The Helgason Conjecture
for non-symmetric domains

Richard C. Penney
Purdue University

July 20, 2005

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1 Introduction

Let $X = G/K$ be a Riemannian symmetric space and let $D_G(X)$ be the algebra of all $G$-invariant differential operators on $X$. Let $\mathcal{I} \subset D_G(X)$ be a co-finite ideal. A $C^\infty$ function $F$ on $X$ is $\mathcal{I}$-harmonic if it is annihilated by every element of $\mathcal{I}$. For example, if $\chi$ is a character of $D_G(X)$ and $F$ satisfies

$$XF = \chi(X)F$$

for all $X \in D_G(X)$, then $F$ is $\mathcal{I}$-harmonic where $\mathcal{I}$ is the kernel of $\chi$.

One of the most beautiful results in the harmonic analysis of symmetric spaces is the “Helgason Conjecture”, which states that on a Riemannian symmetric space of non-compact type, a function satisfies 1 if and only if it is the Poisson integral of a hyperfunction over
the Furstenberg boundary. A companion result, due to Oshima and Sekiguchi, [13] says that the boundary hyperfunction is a distribution if and only if there are positive constants $A$ and $r_o$ (depending on $F$) such that

$$|F(x)| \leq Ae^{r_0\tau(x)}$$

for all $x \in X$ where $\tau(x)$ is the Riemannian distance in $X$ from $x$ to the base point $x_o = eK$.

In this work, we begin work on generalizing these results to general connected, homogeneous, Kähler manifolds $X$. Specifically, we assume that $X = G/K$ where $G$ is the connected component of the holomorphic isometry group of $X$ and $K$ is the isotropy subgroup of a point in $X$. In this context, we hope to

1. Define a collection $G$-invariant differential operators $F$ on $C^\infty(X)$ to play the role of $D_G(X)$.
2. Define an appropriate boundary for $X$.
4. Define a “Poisson” transform which reconstructs $F$ from its boundary distribution.

A result of Dorfmeister and Nakajima [6] (generalizing earlier work of Gindikin and Vinberg [8]) states that the general homogeneous Kähler manifold is a holomorphic fiber bundle whose base is a bounded homogeneous domain in $\mathbb{C}^n$ and whose fiber is the product of $\mathbb{C}^k$ with a compact, complex, homogeneous Kähler manifold. Thus, we assume that $X$ is a bounded homogeneous domain in $\mathbb{C}^n$. In this work, we solve (a)-(d).

Concerning (a), in the non-symmetric case, the group of bi-holomorphisms can be quite small, in which case the algebra $D_G(X)$ can be so large that the space of harmonic functions can consist of little more than the constant functions. In particular, holomorphic functions need not be harmonic. Hence, to produce an interesting theory we need a smaller algebra.

In place of $D_G(X)$ we use an algebra of “geometrically” defined invariant differential operators. Specifically, let $T(X)$ be the tangent
bundle for $X$ and let $g$ be the Riemannian form on $T(X) \times T(X)$. Let
\[ g_{her}(Z, W) = g(Z, \overline{W}) \]
be the corresponding Hermitian form on $T_c(X)$ where $g$ is extended to $T_c(X)$ by bi-linearity.

Let $\Delta_{ij}(\cdot)$ be the torsion free Riemannian connection defined by $g$ and let
\[ R(U, V) = \nabla_U \nabla_V - \nabla_V \nabla_U - \nabla_{[U,V]} \]
be the curvature operator. Then for each $k \in \mathbb{N}$, we define sections $\omega^k$ of $(T^*)^{2k}(X)$ by
\[ \omega^k(X_1, Y_1, X_2, Y_2, \ldots, X_k, Y_k) = (-1)^k \text{Tr} \left( \prod_{j=1}^{k} R(X_j, Y_j) \right) \quad (3) \]
It is clear that $\omega^k$ is invariant under any isometry of $X$. Let $T_{geo}(X)$ be the subalgebra of the full tensor algebra $T^*(X)$ generated by the $\omega^k$, $k \geq 1$.

Let $T^{01}(X)$ denote the bundle of complex tangent vectors of type $(0, 1)$ and let $\{Z_j\}_{j=1}^{n}$ be a (local) frame field for $T^{01}(X)$ which is orthonormal with respect to $g_{her}$.

For $f \in C^\infty(X)$ and $\omega \in T_{geo}(X)$ of degree $2k$, we define
\[ D^w f = \sum_{i,j} \omega(Z_{i_1}, \overline{Z}_{j_1}, \ldots, Z_{i_k}, \overline{Z}_{j_k}) \nabla^{2k} f(\overline{Z}_{i_1}, Z_{j_1}, \ldots, \overline{Z}_{i_k}, Z_{j_k}) \quad (4) \]
where $\nabla^k$ denotes the $k$-fold covariant derivative of $f$ and $i$ and $j$ range over the set of multi-indices of length $k$ with entries between 1 and $n$. It is easily seen that these are real differential operators which are independent of the orthonormal frames and thus define canonical differential operators which commute with all holomorphic isometries of the domain; hence, they belong to $D_G(X)$. We extend this definition to all of $T_{geo}(X)$ by linearity in $\omega$.

