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a) First of all, we prove that the image of a connected space under a continuous map is connected (see Theorem 23.5 - Munkres):

\[ f: X \rightarrow Y \text{ continuous } \Rightarrow f(X) \text{ connected} \]

Because the map \( f: X \rightarrow f(x) \) (restricts the range) is also continuous, we can assume that \( f \) is surjective.

Now \( f: X \rightarrow Y, \text{ continuous, surjective } \Rightarrow Y \text{ connected} \)

\[ X \text{ connected} \]

Suppose \( Y = A \cup B \) a separation of \( Y \)

\[ f^{-1}(A), f^{-1}(B) \text{ open in } X \]

\[ f^{-1}(A), f^{-1}(B) \neq \emptyset \text{ since } f \text{ surjective and } A, B \neq \emptyset. \]

\[ f^{-1}(A) \cap f^{-1}(B) = \emptyset \text{ since } A \cap B = \emptyset \]

and \( X = f^{-1}(A) \cup f^{-1}(B) \) since \( Y = A \cup B \) and \( f \text{ surjective} \)

\[ f^{-1}(A), f^{-1}(B) \text{ is a separation of } X \]

Contradiction.

Now come back to the problem:

If suppose \( X \) is arcwise connected but not connected.

\[ \exists A, B \text{ s.t. } X = A \cup B, A, B \text{ open, } A \neq B \neq \emptyset, A \cap B = \emptyset \]

Let \( a \in A, b \in B. \)

\[ \Rightarrow \exists f: [0,1] \rightarrow X \text{ continuous } \]

\( X \text{ arcwise connected } \) and \( f(0) = a, f(1) = b. \)

But \( [0,1] \text{ connected (Ex34) } \Rightarrow f([0,1]) \text{ connected } \) (above fact)

\( f \text{ continuous} \)
\[ f([0,1]) \subset X = A \cup B \Rightarrow [f([0,1])] \subset A \]
\[ f([0,1]) \text{ connected} \Rightarrow [f([0,1])] \subset CB \]

(I proved in Ex 32)

and this contradicts the fact that \( f(0) = a \in A, f(1) = b \in B \).

Therefore, \( x \) arcwise connected, then \( X \) is connected.

\[ g : (0,1) \to \mathbb{R} \]
\[ x \mapsto \sin \frac{1}{x} \]

\( g \) continuous (clear)

\[ f : (0,1) \to \mathbb{R}^2 \text{ is also continuous} \]
\[ x \mapsto (x, g(x)) \]

\( (0,1) \text{ connected (Ex 34)} \)

\[ \Rightarrow B = \{ (x,y) : y = \sin \frac{1}{x}, 0 < x < 1 \} \text{ is connected } \]

Take a rectangle neighborhood of a point \( E \subset A = \{ (x,y) , x=0, y \in \mathbb{R} \}

(rectangle \sim \text{ circular neighborhood; because they are homeo.})

\[ \exists \, n \text{ large enough s.t. } \frac{1}{2\pi n+1} < \varepsilon \]

\[ \Rightarrow \frac{1}{2\pi n} \leq x < \frac{1}{2\pi n+1} \Rightarrow x \in (0, \varepsilon) \]

\[ \cdots \]

\[ \frac{1}{2\pi n+1} \leq x < \frac{1}{2\pi n+2} \Rightarrow x \in (0, \varepsilon) \]
This observation says that when \( x \in (0, \varepsilon) \), \( \sin \frac{1}{x} \) varies from \([-1, 1]\) infinitely many of times.

This says that the rectangle contains infinitely many points of \( \text{graph}(\sin \frac{1}{x}) = B = \{(x, y) : y = \sin \frac{1}{x}, 0 < x \leq 1\} \).

\[ \Rightarrow A \subset B. \]

Hence \( X = A \cup B = B \) (input \( X = B \), but we don't need).

