a. Give an example to show that the Hahn decomposition need not be unique.

Solution:

Let \( \mathcal{X} = [0, 1] \), \( \mathcal{B} \) be the collection of (Lebesgue) measurable subsets of \([0, 1]\), and \( \mu \) be the Lebesgue measure.

A measure is a special case of a signed measure (p. 271); so \( \mu \) is a signed measure.

Let \( A = [0, 1] \), \( B = \emptyset \)
\( C = (0, 1) \), \( D = [0, 1] \)

Then \( A \cap B = \emptyset \), \( A \cup B = \mathcal{X} \) and \( A \) is positive since it is measurable and \( \mu(B) \leq 0 \) for any subset of \( \mathcal{A} \). \( B \) is negative since \( \mathcal{X} \) is also measurable and every measurable subset of \( \mathcal{B} \) has non-positive measure (\( \mu(\emptyset) = 0 \)).

So, \( A \) and \( B \) are a Hahn decomposition.

Similarly, \( C \cap D = \emptyset \), \( C \cup D = \mathcal{X} \), \( C \) is positive, \( D \) is negative, so \( C \) and \( D \) also form a Hahn decomposition.

Since \( A \neq C \) and \( B \neq D \), we have shown that the Hahn decomposition need not be unique.

Problem (part b)

b. Show that the Hahn decomposition is unique except for null sets.

Solution:

Let \( A, B, C, D \) be two different Hahn decompositions, i.e., \( A \) and \( C \) are positive and \( B \) and \( D \) are negative sets.
\[ A \cap C \text{ is positive (as a subset of } A) \]
\[ B \cup D \text{ is negative (as a subset of } B) \]

\[ A \cup C = A \cap D \implies A \cap D \text{ is positive} \]
\[ A \cap D \subset A \implies A \cap D \text{ is negative} \]

\[ B \setminus D = B \cap C \implies B \cap C \text{ is positive} \]
\[ B \cap C \subset B \implies B \cap C \text{ is negative} \]

The two Kuhn decompositions differ by \( A \cup C \) and \( B \setminus D \), which are sets that are both positive and negative, hence they are null sets.

So, the Kuhn decomposition is unique except for null sets.
\[ v(E) = v(E \cap (A_1 \backslash A_2)) + v(E \cap (A_2 \backslash A_1)) = 0 + 0 = 0 \]

since \((A_2 \backslash A_1) \cap (A_1 \backslash A_2) = \emptyset\)

Because \((A_1 \backslash A_2) \cup (A_2 \backslash A_1)\) is a null set, we can consider \(A_1 = A_2\).

Note that we have to consider both \(A_1 \backslash A_2\) and \(A_1 \backslash A_2\) to avoid the following case.

Similarly, since \((B_1 \backslash B_2) \cup (B_2 \backslash B_1)\) is a null set, \(B_1\) can be considered being equal to \(B_2\).

\[ \Rightarrow \] the Hahn decomposition is unique except null sets.

(We can also take \(X = \{-1, 1\} \cup [0, 1] \cup \{2\}, B = \{\text{measurable set on } X\}\)).

Ex28 - p275

Suppose we have \(v = v_1 - v_2 = v_3 - v_4\)

that means \(\exists A, B, C, D\) measurable

\[ \begin{cases} 
 X = A \cup B = C \cup D; \quad A \cap B = C \cap D = \emptyset \\
 v_1(A) = v_2(B) = 0, \quad v_3(C) = v_4(D) = 0 
\end{cases} \]

\(\forall E \in \mathcal{B}\) we will prove that \(v_1(E) = v_3(E)\) then we are done.

(similarly we can prove \(v_2(E) = v_4(E)\)).

\[ v_1(E) = v_1(E \cap A) + v_1(E \cap B) \quad \text{since} \quad X = A \cup B, \quad A \cap B = \emptyset \]

\[ = v_1(E \cap B) \quad \text{since} \quad 0 = v_1(A) = v_1(E \cap A) > 0 \]

(note that \(v_1\) is measure, not signed measure)

\[ = v_1(E \cap B) \quad (\text{since} v_1 = v_1 - v_2, \quad E \cap B \subseteq B \text{ and } v_2(B) = 0) \]
Similarly, we have $v_3(E) = v(E \cap D)$.

