Ex 34, page 183

a) Let $x, y \in E$. We will prove that $[x, y] \subseteq E$.

$(x < y)$ (*$E > 1$*)

Let $z \in [x, y]$. Suppose $z \notin E$. (in particular $z \neq x, y$)

$\Rightarrow E \subseteq (-\infty, z) \cup (z, +\infty)$

And $(-\infty, z) \cap E \cap (z, +\infty) \cap E$ has the following properties:

- nonempty: $x \in (-\infty, z) \cap E, y \in (z, +\infty) \cap E$
- open in $E$: obvious
- $E \subseteq (-\infty, z) \cup (z, +\infty)\n
So $(-\infty, z) \cap E \cap (z, +\infty) \cap E$ is a separation of $E$.

Contradiction to the fact that $E$ is connected.

- Now let $a = \inf E$, $b = \sup E$.

Let $c \in (a, b)$

$c > a \quad \{ \Rightarrow \exists x \in E \text{ s.t. } a \leq x \leq c \}

a = \inf E \quad \text{(by def. of inf.)}

b = \sup E \quad \text{(we have } E \text{ because for example may be the discrete set, inf \& sup cannot happen when we have } E \text{ in this ex.)}

But $x, y \in E$ then $[x, y] \subseteq E$ (above property)

$\Rightarrow c \in E$

Therefore $(a, b) \subseteq E$.

It is clear that $\forall c \in E$, $c \geq a = \inf E$ and $c \leq b = \sup E$

$\Rightarrow E \subseteq [a, b]$. √

Therefore we have $(a, b) \subseteq E \subseteq [a, b] \Rightarrow E$ is an interval $(E = (a, b), [a, b])$. √
b) Let $I = (a, b)$ and $O$ be a subset of $I$ that is both open and closed in $I$. Suppose $O \neq \emptyset$. Fix $x \in O$.

Suppose $O = (a, b) \cap F$, $F$ closed in $\mathbb{R}$.

Then there exists $y \in (x, y^*) \subseteq O$. It is clear that $c > a$.

$(c, y^*) \subseteq O$ for some $(x, y^*) \subseteq O$. And $c < b$.

Hence, we have $a < c < b$.

Suppose $c < b$. Then $c \in (a, b)$.

And by def. of $c$, there exists $y_n$ s.t. $f(x, y_n) < 0$.

$y_n \in O \Rightarrow y_n \in F$.

$F$ closed and $y_n \to c$.

$\exists c_1, c_2 \in \mathbb{R}$ s.t. $c \in (c_1, c_2) < 0$.

By def. of $c$, $\exists c_3$ s.t. $c_1 < c_3 < c$ and $(x, c_3) \subseteq O$.

Therefore $(x, c_3) \cup (c_1, c_2) \subseteq O$.

$0 < c_3 < c_2 < c$.

Hence, $c \in (a, b) \subseteq O$.
Hence, \( c = b \) \( \sqrt{\) 

- Similarly we can prove that \( \inf \{ z, (z, x) : z \leq 0 \} = a. \)

- Now let \( z \in (a, b) \). If \( z = x \) \( \Rightarrow \) \( z \notin 0 \) \( \checkmark \)

If \( x < z < b \)

then by def. of \( c \), \( \exists y^* \) s.t. \( z < y^* < c \)

and \( (x, y^*) < 0 \)

but \( z \in (x, y^*) \) \( \Rightarrow \) \( z \notin 0 \) \( \checkmark \)

If \( a < z < x \), similarly, \( z \notin 0 \) \( \checkmark \)

So we have if \( 0 \neq \phi \) then \( 0 = (a, b) \)

Therefore \( (a, b) \) is connected.

Because \( \overline{(a, b)} = [a, b] \) and we have

\[
(a, b) \subset [a, b] \subset [a, b] = (a, b)
\]

\[
(a, b) \subset (a, b) \subset [a, b] = (a, b)
\]

\[
\Rightarrow \quad [a, b], (a, b], (a, b) = [a, b] \text{ are also connected.}
\]

* Sketch of the proof of Ex 33

It's not \( \rightarrow \) the main problem and the proof is routine. So I skip it.

**Lemma 1** (Lemma 23.4 - Munkres): If \( Y \) is a subspace of \( X \), a separation of \( Y \) is a pair of disjoint nonempty sets \( A \) and \( B \) whose union is \( Y \), neither of which contains a limit point of the other.

(Proof: just investigate carefully what are closed, open sets in \( Y \).

**Ex 33** (Lemma 23.4 - Munkres): let \( A \) be a connected subspace of \( X \). \( A \subset B \subset A \). Then \( B \) is also connected.

Proof: Suppose \( B = C \cup D \) is a separation of \( B \). I proved in Ex 32 that

\( A \cap C \) or \( A \cap D \). If \( A \cap C \). Then \( \overline{A} \subset C \). But \( \overline{C} \cap D = \phi \) (Lemma 1), \( B \cap D = \phi \) (\( \phi \) is the empty set). Contradiction. 

So \( B \) is also connected.