**Definition 1.** The operator algebra generated over $\mathbb{C}$ by the $D^w$ for $w \in T_{geo}(X)$ is denoted $D_{geo}(X)$.

It should be remarked that a given complex manifold may carry many non-isometric Kähler structures for which the corresponding
group of biholomorphic isometries acts transitively. This is true even if the underlying manifold is bi-holomorphic with a symmetric space. On a bounded homogeneous domain, the Bergman metric yields the largest isometry group since all bi-holomorphisms are automatically isometries. However, the Dorfmeister, Nakajima, Gindikin, Vinberg Theorem does not imply that the induced metric on the base is the Bergman metric. Hence, we are forced to consider more general metrics, even in the symmetric case. The spaces of operators defined above are only guaranteed invariant under the holomorphic isometry group of $X$ which will not typically be the full bi-holomorphism group unless we are actually using the Bergman metric. Fortunately, this all causes only minor complications.

Definition 2. By a “co-finite ideal $\mathcal{I}$”, we mean a co-finite ideal of $D_{\text{geo}}(X)$. In this case we say that $F \in C^\infty(X)$ is $\mathcal{I}$-harmonic if it is annihilated by every element of $\mathcal{I}$.

We use the concept of $\mathcal{I}$-harmonic as a replacement for the harmonicity studied in the semi-simple case.

The next question is, “What should play the role of the Furstenberg boundary in the non-symmetric case?” There seems, in general, to be no way of constructing an analogue of the Furstenberg boundary. We can, however construct what, in the symmetric case, is an open subset of the Furstenberg boundary. Specifically, in general, $G$ is algebraic and has an “Iwasawa” decomposition

$$G = AN_SK$$

where $A$ is an $\mathbb{R}$ split algebraic torus, $N_S$ is a unipotent subgroup normalized by $A$, and $K$ is a maximal compact subgroup. Then $S = AN_S$ acts simply-transitively on $X$.

We identify $X$ with $S$.

As an algebraic variety,

$$S = N_S \times (\mathbb{R}^+)^d \subset N_S \times \mathbb{R}^d$$

where $d$ is the rank of $X$. Under this identification, $N_S$ is contained in the topological boundary of $AN_S$. We use $N_S$ as a substitute for the Furstenberg boundary. In the semi-simple case this amounts to
restricting to a dense, open, subset of the Furstenberg boundary. We refer to $N_S$ as the naive boundary.

We prove the following result in the Hermitian-symmetric case. Our proof carries over to the non-symmetric case. However, in the non-symmetric case, our operator algebras are non-abelian so the concept of regular singularities requires some conditions on the commutators of the operators which we have not been able to verify. (See [10] and [12].) Our techniques do, however, provide an especially simple way of proving the regular singularity property for Hermitian symmetric spaces.

**Theorem 1.** Let $X$ be a Hermitian symmetric space and let $\mathcal{I}$ be a co-finite ideal. There are elements $D_i \in \mathcal{I}$ and elements $Q_i \in (S)$, $1 \leq i \leq d$, such that the system $R(Q_i)D_i$ has regular singularities in the weak sense along the walls $t_i = 0$ with edge $N_S$ where $R$ is the right action of $S$ on $C^\infty(X) = C^\infty(S)$.

Without the regular singularity property, we cannot appeal to the general theory of hyperfunctions to define the boundary values. Instead, we use ideas due to Wallach [16] as extended by van den Ban and Schlichtkrull [1] to construct a family of boundary values on the naive boundary for $F$. To describe these ideas we require some notation. Our basic references for the structure of bounded homogeneous domains are [7] and [15], although we will at times refer the reader to some of our papers where the results are presented in similar notation to our current needs. In particular, the summary given on p. 86-91 and p. 94-97 of [4] covers many of the essentials.

*Throughout this work, we will usually denote Lie groups by upper case Roman letters, in which case the corresponding Lie algebra will automatically be denoted by the corresponding upper case script letter.*

Since the elements of $D_{geo}(X)$ commute with the left action of $S$ on $X = S$, we may consider $D_{geo}(X) \subset (S)$ where the universal enveloping algebra is identified with the left invariant differential operators, in which case we will usually set $D_{geo}(X) = \mathcal{I}_{geo}$.

Let $\mathcal{I} \subset \mathcal{I}_{geo}$ be a co-finite ideal. Let $\mathcal{J} = (S)\mathcal{I} \subset (S)$

be the left ideal generated by $\mathcal{I}$ and

$$\mathcal{P} = (S)/(\mathcal{J} + (S)N_S)$$
Since 
\[(\mathcal{S}) = (A) + (\mathcal{S})\mathcal{N}_S,\]  
(6)

it follows that 
\[\mathcal{P} = (A)/(A) \cap (J + (\mathcal{S})\mathcal{N}_S)\]  
(7)

In particular, \(\mathcal{P}\) is an abelian algebra over \(\mathbb{R}\) which is also an \((\mathcal{S})\)-module. The following result, which is proved in Section 2, is central:

**Proposition 1.** \(\mathcal{P}\) is finite dimensional.

An element \(\tilde{\lambda} \in \mathcal{P}_c^*\) is a root of \(\mathcal{P}\) if there is a non-zero \(X \in \mathcal{P}\) such that 
\[AX = \tilde{\lambda}(A)X\]
for all \(A \in \mathcal{P}\). The roots are characters on \(\mathcal{P}\). In particular, \(\tilde{\lambda}\) is determined by its lift \(\lambda \in \mathcal{A}_c^*\). The set of such functionals in \(\mathcal{A}_c^*\) is denoted \(\mathcal{E}_o\) and is referred to as the set of *characteristic exponents*.

