Math 544  HW #7.

2/27/2006

1)  Pro 19. Let $D$ be dense in $\mathbb{R}$, $f$ be an extended real-valued fn on $\mathbb{R}$ \text{s.t.} \( f(x) \geq x^3 \) is measurable for each $x \in D$. Then $f$ is measurable.

**Proof:** Since $D$ is dense in $\mathbb{R}$, for $A \subseteq \mathbb{R}$ \text{ s.t. } \( d_n < y \) and $A - d_n < d_n$ \text{ s.t. } \( f(x) \geq x^3 \Rightarrow \bigwedge_{i=1}^n f(x) > d_n^3 \).

Since \( \bigwedge_{i=1}^n f(x) > d_n^3 \) is measurable for all $n$.

\[ \Rightarrow \bigwedge_{i=1}^n f(x) > d_n^3 \text{ is measurable for each } A \subseteq \mathbb{R}. \]

2) Pro 20. Show that the sum and product of two simple fns are simple.

Show that $X_{A \cap B} = X_a \cdot X_b$, $X_{A \cup B} = X_a + X_b - X_a \cdot X_b$, $X_{\overline{A}} = -X_a$.

**Proof:**

1. Suppose $f$ and $g$ are simple fns. Then $f + g$ and $f \cdot g$ are measurable by proposition 19.

Since $f = \sum_{i=1}^m a_i X_{A_i}$, $g = \sum_{j=1}^n b_j X_{B_j}$

\[ f + g = \sum_{i=1}^m a_i X_{A_i} + \sum_{j=1}^n b_j X_{B_j} = \sum_{i=1}^m \sum_{j=1}^n (a_i + b_j) X_{A_i \cap B_j} \]

$\Rightarrow$ $f + g$ is simple since only assumes finite number of values.

And $f \cdot g = \left( \sum_{i=1}^m a_i X_{A_i} \right) \left( \sum_{j=1}^n b_j X_{B_j} \right) = \sum_{i=1}^m \sum_{j=1}^n (a_i b_j) X_{A_i \cap B_j}$

\[ \Rightarrow f \cdot g \text{ is simple} \]

2. $X_{A \cap B} = \sum_{i=0}^1 X_{A_i} \cdot X_{B_i}$. $X_A = \sum_{a=0}^1 X_{A_a} \cdot X_B = \sum_{a=0}^1 X_{A_a} \cdot X_{A_a} = 0$.

\[ \Rightarrow X_A \cdot X_B = \sum_{a=0}^1 X_{A_a} \cdot X_{B_a} \Rightarrow X_{A \cap B} = X_A \cdot X_B \]

$X_{A \cup B} = \sum_{i=0}^1 (X_{A_i} + X_{B_i}) \cdot X_{A_i} \cdot X_{B_i}$.

$X_{A + X_B - X_A \cdot X_B} = \left\{ \begin{array}{ll} 1 & \text{if } X_{A \cap B} \\ 0 & \text{otherwise} \end{array} \right.$

$X_{A \cap B} = X_A + X_B - X_A \cdot X_B$.

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$X_{A \cap B} = \left\{ \begin{array}{ll} 1 & \text{if } X_{A \cap B} \\ 0 & \text{otherwise} \end{array} \right.$

$X_A = \left\{ \begin{array}{ll} 1 & \text{if } X_{A \cap B} \\ 0 & \text{otherwise} \end{array} \right.$

$X_{A \cap B} = X_A \cdot X_B$.

$X_{A \cup B} = \left\{ \begin{array}{ll} 1 & \text{if } X_{A \cup B} \\ 0 & \text{otherwise} \end{array} \right.$

$X_{A \cup B} = X_A + X_B - X_A \cdot X_B$.

$X_{A \cup B} = \left\{ \begin{array}{ll} 1 & \text{if } X_{A \cup B} \\ 0 & \text{otherwise} \end{array} \right.$

$X_{A \cup B} = X_A + X_B - X_A \cdot X_B$. 

$X_{A \cap B} = X_A \cdot X_B$.
Pro 2.1 (a) D, E measurable. $f : D \times E \to \mathbb{R}$

Show that $f$ measurable $\iff$ $f$'s restrictions to $D$ and $E$ are measurable.

(b) $D$ measurable. $f : D \to \mathbb{R}$. Show that $f$ measurable $\iff \forall x \in D : f(x) = g(x)$ for all $x \in D$ and $g(x) = 0$ for $x \notin D$ is measurable.

**Proof:**

(a) (⇒) $f$ measurable $\implies \forall x : f(x) > a \land D$ is measurable

$\implies \forall x : f(x) > a \land D$ is measurable.

and $\forall x : f(x) > a \land E$ is measurable. \checkmark

(⇐) Since $\forall x : f(x) > a \land D$.

$\implies \forall x : f(x) > a \land D = \left( \forall x : f(x) > a \land D \right) \cup \left( \forall x : f(x) > a \land E \right)$.

is measurable. \checkmark

(b) (⇒) $f$ measurable, $D$ is a measurable subset, $f = g$ on $D$.

Then $f$ and $g$ restricted to $D$ $\implies f$ is measurable.

(⇐) For $x \in D$.

$\forall x : f(x) > a \land D$ = \left( \forall x : f(x) > a \land D \right) \cup \left( \forall x : f(x) > a \land \neg (\epsilon D) \right)$

= \left( \forall x : f(x) > a \land D \right) \cup \left( \forall x : 0 > a \land \neg (\epsilon D) \right)$

is measurable.

$\forall x : 0 > a = \left\{ 0 \right\}$ if $a > 0$.