Now we use the fact that $B$ and $D$ differ by a null set to imply that $v(E \cap B) = v(E \cap D)$. And then we will have $v_1(E) = v_3(E)$.

In fact, $E \cap B = (E \cap B \cap D) \cup (E \cap B \cap (B \setminus D))$ (by Ex. 27)

$\downarrow$ disjoint

$\Rightarrow v(E \cap B) = v(E \cap B \cap D) + v(E \cap B \cap (B \setminus D))$

$= v(E \cap B \cap D)$ (since $B \setminus D$ is a null set)

Similarly, $v(E \cap D) = v(E \cap B \cap D)$

Hence $v(E \cap B) = v(E \cap D)$

So $v_1(E) = v_3(E)$ \quad $\forall E \in B$

$\Rightarrow$ uniqueness

(from this ex, we can also see that it is good enough to consider $A = B$ when $A \setminus B$ and $B \setminus A$ are null sets.)

Ex 31 - p275

Note that $v^+$, $v^-$ are measures (not signed measures).

Since $|f| \leq M \Rightarrow 0 \leq f^+, f^- \leq M$. By definition of integrable of nonnegative function, we have

$$\int_E f^+ \, dv^+ \quad \int_E f^- \, dv^+ < \infty$$

(note that by def., measure is a nonnegative set function, not extended func.)

$$\Rightarrow -\infty < \int_E f \, dv^- \leq \int_E f^+ \, dv^+ - \int_E f^- \, dv^- < \infty \quad \Rightarrow v^+(E) < \infty$$

So we have $f$ is integrable with respect to $v^+$ (over $E$)

Similarly, $f$ is integrable with respect to $v^-$ (over $E$)
We define integration with respect to a signed measure $\nu$ by defining

$$\int f d\nu = \int f d\nu^+ - \int f d\nu^-$$

If $|f| \leq M \Rightarrow \int_E |f| d\nu \leq M |\nu|(E)$

Moreover there is a measurable function $|f| \leq M$ s.t. $\int_E f d\nu = |\nu|(E)$

(proof)

Notice the first statement follows from $A$ inequality:

$|f| \leq M \Rightarrow \int_E |f| d\nu = |\int_E f d\nu^+ - \int_E f d\nu^-|

\leq |\int_E f d\nu^+| + |\int_E f d\nu^-|

\leq \int_E |f| d\nu^+ + \int_E |f| d\nu^-

\leq M \left( \int_E d\nu^+ + \int_E d\nu^- \right)

= M \left( \nu^+(E) + \nu^-(E) \right)

= M |\nu|(E)$

If $|f| \leq M \Rightarrow \int_E |f| d\nu \leq M |\nu|(E)$

To see the second statement consider the function

$f(x) = \chi_A(x) - \chi_B(x)$, where $\mathcal{F}$ is the Hahn

Decomposition of $\mathbb{X}$ with $A$ positive & $B$ negative.

Since $A \cap B = \emptyset$ if $x \in \mathbb{X} \Rightarrow f(x) = \chi_A = 1$

or $f(x) = \chi_B = 1$

$\Rightarrow |f(x)| \leq 1 \quad \forall x \in \mathbb{X}$
Next observe that $\int_E f(x) \, d\nu(x) =$

$\int_E \chi_A(x) - \chi_E(x) \, d\nu = \int_E \chi_A(x) \, d\nu^+ - \int_E \chi_E(x) \, d\nu^-$

$= \int_E \chi_a(x) \, d\nu^+ - \int_E \chi_b(x) \, d\nu^+ - \int_E \chi_a(x) \, d\nu^- + \int_E \chi_b(x) \, d\nu^-$

$= \gamma(E \setminus A) - \gamma(B \setminus E) - \gamma(A \setminus E) + \gamma(E \setminus B)$

$= \gamma((E \setminus A) \cap A) - \gamma((B \setminus E) \cap A) - \gamma((A \setminus E) \cap B)$

$+ \gamma((E \setminus B) \cap B)$

$= \gamma(E \setminus A) - \gamma(B) - \gamma(A) + \gamma(E \setminus B)$

$= \gamma^+(E) - \gamma^-(E)$

$= \gamma(E)$

$\therefore \exists f \text{ s.t. } |f(x)| \leq 1 \Rightarrow \int_E f(x) \, d\nu = |\gamma(E)|$
We first prove the uniqueness.