Since \(\mathcal{P}\) is abelian, there is a direct sum decomposition
\[\mathcal{P}_c = \sum_{\alpha \in \mathcal{E}_o} \mathcal{P}_\alpha\]
(8)

where each \(\mathcal{P}_\alpha\) is an ideal in \(\mathcal{P}_c\) and for all \(A \in \mathcal{A}\)
\[(L_A - < A, \alpha >)^{n_\alpha}|_{\mathcal{P}_\alpha} = 0\]
(9)

where \(n_\alpha = \dim \mathcal{P}_\alpha\) and \(L_A\) denotes the action of \(A\) on \(\mathcal{P}\). The roots of \(\alpha\) is, by definition, \(n_\alpha\). Let \(\Sigma \subset \mathcal{A}_c^*\) be the set of roots of \(\mathcal{A}\) on \(\mathcal{N}_S\)–i.e. \(\lambda \in \Sigma\) if and only if there is a non-zero vector \(X \in \mathcal{N}_S\) such that
\[[A, X] = < A, \lambda > X\]

There is an ordered basis \(\lambda_1, \lambda_2, \ldots, \lambda_d\) for \(\mathcal{A}_c^*\) consisting of roots for which the root space of \(\lambda_i\) is a one dimensional subspace \(\mathcal{M}_{ii}\) of \(\mathcal{N}_S\). All of the other roots are one of the following types

1. \(\beta_{ij} = (\lambda_i - \lambda_j)/2\) where \(i < j\),
2. \(\tilde{\beta}_{ij} = (\lambda_i + \lambda_j)/2\) where \(i \leq j\),
3. \(\lambda_i/2\).
The root spaces are denoted, respectively, (a): $\mathcal{S}_{ij}$, (b): $\mathcal{M}_{ij}$, and (c): $\mathcal{Z}_i$. We let $\nu_{ij} = \dim \mathcal{S}_{ij} = \dim \mathcal{M}_{ij}$ and $\nu_i = \dim \mathcal{Z}_i$. Note that some of these dimensions may be 0.

The ordered basis of $\mathcal{A}$ that is dual to the basis formed by $\{\lambda_i\}$ is denoted $\{A_i\}$. Let $\nu_{ij} = \dim \mathcal{S}_{ij} = \dim \mathcal{M}_{ij}$ and $\nu_i = \dim \mathcal{Z}_i$. Note that some of these dimensions may be 0.

The ordered basis of $\mathcal{A}$ that is dual to the basis formed by $\{\lambda_i\}$ is denoted $\{A_i\}$. Let $W^+ = \text{span}_{\mathbb{R}^+} \Sigma$.

Then, $W^+$ is an open cone in $\mathcal{A}^*$ which plays the role of a positive Weyl chamber. Let $\mathcal{A}^+ = \{A \in \mathcal{A} | <A, \lambda> > 0, \lambda \in \Sigma\}$

Finally, let $\mathcal{E} = \mathcal{E}_o + \text{span}_N(\Sigma)$ where $N_0 = N \cup \{0\}$

Now for $r \in \mathbb{R}$, let

$L^1_r(S) = L^1(S, e^{\tau(x)} dx)$

where $dx$ is a choice right-invariant of Haar measure on $S$ and $\tau(x)$ is the Riemannian distance from $x$ to $e$ in $S = X$. Since $S$ acts on $X$ by isometries, it is easily seen that

$\tau(xy) \leq \tau(x) + \tau(y)$

for all $x, y \in S$. It follows that $L^1_r(S)$ is invariant under right translation by elements of $S$. Let $\pi_r$ be the right-regular representation of $S$ in $L^1_r(S)$. Let $\mathcal{H}_\omega(\pi_r)$ (resp. $\mathcal{H}^\infty(\pi_r)$) be the space of analytic vectors (resp. $C^\infty$ vectors) for $\pi_r$—i.e. the space of functions $f \in L^1_r(S)$ for which $g \to \pi_r(g)f$ extends holomorphically to a neighborhood of $e$ in the complexification $S_\mathbb{C}$ of $S$ (resp. is $C^\infty$ on a neighborhood of $e$ in $S$). It follows from Theorem 4 of [11] that $\mathcal{H}_\omega(\pi_r)$ is dense in $L^1_r(S)$. The topology on $\mathcal{H}_\omega(\pi_r)$ is of particular importance to us. Let $\rho(\cdot, \cdot)$ be some metric on the complexification $S_\mathbb{C}$ of $S$ which defines the topology of $S_\mathbb{C}$ and, for $s > 0$, let $B_s \subset S_\mathbb{C}$ be the closed $\rho$-ball of radius $s$ centered at $e$. For each $s > 0$, let $\mathcal{H}_\omega^s(\pi_r)$ be the set of $v \in \mathcal{H}_\omega(\pi_r)$ such that $g \to \pi_r(g)v$ extends continuously to $B_s$ and
holomorphically to the interior of this set. This space is non-zero for all sufficiently small \( s \). For \( v \in \mathcal{H}_s^s(\pi_r) \) let