$\forall x : 0 > a = \mathbb{R}$ if $a < 0$.

$\implies \forall x : g(x) > a$ is measurable. \checkmark
3) P257. Let \( \{ A_n \} \) be a countable collection of measurable sets. Then \( \mu \left( \bigcup_{k=1}^{n} A_k \right) = \sum_{k=1}^{n} \mu(A_k) \).\\

**Proof.** Let \( B_n = \bigcap_{k=1}^{n} A_k \), then \( B_n \in \mathcal{B}_n \). Let \( B_0 = \emptyset \).

Define \( C_n = B_n - B_{n-1} \). Then \( \bigcup_{n=1}^{\infty} C_n = \bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n \).

Therefore, \( \mu \left( \bigcup_{n=1}^{\infty} C_n \right) = \mu \left( \bigcup_{n=1}^{\infty} B_n \right) = \mu \left( \bigcup_{n=1}^{\infty} \bigcap_{k=1}^{n} A_k \right) \).

Since \( \mu \) is countably additive, we have
\[
\mu \left( \bigcup_{n=1}^{\infty} C_n \right) = \sum_{n=1}^{\infty} \mu(C_n) = \sum_{n=1}^{\infty} \left( \mu(B_n) - \mu(B_{n-1}) \right) = \sum_{n=1}^{\infty} \mu(B_n) = \lim_{n \to \infty} \mu \left( \bigcap_{k=1}^{n} A_k \right)
\]

4) See next page.

5) Prove that the set of pts at which a sequence of measurable fn converges is measurable.

**Proof.** Let \( X = \{ x : \lim_{n \to \infty} f_n(x) \exists \} = \{ x : \liminf_{n \to \infty} f_n(x) = \limsup_{n \to \infty} f_n(x) \} \)

Since \( f_n \) are measurable, \( \liminf_{n \to \infty} f_n \) and \( \limsup_{n \to \infty} f_n \) both measurable.

Claim: \( \{ x : f(x) = g(x) \} \) is measurable if both \( f \) and \( g \) are measurable.

**Proof:** \( \{ x : f(x) = g(x) \} = \{ x : f(x) > g(x) \} \cup \{ x : f(x) < g(x) \} \). \( \{ x : f(x) > g(x) \} = \{ x : f(x) < g(x) \} \overline{\cap} \{ x : f(x) < g(x) \} \).

\( \{ x : f(x) < g(x) \} = \left( \bigcap_{n=1}^{\infty} \{ x : f(x) < g(x) \} \right) \cap \{ x : f(x) < g(x) \} \).

where \( n \)'s are enumerated. \( n \)'s numbers.

Since \( f \) and \( g \) are measurable, \( \{ x : f(x) < g(x) \} \) is measurable.

\( \Rightarrow \{ x : f(x) = g(x) \} \) measurable.

Similarly, \( \{ x : f(x) < g(x) \} \) measurable.

Result follows.
4. Page 13 (a) (b): (a). Sequence \( \{f_n: (f_n) \} \) of measurable real-valued functions converges in measure to \( f \) if there exists a subsequence \( f_{n_k} \) converges to \( f \) a.e.

(b). \( \{f_n\} \) sequence of measurable functions vanishes outside a fixed measurable set \( A \) with \( \mu(A) < \infty \), then \( f_n(x) \to f(x) \) a.e. \( \implies f_n \to f \) in measure.

Proof: (a) \( f_n \to f \) in measure \( \implies \frac{\mu(\{x: |f_n(x) - f(x)| > \varepsilon \})}{\mu(A)} \to 0 \) for all \( \varepsilon > 0 \), and all \( n \geq N_n \), which includes \( n \geq N_n \), which includes

\[\mu(\{x: |f_{n_k}(x) - f(x)| > \varepsilon \}) < \frac{1}{2^n} \text{ for all } n \geq N_n, \text{ which includes} \]

Let \( E_n = \{x: |f_{n_k}(x) - f(x)| > \varepsilon \} \).

\[\mu(\bigcap_{n=1}^{\infty} E_n) = \lim_{M \to \infty} \mu(E_n) = \lim_{M \to \infty} \frac{1}{M} = 0 \text{ as } M \to \infty.\]

Let \( E = \bigcap_{n=1}^{\infty} E_n \), for \( x \in E \), \( \Rightarrow x \in \bigcap_{n=M}^{\infty} E_n \) for some \( M \).

\[\Rightarrow x \in \{x: |f_{n_k}(x) - f(x)| < \varepsilon \}\]

\[\Rightarrow \text{if } f_{n_k}(x) \to f(x) \quad \checkmark\]

\[\Rightarrow \mu(\{f_n \to f\}) = \mu(E) = 0 \quad \text{result follows.} \]

(b) \( f_n \to f \) a.e. \( \Rightarrow \mu(\{f_n \to f\}) = 0 \). \( \Rightarrow \) For all \( \varepsilon > 0 \),

\[\mu(\bigcap_{n=1}^{\infty} \{x: |f_n(x) - f(x)| > \varepsilon \}) = 0 \]

\[\Rightarrow \lim_{M \to \infty} \mu(\bigcup_{n=M}^{\infty} \{x: |f_n(x) - f(x)| > \varepsilon \}) = 0 \]

since \( \mu(\{x: |f_n(x) - f(x)| > \varepsilon \}) \leq \mu(\bigcup_{n=M}^{\infty} \{x: |f_n(x) - f(x)| > \varepsilon \}) = 0 \),

\[\Rightarrow \lim_{M \to \infty} \mu(\{x: |f_n(x) - f(x)| > \varepsilon \}) = 0 \]

Result follows. \( \checkmark \)