AC on \([\varepsilon, 1] \) \( \Rightarrow \) \( g \) and \( h \) AC

\( g, h \) AC, \( f = g - h \) on \([\varepsilon, 1] \)

Then we can prove for the case \( f = g \) monotone increasing.

(since if \( g, h \) are AC on \([0, 1] \) then \( f = g - h \) AC on \([0, 1] \)

directly from definition)
Now we have $f \in AC \text{ on } [\varepsilon, 1] \quad \forall \varepsilon > 0$

$f$ continuous at $0$.

We will show that $f$ is $AC$ on $[0, 1]$, then we are done.

(By adding a constant, we can assume that $f(x) \geq 0 \quad \forall x \in [0, 1]$)

Since $f$ is $AC$ on $[\varepsilon, 1]$, \quad $(f' = \exists f'(t), f'(t) > 0)$ \quad (Thm 3, and def. of $f$)

$$f(x) - f(\varepsilon) = \int_{\varepsilon}^{x} f'(t) \, dt$$

$$= \int_{0}^{x} f'(t) \cdot 1_{[\varepsilon, x]} \, dt$$

Choose $\varepsilon > 0$ we have

$$f(x) - f(\varepsilon) = \int_{0}^{x} f'(t) \cdot 1_{[\varepsilon_n, x]} \, dt$$

Since $f'(t) > 0 \quad \forall t$ \quad (we can drop set with measure 0)

$\forall t$, $f'(t) \cdot 1_{[\varepsilon_n, x]} \geq 0$, \quad $g_n(t) > 0$ \quad (note that $\varepsilon_n$ decreasing)

and it is clear that when $n \to \infty$ \quad ($\varepsilon_n \to 0$) \quad $g_n(t) \to f'(t)$

So by Monotone Convergence Thm,

$$\lim_{n \to \infty} [f(x) - f(\varepsilon_n)] = \lim_{n \to \infty} \int_{0}^{x} f'(t) \cdot 1_{[\varepsilon_n, x]} \, dt$$

$$= \int_{0}^{x} \lim_{n \to \infty} (f'(t) \cdot 1_{[\varepsilon_n, x]}) \, dt$$

$$= \int_{0}^{x} f'(t) \, dt$$

but $\lim_{n \to \infty} f(\varepsilon_n) = f(0)$ \quad by hypothesis, we have

$$f(x) - f(0) = \int_{0}^{x} f'(t) \, dt$$

$$= \int_{0}^{x} f'(t) \, dt$$

$$= \int_{0}^{1} f'(t) \, dt$$

and we are done.
Ex13 - page 110

10/10

a) \( T^b_a(f) = \int_a^b |f'| \)

- For one direction, we just need BV. In fact:

\[ f \in BV[a, b] \Rightarrow \int_a^b |f'(x)| \, dx \leq T^b_a(f) \]

By Lemma 4:

\[ f(x) = f(a) + \int^x_a \frac{p_x^b(f) - N_x^b(f)}{g(x) - h(x)} \, dx, \quad g, h \text{ R} \]

\[ \Rightarrow f'(x) = \frac{g'(x) - h'(x)}{g(x) - h(x)} \quad \text{a.e.} \]

\[ \Rightarrow |f'(x)| \leq g'(x) + h'(x) \quad (g, h \text{ R} \Rightarrow g'(x), h'(x) \geq 0) \]

\[ \Rightarrow \int_a^b |f'(x)| \, dx \leq \int_a^b g'(x) \, dx + \int_a^b h'(x) \, dx \leq g(b) - g(a) + h(b) - h(a) \quad (\text{Thm 3}) \]

\[ P^b_a(f) - N^b_a(f) \]

(by def. of \( g \) and \( h \))

\[ \Rightarrow \int_a^b |f'(x)| \, dx \leq TV^b_a(f). \]

- Now \( \forall a = x_0 < x_1 < \ldots < x_k = b \)

\[ t = \sum_{i=1}^k |f(x_i) - f(x_{i-1})| = \sum_{i=1}^k \int_{x_{i-1}}^{x_i} |f'(x)| \, dx \leq \sum_{i=1}^k \int_{x_{i-1}}^{x_i} |f'(x)| \, dx \]

\[ \left( f \in AC[a, b] \Rightarrow f \in AC[x_{i-1}, x_i] \right) \quad \text{and Thm 14} \]

\[ \Rightarrow TV^b_a(f) \leq \int_a^b |f'(x)| \, dx \]

\[ \square \]

b) \( p^b_a(f) = \int_a^b |f'|^+ \, dx \quad (r^+ = r^+ r > 0, \quad r^0 \leq r < 0) \)

We know that \( r = r^+ - r^- \)

\[ |r| = r^+ + r^- \]

So \( \int_a^b |f'|^+ - \int_a^b |f'|^- = \int_a^b f(b) - f(a) \quad (f \in AC) \)
On the other hand
\[ \int_a^b |f'|^+ + \int_a^b |f'|^- = \int_a^b |f'| = TV_a^b (f) \text{ by part a.} \]

So
\[ \int_a^b |f'|^+ = \frac{1}{2} \left[ TV_a^b (f) + f(b) - f(a) \right] = PV_a^b (f) \checkmark \text{ (in the proof of lemma 4)} \]

and we are done \( \Box \).