\[
\|v\|_s = \sup_{g \in B_s} \|\pi_r(g)v\|
\]

where we use the \( L^1_r(S) \) norm on the right. Then \( \mathcal{H}_s^s(\pi_r) \) is a Banach space in this norm. Furthermore, for \( s < t \), there is an obvious injection of \( \mathcal{H}_s^s(\pi_r) \) into \( \mathcal{H}_t^s(\pi_r) \) where the norm of the injection mapping is \( \leq 1 \). The \( \mathcal{H}_s^s(\pi_r) \) topology is defined by the equality

\[
\mathcal{H}_s^s(\pi_r) = \text{Dir lim } \mathcal{H}_s^s(\pi_r)
\]

The dual topology is defined by

\[
\mathcal{H}_{-s}(\pi_r) = \text{Inv lim}(\mathcal{H}_s^s(\pi_r))^*
\]

(See p. 155 and p. 174 of [9] for notation.)

Now let \( F \) be \( \mathcal{I} \)-harmonic on \( X = S \) and satisfy 2. For each \( A \in \mathcal{A} \), let \( F_A \in L^1_r(S)^* \) be defined by

\[
< \phi, F_A > = \int_S \phi(x) F(x \exp A) \, dx.
\] (11)

By restriction, we may also consider \( F_A \) as an element of either \( \mathcal{H}_{-\infty}^s(\pi_r) \) or \( \mathcal{H}_{-s}(\pi_r) \).

The following result is a strengthening of Theorem 3.5 of [1]. The convergence of this expansion, which seems to be new even in the symmetric case, is one of our main results. Our arguments are based on techniques of Baouendi and Goulaouic [3]. (In both [1] and [16], only asymptotic convergence, similar to (a) below, was proven.)

**Theorem 2.** Assume that \( F \in C^\infty(X) \) satisfies 2 and is \( \mathcal{I} \)-harmonic where \( \mathcal{I} \) is either a co-finite ideal in \( D_{\text{geo}}(X) \) or \( X \) is a Riemannian symmetric space and \( \mathcal{I} \) is a co-finite ideal of \( D_G(X) \). Let \( s > 0 \). Then for each \( \alpha \in \mathcal{E} \), there exists a unique \( \mathcal{H}_{-s}(\pi_r) \)-valued polynomial \( F_\alpha \) on \( \mathcal{A} \) (independent of \( s \)) and a \( t_\alpha > 0 \) (which may depend on \( s \)) such that

\[
F_A = \sum_{\beta \in <A,\mathcal{E}>} \left( \sum_{\alpha \in \mathcal{E}, <A,\alpha> = \beta} F_\alpha(A)e^{<A,\alpha>} \right)
\] (12)

for all \( A \in \mathcal{A}_K \), \( |A| > t_\alpha \), where the convergence is in \( (\mathcal{H}_s^s(\pi_r))^* \).

(The inner sum is finite and the outer is countable.) Furthermore
1. For all $A$ and $\alpha$, $F_\alpha(A) \in \mathcal{H}^{-\infty}(\pi^r)$. Hence the $F_\alpha(A)$ define distributions on $S$. Also, for all $s \in \mathbb{R}$ and $A \in \mathcal{A}^-$ there is a finite set $J_s \subset \mathcal{E}$ such that the following set is bounded in $\mathcal{H}^{-\infty}(\pi^r)$:

$$\{e^{-st}(F_{tA} - \sum_{\alpha \in J_s} F_\alpha(A)e^{<A,\alpha>t}) \mid t \in \mathbb{R}^+\}$$

2. The $F_\alpha$ have bounded homogeneous degree. Specifically, for $\alpha \in \mathcal{E}_o$, $\deg F_\alpha < n_\alpha$ where $n_\alpha$ is the multiplicity of $\alpha$.

3. For all $A, B \in \mathcal{A}^-$

$$\pi_r(\exp B)F_\alpha(A)e^{-<A,\alpha>} = F_\alpha(A + B).$$

Remark: Since $C^\infty_c(S) \subset \mathcal{H}^{-\infty}(\pi^r)$, part (a) of Theorem 2 implies that the expansion 12 converges in the sense of distribution valued asymptotic expansions. Hence, our result implies Theorem 3.5, part (i), of [1].

In [1], the asymptotic expansions are over $\mathcal{A}^+$ as $A \to \infty$. This is because they use the parabolic opposite to ours—i.e., they use $\overline{\mathcal{N}}_S$ rather than $\mathcal{N}_S$. The difference is really just a matter of notation. If we think of $\mathcal{N}_S$ as being in the opposite parabolic, then we should call the roots $-\lambda_i$ rather than $\lambda_i$, in which case our $\mathcal{A}^-$ becomes their $\mathcal{A}^+$.

Definition 3. The boundary values of $F$ are the set of polynomials $F_\alpha(A)$ for $\alpha \in \mathcal{E}_o$.