Suppose we have \( f, g \) measurable \( \mathbb{X} \to [0, \infty] \) s.t.

\[
\nu(E) = \int_E \mu^\text{d}\mu = \int_E f \, d\mu \quad \forall E \in B
\]

Let \( B = \{ x \in \mathbb{X} : f(x) \neq g(x) \} \). We will prove that \( \mu(B) = 0 \).

Suppose \( \mathbb{X} = \bigcup_{n} X_n \), \( M(X_n) < \infty \ \forall n \).

Since \( \mu(B) = \sum_{n} \mu(B \cap X_n) \) \((\text{Proposition 3})\),

if we prove that \( \mu(B \cap X_n) = 0 \ \forall n \), we will have \( \mu(B) = 0 \).

* Fix \( n \). Suppose \( B \cap X_n = D \cup E \),

with \( D = \{ x \in B \cap X_n : f(x) < g(x) \} \), \( E = \{ x \in B \cap X_n : f(x) \geq g(x) \} \).

Set \( D_n = \{ x \in B \cap X_n : f(x) + \frac{1}{n} < g(x) \} \),

\[
E_n = \{ x \in B \cap X_n : f(x) \geq g(x) + \frac{1}{n} \}.
\]

Then it is clear that \( B \cap X_n = D \cup E \cup \bigcup_{n=1}^{\infty} (D_n \cup E_n) \).

* By Proposition 3 again, to prove that \( \mu(B \cap X_n) = 0 \), it is enough to prove that \( \mu(D_n \cup E_n) = 0 \), or it is enough to prove that \( \mu(D_n), \mu(E_n) \) both equal to 0 \( \forall n \geq 1 \).

* Fix \( n \geq 1 \). \( \forall x \in D_n \) we have \( f(x) + \frac{1}{n} < g(x) \)

\[
\Rightarrow f(x) < \infty \quad \text{(if } f(x) = \infty \Rightarrow g(x) = \infty \Rightarrow f(x) = g(x))
\]

Therefore we have

\[
D_n = \bigcup_{k=1}^{\infty} D_n^k = \bigcup_{k=1}^{\infty} \{ x \in D_n : f(x) \leq k \}
\]

We will prove that \( \mu(D_n^k) = 0 \) and we are done.
Fix $k > 1$. Show that $\mu(D_n^k) = 0$.

On $D_n^k$, $f(x) \leq k$. We have
\[
\nu(D_n^k) = \int_{D_n^k} f \, d\mu \leq \int_{D_n^k} k \, d\mu = k \mu(D_n^k) \leq k \mu(X_\infty) < \infty
\]
\[\text{(since $D_n^k \subset X_\infty$, note that $B \cap X_\infty = \emptyset$)}\]

We also have $g(x) \geq f(x) + \frac{1}{n}$ $\forall x \in D_n^k$ (note that $D_n^k = \bigcup_{k=1}^{\infty} D_n^k$).

So we have
\[
\nu(D_n^k) = \int_{D_n^k} g \, d\mu \geq \int_{D_n^k} (f + \frac{1}{n}) \, d\mu
\]
\[
= \nu(D_n^k) + \frac{1}{n} \mu(D_n^k)
\]
But by (1), $\nu(D_n^k) < \infty$, so (2) $\implies$ $\mu(D_n^k) = 0$ and we are done.

a) $X = X_1 U X_2 U \ldots$

$M(X_\infty) < \infty$

$A_1 = X_1$, $M(A_1) < \infty$

$A_2 = X_1 U X_2$, $M(A_2) \leq M(X_1) + M(X_2) < \infty$

\[\ldots\]

we have $X = A_1 U A_2 U A_3 U \ldots$, $M(A_i) < \infty$

$A_n \subset A_{n+1}$ $\forall n$.

