Let $B = A + \mathbb{R}$. Then $B$ is a subalgebra of $C(X)$ which separates points and contains the constants. Thus, from the Stone-Weierstrass Theorem $\overline{B} = C(X)$. Suppose that there is a function $f \in A$ such that $f(x) \neq 0 \forall x$. Then $1/f \in C(X) = \overline{B}$. Hence

$$
\frac{1}{f} = \lim_{n \to \infty} (a_n + g_n)
$$

where $g_n \in A$ and $a_n \in \mathbb{R}$. Thus

$$
1 = \lim_{n \to \infty} (a_n f + g_n f)
$$

showing $1 \in \overline{A}$. Hence $\overline{A} = C(X)$.

It follows that if $\overline{A} \neq C(X)$, then every element of $\overline{A}$ has at least one zero. Suppose that there is an $x_o$ such that $f(x_o) = 0 \forall f \in \overline{A}$. We claim that then

$$
\overline{A} = \{ f \in C(X) \mid f(x_o) = 0 \}.
$$

To see this, call the set of functions on the right $D$ and let $f \in D$. Since $\mathbb{R} + \overline{A} = C(X)$, $f = g + a$ for some $g \in \overline{A}$. But then $0 = f(x_o) = a + g(x_o) = a$, showing $a = 0$; hence $f \in \overline{A}$. Thus, $D \subset \overline{A}$. Since, by hypothesis $\overline{A} \subset D$ we see that $\overline{A} = D$, as claimed.

Thus, to finish the proof we need only prove the existence of $x_o$. If no such point exists, then for all $x \in X$ there is an element $f_x \in \overline{A}$ such that $f_x(x) \neq 0$. Replacing $f_x$ with $f_x^2$ allows us to assume that $f_x(y) \geq 0$ for all $y \in X$. For each $x$ there is a ball $B_{\delta_x}(x)$ such that $f_x(y) > 0$ for all $y \in B_{\delta_x}(x)$. Since

$$
X = \bigcup_{x \in X} B_{\delta_x}(x)
$$

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there is a finite sequence $x_i$ such that

$$X = \bigcup_{x_i \in X} B_{\delta_i}(x_i)$$

where $\delta_i = \delta_{x_i}$. But then $g = \sum f_{x_i}$ is an element of $A$ which is nowhere equal to zero, which is a contradiction.