According to the preceding definition, the boundary values are distributions on $S \times \mathcal{A}$. It appears that we have made describing the harmonic functions more difficult in that we have replaced functions on $S$ with distributions on $S \times \mathcal{A}$. It turns out, however, that each boundary function is uniquely determined by a distribution on the $C^\infty_c$ sections of a finite dimensional line bundle over $\mathcal{N}_S$.

To describe this, let $F$ satisfy the same conditions as $F_\alpha$ in conditions (b) and (c) in Theorem 2. For $n = n_\alpha$, let $\mathcal{W}_n$ be the space of polynomial functions on $\mathcal{A}$ of total degree $\leq n$ and let $\rho_n$ be the
representation of $A$ in $\mathcal{W}_n$ defined by right translation. $F$ defines an element of $\mathcal{H}^{-\infty}(\pi^r) \otimes \mathcal{W}_n$, which is the dual space of $\mathcal{H}^{\infty}(\pi^r) \otimes \mathcal{W}_n^*$. The covariance condition becomes

$$\pi_r(a)F = \rho_n(a^{-1})F$$

(13)

for all $a \in A$. (With obvious abuse of notation, we denote $\pi_r \otimes I$ and $I \otimes \rho_n$ by $\pi_r$ and $\rho_n$, respectively.)

Formally, for $\phi \in C^\infty_c(S) \otimes \mathcal{W}_n^*$,

$$< \phi, F > = \int_S < \phi(x), F(x) > \, dx$$

$$= \int_{N_S} \int_A < \phi(na), F(na) > \, d\alpha \, dn$$

(14)

$$= \int_{N_S} < T\phi(n), F(n) > \, dn$$

where

$$T\phi(x) = \int_A \rho_n^*(a)\phi(xa) \, da$$

(15)

and $\rho_n^*$ is the contragredient representation to $\rho_n$ in $\mathcal{W}_n^*$. Then

$$T\phi(xa) = \rho_n^*(a^{-1})T\phi(a).$$

Hence, $T\phi$ is a section of the homogeneous line bundle $L_n$ over $N_S$ defined by

$$L_n = (S \times \mathcal{W}_n^*) / A$$

where the $A$-action is defined by

$$(x, p)a = (xa, \rho_n^*(a^{-1})p)$$

As is well known, and easily shown, $T$ maps $C^\infty_c(S, \mathcal{W}_n)$ onto the space $\Gamma^\infty_c(L_n)$ of $C^\infty$, compactly supported sections of $L_n$. These calculations suggest the following proposition.

**Proposition 2.** For any $F \in \mathcal{H}^{-\infty}(\pi^r) \otimes \mathcal{W}_n$ which satisfies 13, there is a unique element $\tilde{T}F \in \Gamma^\infty_c(L_n)^*$ such that for all $\phi \in C^\infty_c(S) \otimes \mathcal{W}_n^*$,

$$< \phi, F > = < T\phi, \tilde{T}F >$$
Proof For the functional $\hat{T}F$ to be well defined it suffices to show that $T\phi = 0$ implies $<\phi, F> = 0$. If $F$ is a function, this follows from 14. The general case follows by convolving $F$ on the left with a $C_c^\infty$ approximate identity. The continuity of $\hat{T}F$ is due to the observation that kernel of $T$ is a closed subspace of $C^\infty_c(S, \mathcal{W}_n)$. □

It is clear that $\Gamma_c^\infty(L_n) = C_c^\infty(N_S) \otimes \mathcal{W}_n$. Hence, $\Gamma_c^\infty(L_n)^* = \mathcal{D}'(N_S) \otimes \mathcal{W}_n$, implying that each boundary function is uniquely determined by a finite family of distributions on $N_S$.

In the Hermitian symmetric case, our boundary values are the restrictions of those of [1] to the naive boundary. In [2] it was shown that in the Hermitian symmetric case, the function $F$ is uniquely determined by the restrictions of its boundary values to any open subset of the Furstenberg boundary. It is a consequence of our convergence result mentioned above that the same holds in the general case for restrictions to the naive boundary. As in the symmetric case we require all of the boundary values. (For the Furstenberg boundary, there is a distinguished boundary distribution that uniquely determines the solution. This, however, is not true for restrictions to open subsets of the boundary.) We also describe an algorithm for reconstructing all of the $F_\alpha$ from the boundary distributions. From the convergence result mentioned above, this then reconstructs $F$, producing a kind of Poisson transformation.

2 Abstract Asymptotic Expansions

Here we prove the existence and convergence of general asymptotic expansions. The existence, but not the convergence nor the boundedness of the degrees, was already proven in [14].

Let $\mathcal{V}$ be a locally convex, topological vector space over $\mathbb{C}$. For $r \in \mathbb{R}$, let $\mathcal{C}_r$ be the set of $F : (-\infty, 0] \to \mathbb{R}$ such that

$$\{e^{-rt}F(t) \mid t \in (-\infty, 0]\}$$

is bounded in $\mathcal{V}$. Let

$$\mathcal{C} = \cup_r \mathcal{C}_r$$

Let $I \subset \mathbb{C}$ be countable. An exponential series with exponents from $I$ is a formal sum

$$\tilde{F}(t) = \sum_{\gamma \in I} e^{\gamma t}F_\gamma(t)$$

(17)
where $F_\gamma$ is an $V$-valued polynomial. If $I$ is finite the above sum (which is now an element of $C$) is referred to as an exponential polynomial.