Since $A_n$ finite, we have $f_n$ non-negative measurable function on $A_n$ s.t. $\forall E \in B$, $E \subset A_n$, we have
\[
\nu(E) = \int_{A_n} f_n \, d\mu
\]

By uniqueness, $\forall m > n$, $f_m = f_n \mu$-almost everywhere.
So we can define \( f : X \to [0, \infty] \) as follows.

\[
f = f_1 \quad \text{on } A_1
\]

\[
f = f_n \quad \text{on } A_n \setminus A_{n-1}
\]

\( = f_n \quad \text{on } A_n \setminus A_{n-1} \quad \text{(it is well defined since)}
\]

\[
\forall m > n \quad f_m = f_n \quad \text{on } A_n
\]

Now we need to show that \( f \) is measurable and \( \forall E \subset X \) we have \( \mu(E) = \int_E f \, d\mu \).

(i) \( \forall x \in \mathbb{R} \)

\[
\{ f(x) < \alpha \} = \bigcup_{n=1}^{\infty} \{ x \in A_n \mid f(x) < \alpha \} \quad \text{since } X = \bigcup_{n=1}^{\infty} A_n
\]

\[
= \bigcup_{n=1}^{\infty} \{ f_n(x) < \alpha \}
\]

measurable since \( f_n \) measurable on \( A_n \).

\( \Rightarrow \) \( \{ f(x) < \alpha \} \) measurable.

\( \Rightarrow \) \( f \) is measurable.

(ii) \( \forall E \subset X \). Let \( g_n = f_n \quad \text{on } A_n \)

\[
0 \quad \text{on } \Omega \setminus A_n
\]

We can assume \( f_n \mid_{A_n} = f_{n-1} \quad \forall n \) (since \( f_m = f_n \quad \text{a.e. for every } m > n \), and the integral taken over a set with measure zero is zero.)

Since \( f_n \geq 0 \), from the definition of \( g_n \) we have \( g_n \geq 0 \) and \( g_n \uparrow f \).

It is clear that \( g_n \geq 0 \) and \( g_n \uparrow f \), \( g_n \leq f \quad \forall n \).

\( \Rightarrow \) we can apply the Monotone Convergence Theorem.
We have \( \nu(E \cap A_n) = \int_{E \cap A_n} f \, d\mu \)

\[= \int_{E \cap A_n} g_n \, d\mu \quad \text{(we pass from } f_n \rightarrow g_n \text{ because we want all the domains are the same)} \]

\[= \int_{E \cap A_n} g_n \chi \, d\mu \]

but \( g_n \chi_{E \cap A_n} \rightarrow f \chi_E \) (since \( X = UA_n \)),

\[\quad \text{since } x \in UA_n, \quad x \in E \Rightarrow \exists n_0 \text{ s.t. } x \notin A_n \forall n \geq n_0 \]

\[= \chi_{E \cap A_n}(x) = \chi_E(x) = 1 \forall n \geq n_0 \]

\[= \chi_{E \cap A_n}(x) = \chi_E(x) = 1 \forall n \geq n_0 \]

\[0 \leq g_n \chi_{E \cap A_n} \leq f \chi_E \]

Apply the monotone convergence theorem, we have

\[\lim_{n \to \infty} \nu(E \cap A_n) = \int_{E \cap A_n} \chi \, d\mu \]

\[= \int_{X} f \chi_E \, d\mu = \int_{E} f \, d\mu \]

Now we will show \( \lim_{n \to \infty} \nu(E \cap A_n) = \nu(E) \) then we are done.