Now we have \( B \) connected \( \frac{1}{2} X \) connected (Ex 33 - Improved in Ex 34).

\[ \checkmark \]

* \( X \) is not arcwise connected because there is no function \( p : [0, 1] \rightarrow X \)

s.t. \( p(0) = (0, 0) \), \( p(1) = \), \( P \in B = \text{graphg} = \text{graph}(\sin \frac{1}{x}) \)

since \( p : [0, 1] \rightarrow R^2 \) continuous, \( p([0, 1]) \subset X = A \cup B \)

\[ \Rightarrow p([0, 1]) \subset B ; \text{ and } \tilde{p} = \text{pr}_2 \circ p \text{ continuous, } (\text{pr}_1 : R \times R \rightarrow R) \]

If \( p(0) = (0, 0) \)

\[ \Rightarrow \tilde{p}(0) = \text{pr}_2 \circ p(0) = 0 \]

since \( p((0, 1]) \subset B = \text{graphg} \) and for all \( x > 0 \)

\[ \tilde{p}(x) = \text{pr}_2 \circ p(x) = \text{pr}_2(x, g(x)) = g(x) = \sin \frac{1}{x} \]

because \( \tilde{p} \) continuous, \( \forall x > 0 \) then there exist a nbhd \( V \) of \( 0 \) s.t.

\[ \tilde{p}(x) < \varepsilon \forall x \in V \]

But this is a contradiction since \( \sin \frac{1}{x} \) can receive every value between \([-1, 1]\) in every nbhd of \( 0 \) (as we showed above).

Therefore \( X \) is not arcwise connected.

\[ \checkmark \]

b) \( \exists x \in G \)

\[ \exists x \in G \text{ s.t. } \exists \text{ } f \text{ continuous } : [0, 1] \rightarrow G \]

such that \( f(0) = x \), \( f(1) = y \).

We prove that \( H \) is open, closed, \( \neq \emptyset \). Therefore \( H = G \), since \( G \) connected.

c) \( \exists x \in G \)

\[ \exists x \in G \text{ s.t. } \exists \text{ } f \text{ continuous } : [0, 1] \rightarrow G \]

such that \( f(0) = x \), \( f(1) = y \).

We prove that \( H \) is open, closed, \( \neq \emptyset \). Therefore \( H = G \), since \( G \) connected.
\[ H \neq \emptyset \quad \text{x} \in H \quad \text{because } f : [0,1] \rightarrow G \quad \text{connects } x \text{ to } x \\

\text{H open}:

\[
\begin{aligned}
\text{let } y \in H \Rightarrow \exists f : [0,1] \rightarrow & \quad \text{s.t.} \\
& \begin{cases}
  f(0) = x \\
  f(1) = y
\end{cases}
\end{aligned}
\]

\[ \text{let } B_\varepsilon(y) \subset G \quad \text{(since } G \text{ is open)} \]

\[ \text{let } z \in B_\varepsilon(y) \quad B_\varepsilon(y) \text{ is arcwise connected (because we can connect any two points by a straight line - "convex property")} \]

\[ \Rightarrow \exists g : [0,1] \rightarrow B_\varepsilon(y) \subset G \text{ cont.} \]

\[ \text{and } g(0) = y ; g(1) = z \]

\[ \text{Attach } f \text{ and } g \text{ as follow:} \]

\[
\begin{aligned}
h = & \begin{cases}
    f(2s) & \text{s} \in [0, \frac{1}{2}] \\
    g(2s - 1) & \text{s} \in [\frac{1}{2}, 1]
  \end{cases} \\
\text{g(2s-1), f(2s) cont.} \Rightarrow n \text{ is continuous}
\end{aligned}
\]

\[ \text{so by Attaching lemma (Munkres)} \]

\[ \text{and clearly } h(0) = f(0) = x \]

\[ h(1) = g(1) = z \]

\[ \Rightarrow z \in H \Rightarrow B_\varepsilon(y) \subset H \]

\[ \Rightarrow H \text{ is open} \]

\[ \text{H closed: If } y \notin G \setminus H \]

\[ \Rightarrow \exists B_\varepsilon(y) \subset G \text{ s.t. } B_\varepsilon(y) \cap H = \emptyset \]

\[ \text{(if } 2 \in B_\varepsilon(y) \setminus H \text{ then we can connect } x, z \text{ by } f \text{ (by def. of } H) \]

\[ \text{and connect } z \text{ and } u \text{ by } f ; (u, y) \subset G \rightarrow \text{similar to the above proof} \]