Let $\mathcal{F}$ be the family of finite subsets of $I$, directed by inclusion.

**Definition 4.** Let $F \in C$. Given a topology $\mathcal{T}$ on $C$, we say that the exponential series 17 equals $F(t)$ in $\mathcal{T}$ if

$$F(t) = \lim_{J \in \mathcal{F}} \sum_{\gamma \in J} e^{\gamma t} F_\gamma(t)$$

where the limit is in the sense of nets.

The two topologies of interest are

1. The topology of point-wise convergence.
2. The locally convex TVS-topology for which the spaces $C_r$ form a base of neighborhoods of 0. We refer to this as the asymptotic topology. Convergence in this topology is called asymptotic convergence. It is a $T_1$ topology.

Let $D = \frac{d}{dt}$. We consider a differential equation on $C$ of the form

$$P(D)F(t) = N F(t) + G(t) \quad (18)$$

where:

1. $P$ is a polynomial of degree $d$.
2. 

$$N = \sum_{i=1}^{k} e^{\beta_i t} N_i \quad (19)$$

where the $N_i$ are continuous linear operators on $V$ and

$$\text{re } \beta_i > b > 0$$

for all $1 \leq i \leq k$.

3. $G$ is an exponential polynomial with exponents from $\mathcal{E}_1 \subset \mathbb{C}$. 
Note that under these assumptions,
\[ N : C_r \to C_{r+b} \]  
(20)

We factor \( P(D) \) as
\[ P(D) = (D - \alpha_1)(D - \alpha_2) \ldots (D - \alpha_d) \]  
(21)
where some of the roots may be repeated. Let \( a_i = \text{re } \alpha_i \). We assume that the \( \alpha_i \) are ordered so that
\[ a_i \leq a_{i+1}. \]

Let
\[ I = \{ \alpha + \sum_j \beta_j k_j | \alpha \in E_o \cup E_1, k_j \in \mathbb{N}_0 \} \]
where
\[ E_o = \{ \alpha_1, \ldots, \alpha_d \}. \]

**Theorem 3.** Let \( F \in \mathcal{C} \) satisfy 18. Then \( F \) has an asymptotically convergent expansion with exponents from \( I \). Furthermore, the \( F_{\alpha} \) have degrees bounded independently of \( \alpha \).

**Proof** From Corollary 1.7 of [14], it suffices to show that for all \( r \) there is an exponential polynomial \( F_r \) with exponents from \( I \) such that \( F - F_r \in C_r \).

Let \( \Lambda_o^\alpha \) be the integral operator on \( \mathcal{C} \) defined by
\[ \Lambda_o^\alpha(F)(t) = e^{\alpha t} \int_0^t e^{-\alpha x} F(x) \, dx \]
It is easily checked that \( \Lambda_o^\alpha \) is a right inverse for \( D - \alpha \) and
\[ \Lambda_o^\alpha : C_r \to C_{r_o} \]  
(22)
where \( r_o = \min\{ r, \text{re } \alpha \} \). Let
\[ \Lambda^{(0)} = \prod_{i=1}^d (\Lambda_o^{\alpha_i}) \]  
(23)
Then, \( \Lambda^{(0)} \) is a right inverse for \( P(D) \) and

\[
\Lambda^{(0)} : C_r \to C_{r_1}
\]

where \( r_1 = \min\{r, a_1\} \).

Let \( H_0 = \Lambda^{(0)}NF \). Then from 18

\[
P(D)(F - H_0) = NF - NF + G = G
\]

Hence

\[
F - H_0 = \Lambda^{(0)}G + F_0 \equiv H_1
\]

where \( P(D)F_0 = 0 \). Thus, \( F_0, \Lambda^{(0)}G, \) and, \( H_1 \) are exponential polynomials with exponents from \( I \).

We write equation 25 as

\[
(I - \Lambda^{(0)}N)F = H_1
\]

Let

\[
F_n = \sum_{k=0}^{n} (\Lambda^{(0)}N)^k H_1 = (I - (\Lambda^{(0)}N)^{n+1})F
\]

so that

\[
F - F_n = (\Lambda^{(0)}N)^{n+1}F.
\]

\( F_n \) is an exponential polynomial with exponents from \( I \). Also, from 24 and 20, for \( r + nb \geq a_1 \),

\[
\tilde{F} = F - F_n \in C_{a_1}.
\]

Note that

\[
P(D)\tilde{F} = P(D)F - P(D)F_n
\]

\[
= NF + \hat{G}
\]

where

\[
\hat{G} = NF_n - P(D)F_n + G
\]

is an exponential polynomial. Equation 27 has precisely the same form as 18. Hence, we are reduced to the case where \( F \in C_{a_1} \).