But by Ex 1, p 258 (previous HW), we have

\[\nu \left( \bigcup_{k=1}^{\infty} \nu(E \cap A_k) \right) = \lim_{n \to \infty} \nu \left( \bigcup_{k=1}^{n} \nu(E \cap A_k) \right) \]

\[= \lim_{n \to \infty} \nu(E \cap A_n) \quad \text{since } A_n \uparrow \]

\[= \nu(E) = \lim_{n \to \infty} \nu(E \cap A_n) \quad \text{since } X = UA_n \]

So we have \( \nu(E) = \int_{E} f \, d\mu \), i.e., we can pass from finite to \( \nu^* \)-finite.
Ex 3.4 – p239

a) We know that \( \phi(E) = \int_E \, dv \int_{\mu} \, d\mu \)

\[ \Rightarrow \int_E \, dv = \int_E \left( \int_{\mu} \, d\mu \right) \, d\mu \]

\[ \Rightarrow \phi_{cv}(E) = \left( \int_E \, dv = \int_E \left( \int_{\mu} \, d\mu \right) \, d\mu \right) \text{ for any constant } c_i \]

\( \Rightarrow \) we can see that we can prove the result first for the simple functions and then use the Monotone Convergence Theorem.

For simple functions, \( \Phi = \sum_{i=1}^{k} c_i \chi_{E_i} \), good representation, i.e., \( E_i \cap E_j = \emptyset \)

\[ \int_E \phi \, d\mu = \sum_{i=1}^{k} c_i \int_{E_i \cap E_j} \, dv \, (E_i \cap E_j) \Phi \geq 0 \]

\[ \int_E \left( \int_{\mu} \, d\mu \right) \, d\mu = \sum_{i=1}^{k} c_i \int_{E_i \cap E_j} \, d\mu \]

\[ = \sum_{i=1}^{k} c_i \left( \int_{E_i \cap E_j} \, dv \right) \, d\mu \]

\[ = \sum_{i=1}^{k} c_i \int_{E_i \cap E_j} \, dv \, (E_i \cap E_j) \]

\( \Rightarrow \int_E \, dv = \int_E \left( \int_{\mu} \, d\mu \right) \, d\mu \)

\( \Phi \geq 0 \)

Now consider \( f \geq 0 \).

By proposition 7, \( \exists \Phi_n \text{ simple } \Phi_n \uparrow \Phi \)

(If \( f \) defined on \( \sigma \)-finite measure space, \( \Phi_n \) can be chosen s.t. it vanishes outside a set of finite measure)

Since \( \Phi_n \uparrow \Phi \Rightarrow \Phi_n \leq \Phi \)

Now by the Monotone Convergence Theorem we have

\[ \int_E \, dv = \lim_{n \to \infty} \int_E \phi \, d\mu = \int_E \phi \, d\mu \]

(Using the above result)
But \( \frac{dv}{dm} > 0 \), so \( \phi_n \frac{dv}{dm} \leq \frac{dv}{dm} \) for all \( n \), \( \phi_n \to f \) since \( \phi_n \to f \).

Apply the Monotone Convergence Theorem again we have

\[
\int_E dv = \lim_{n \to \infty} \int_E \phi_n \frac{dv}{dm} d\mu = \int_E f \frac{dv}{dm} d\mu \quad \checkmark
\]

and we are done.

b) It is obvious from the uniqueness in the Radon-Nikodym Thm.

We have

\[
(v_1 + v_2)(E) = v_1(E) + v_2(E)
\]

\[
= \int_E \frac{dv_1}{dm} d\mu + \int_E \frac{dv_2}{dm} d\mu
\]

\[
= \int_E \left( \frac{dv_1}{dm} + \frac{dv_2}{dm} \right) d\mu
\]

Uniqueness \( \Rightarrow \) \( \frac{d(v_1 + v_2)}{dm} = \frac{dv_1}{dm} + \frac{dv_2}{dm} \quad \checkmark \)

c) \( v(E) = \int_E \frac{dv}{dm} d\lambda = \int_E \frac{dv}{dm} d\mu \)

\( (v \ll \lambda) \quad (v \ll \mu) \)

But from a) we have \( \int_E \frac{dv}{dm} d\mu = \int_E \frac{dv}{dm} \frac{d\mu}{d\lambda} d\lambda \)

\( (\mu \ll \lambda) \quad (uniqueness) \quad \checkmark \)

d) Apply c) for \( v = \lambda \) we have \( \frac{dv}{dm} = \frac{dv}{d\mu} \frac{d\mu}{d\lambda} \), \( \frac{dv}{d\lambda} = \frac{d\mu}{d\lambda} \).