For \( r > \text{re} \alpha \), the operator \( \Lambda_\alpha \) defined by

\[
\Lambda_\alpha(F)(t) = e^{\alpha t} \int_{-\infty}^{t} e^{-\alpha x} F(x) \, dx
\]
maps \( C_r \) into itself and is the (two sided) inverse of \( D - \alpha \) on \( C_r \). Also, from 20, for \( r > \text{re} \alpha - b \)

\[
\Lambda N : C_r \to C_{r+b}.
\]  

(29)

We repeat the argument leading up to 27 with \( \Lambda^{(0)} \) replaced by \( \Lambda^{(1)} \) where

\[
\Lambda^{(i)} = \prod_{1}^{i}(\Lambda_{\alpha_{i}}) \prod_{i+1}^{d}(\Lambda_{\alpha_{i}}^{o})
\]  

(30)

Note that from 24 and 29, for \( r > a_j \)

\[
\Lambda^{(j)} : C_r \to C_{r_{j}}
\]  

(31)

where \( r_{j} = \min\{r, a_{j+1}\} \). In particular, \( \Lambda^{(1)}NF \) is defined since \( F \in C_{a_{2}} \). Precisely as before we are able to replace 18 with a similar differential equation where \( F \in C_{a_{2}} \).

We continue, using each of the operators \( \Lambda^{(i)} \) in succession, eventually reducing to the case where \( F \in C_{a_{2}} \). In this case, the asymptotic series is produced by 26 with \( \Lambda^{(0)} \) replaced by \( \Lambda^{(d)} \). The boundedness of the degrees follows from the observation that \( \Lambda_{\alpha} \) preserves degrees of exponential polynomials.

For future reference, we note the following lemma:

**Lemma 1.** If \( F \) satisfies 18 where \( F \) and \( G \) both belong to \( C_r \) then \( F^{(n)} \in C_{r_{o}} \) for all \( n \) where \( r_{o} = \min\{r, \text{re} \alpha_{1}\} \).

*Proof*

Suppose that \( F \in C_r \) is such that \( P(D)F = H \in C_r \). Then

\[
F = \Lambda^{(0)}H + K
\]

where \( P(D)K = 0 \). In particular, \( K \) is an exponential polynomial with exponents from the \( \alpha_{i} \). Hence

\[
(D - \alpha_{1})(D - \alpha_{2}) \ldots (D - \alpha_{k})F = \prod_{i=k+1}^{d}(\Lambda_{\alpha_{i}}^{o}) H + K_k
\]

where \( K_k \) is an exponential polynomial with exponents from the \( \alpha_{i} \). It follows easily by induction on \( k \) that \( F^{(k)} \in C_{r_{o}} \) for \( 0 \leq k \leq \deg P(D) \). Our result now follows by repeated differentiation of 18.

\[\blacksquare\]
Remark: We can actually generalize Theorem 3 to apply to equations 18 where

\[ N = \sum_{i=1}^{k} e^{\beta_i t} P_i(D) N_i \]

where the \( P_i \) are polynomials. In fact, suppose that \( F \) satisfies such an equation where \( F \in C_r \). Let \( \tilde{F} : (-\infty, 0] \to C_r \) be defined by

\[ \tilde{F}(t)(s) = F(t + s) \]

Then \( \tilde{F} \) satisfies

\[ P(D_t) \tilde{F}(t) = \sum_{i=1}^{k} e^{\beta_i t} e^{\beta_i s} P_i(D_s) N_i \tilde{F}(t) + \tilde{G}(s) \]

where \( \tilde{G} \) is a \( C_r \) valued exponential polynomial. Hence, the more general result follows from Theorem 3. We leave the details to the reader as we don’t require the more general result.

Next we define a “Poisson transformation” for 18 with \( G = 0 \). For the remainder of this section, we assume that there is an increasing family of Banach spaces \((\mathcal{V}(s), \| \cdot \|_s)\) for \( s \in \mathbb{R}^+ \), with continuous and dense, injections, such that

\[ \mathcal{V} = \text{Inv lim} \mathcal{V}(s). \]

For example, if \( \mathcal{V}(s) = \mathcal{H}_{-\omega}(\pi_r) \), then \( \mathcal{V} = \mathcal{H}_{-\omega}(\pi_r) \). (See the discussion below for the notation.) The theory also works, with only slight modifications, for inverse limits. We do not, however, require this case.

We say that an operator \( N : \mathcal{V} \to \mathcal{V} \) has degree \( \leq d \) if for all \( 0 < a \leq u < v \leq b \), \( N : \mathcal{V}(u) \to \mathcal{V}(v) \), and there is a constant \( C_N(a, b) \) such that

\[ \| Nw \|_v \leq \frac{C_N(a, b)}{|v - u|^d} \| w \|_u \tag{32} \]

for all \( w \in \mathcal{V}(u) \).

Assume that \( F \) satisfies 18 with \( G = 0 \). We assume also that

\[ \text{degree}N_i \leq d \]
for all $i$ where $d$ is the degree of $P(D)$. By definition, the Poisson transformation maps the $F_\alpha$, $\alpha \in \mathcal{E}_o$, into $F$. We showed in [14] that asymptotic expansions may be differentiated term-by-term. Substitution of 17 into 18 and equating terms with the same exponent shows that

$$P(D)(e^{\gamma t} F_\gamma(t)) = e^{\gamma t} \sum_{i=1}^{k} N_i F_{\gamma - \beta_i}(t). \quad (33)$$

Let $\alpha \in \mathcal{E}$ have $\text{re } \alpha$ minimal with respect to $F_\alpha \neq 0$. Equation 33 shows that $P(D)(e^{\alpha t} F_\alpha(t)) = 0$. Hence, $\alpha$ is a root of $P$. Let the distinct roots of $P$ be $\tilde{\alpha}_1, \ldots, \tilde{\alpha}_l$ so $\alpha = \tilde{\alpha}_j$ for some $j$.