\( \checkmark \)
37. (a) It is plain that Rev and Inv are signed measures. Hence, Rev = \mu_1 - \mu_2 for \mu_1, \mu_2 the finit measures of the Jordan decomposition. Inv = \mu_3 - \mu_4 + \mu_5, as finite measures. This shows that \nu = \mu_1 + (\mu_2 + \mu_3) - \mu_4 + \mu_5.

(b) It is clear that \nu, \mu_1, \mu_2, \mu_3, \mu_4, \mu_5 are measures and \mu_i < \infty for all i. Hence
\[ \nu(E) = \int_E f \, d\nu. \]

Part (a) is verified by 34a.

b) Suppose \nu(E) = \int_E f \, d\nu = \int_E g \, d\mu. Then \int_E (f-g) \, d\mu = 0 for all measurable E.

Let A be the set for which \nu(E) > 0. Then \int_A f \, d\mu = 0, so
\[ f \equiv 0 \text{ a.e. on } A, \text{ by prop. 13, p. 280. Similarly, } f \equiv 0 \text{ a.e. on } \mathbb{B} \text{ also. As } f \text{ is a.e. on } \mathbb{B} \text{ non-negative or non-positive, we know that } \nu(E) = \int_E f \, d\mu = 0. \]
37c. Suppose \( v = \int_0^1 \phi \, dx = \int_0^1 \phi' \, dx \). Let \( \lambda = \mu + \mu' \). Then \( \lambda \leq \phi \leq \mu \), so

\[
\int_0^1 \phi \left( \frac{d\mu}{dx} \right) \, dx = \int_0^1 \phi' \left( \frac{d\mu}{dx} \right) \, dx = v.
\]

By a process analogous to proving the Riemann–Lebesgue lemma, we have \( \phi \left( \frac{d\mu}{dx} \right) = \phi' \left( \frac{d\mu}{dx} \right) \) a.e. \( \lambda \).

But \( |p| = 1 \), so \( \frac{d\mu}{dx} = \frac{1}{|p|} \frac{dp}{dx} \) a.e. \( \lambda \). But \( \frac{dp}{dx} \) is non-negative everywhere, so \( \frac{dp}{dx} \) is real and non-negative whenever the above equality is true. But this implies \( \phi' / \phi = 1 \) a.e. \( \lambda \). Hence \( v = \int_0^1 \frac{d\mu}{dx} \, dx = \int_0^1 \frac{dp}{dx} \, dx = v \). This shows that \( \mu \) is uniquely determined. Now, if \( v = \int_0^1 \phi \, dx = \int_0^1 \phi' \, dx \) then we must have \( \phi = \phi' \) a.e. \( \lambda \); for if \( \phi = \phi' \) a.e. \( \lambda \), then \( \int_0^1 \phi' \, dx = 0 \) and \( \int_0^1 \phi \, dx = 0 \) a.e. \( \lambda \). (Evidently, to be put and complete, see pp. 263, 264.)

31. a) \[
\int_E \mu = \sup \left\{ \sum a_i \mu(E_i) : \sum a_i x_{E_i} \leq f \right\}
\]

where the sup is taken over all simple \( \sum a_i x_{E_i} \leq f \). The last line follows by Lebesgue's dominated convergence theorem.

b) \[
\int E \left[ \frac{d\mu}{dx} \right] \left[ \frac{d\nu}{dx} \right] \, dx = \nu_1 (E) + \nu_2 (E) \leq \int E \left[ \frac{d(\mu + \nu)}{dx} \right] \, dx
\]

so the result follows by uniqueness.

c) \[
\sigma (E) = \int E \left[ \frac{d\mu}{dx} \right] \, dx = \int E \left[ \frac{d\nu}{dx} \right] \, dx \leq \int E \left[ \frac{d\mu + \nu}{dx} \right] \, dx
\]

by \( \phi \) and \( \psi \) non-negative.

d) Using (c), \[
\left[ \frac{d\mu}{dx} \right] \left[ \frac{d\nu}{dx} \right] = \left[ \frac{d\mu}{dx} \right] = \left[ \frac{d\nu}{dx} \right] = \left[ \frac{d\mu + \nu}{dx} \right] = 0
\]