Then $\text{deg } F_\alpha \leq n_j$ where $n_j$ is the multiplicity of $\tilde{\alpha}_j$ as a root of $P$. Let the $\tilde{\alpha}_i$ be ordered so that $\text{re } \tilde{\alpha}_i \leq \text{re } \tilde{\alpha}_j$ for $i \leq j$. For each multi-index $n$ of length $k$, let

$$\gamma(n) = \tilde{\alpha}_j + n_1 \beta_1 + \cdots + n_k \beta_k.$$

Given a $\mathcal{V}(s)$-valued polynomial $H$, we inductively define for each $n \in \mathbb{Z}^k$ a polynomial $H_n$ (which also depends on $j$ through $\gamma(n)$) by the stipulations

1. $H_0(t) = H(t)$.
2. $H_n = 0$ if any of the components of $n$ are negative.
3. $P(D)(e^{\gamma(n)t} H_n)(t) = e^{\gamma(n)t} \sum_{i=1}^{k} N_i H_{n-e_i}(t) \quad (34)$

where $e_i$ is the $i$th standard basis element in $\mathbb{R}^k$.
4. If for some $n \neq 0$, $\gamma(n) = \tilde{\alpha}_k \in \mathcal{E}_o$, then

$$D^m(e^{\gamma(n)t} H_n)(0) = 0 \quad 0 \leq m < n_k$$

where $n_k$ is the multiplicity of $\tilde{\alpha}_k$ as a root of $P$.

We remind the reader that $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.
Proposition 3. Conditions (1)-(4) uniquely determine polynomials $H_n$ which are valued in $\mathcal{V}(u)$ for all $u > s$. Furthermore, for all $u > s$, there is a $t_0 \leq 0$ such that

$$\pi^j(H)(t) = \sum_{n \in \mathbb{N}_0^k} e^{\gamma(n)t} H_n(t)$$

converges in the $\mathcal{V}(u)$ topology for $t \leq t_0$. If $P(D)(e^{\gamma(n)t}H)(t) = 0$, then $\pi^j(H)$ is a $\mathcal{V}(u)$-valued solution to 18 for $t < t_0$.

Proof

Let

$$P^\alpha(D) = P(D + \alpha) = e^{-\alpha t} P(D) e^{\alpha t}$$

Equation 34 can be written

$$P^{\gamma(n)}(D) H_n = Q_n$$

where $Q_n$ is may be assumed (by induction) to be a known polynomial, valued in $\mathcal{V}(u)$ for all $u > s$.

If $\gamma(n) \notin \mathcal{E}_o$, then $P^{\gamma(n)}(D)$ has trivial kernel in the space of polynomials. Hence, in this case, $P(D)^{\gamma(n)}$ maps the space of polynomials of a given degree injectively onto itself. Thus equation 34 has a unique solution $H_n$ in the space of polynomials. It is clear that $H_n$ is valued in $\mathcal{V}(u)$ for all $u > s$.

If $\gamma(n) = \tilde{\alpha}_i \in \mathcal{E}_o$, then

$$P^{\gamma(n)}(D) = D^{\tilde{\alpha}_i} D_o$$

where $D_o$ is bijective on the space of polynomials of a given degree. It follows that 34, together with condition (4), uniquely determines $H(n)$.

To prove convergence, for $r \in \mathbb{R}$, let $\mathcal{C}(u)_r$ be the set of $F \in C^\infty((-\infty, 0], \mathcal{V})$ for which the set 16 is a bounded subset of $\mathcal{V}(u)$. For such $F$, we define

$$\|F\|_{u,r} = \sup_{t \in (-\infty, 0]} e^{-rt}\|F(t)\|_u$$

Let $n_0 \in \mathbb{N}_0^k$ be such that $\text{re } \gamma(n) > \text{re } \tilde{\alpha}_i$ for all $|n| \geq n_0$ where $n \in \mathbb{N}_0^k$ and $|n| = \sum n_j$. (Note that due to the ordering of the roots, this implies that $\text{re } \gamma(n) > \text{re } \tilde{\alpha}_i$ for all $i$.) Let

$$K_n(t) = e^{\gamma(n+n_0)t} H_{n_0+k}(t).$$
Then $K_0 \in C_r$ where $r = \text{re} \gamma(n_o)$ and equation 34 implies that

$$K_n(t) = \sum_{i=1}^{k} \Lambda^{(d)} N_i e^{\beta_i t} K_{n-e_i}(t). \quad (36)$$

For $q \in \mathbb{N}_0$, let

$$\rho(q, u, s) = \sup_{|n|=q} \| K_n \|_{u,s}.$$  

**Lemma 2.** For all $n \in \mathbb{N}_0$ and all $0 < u_o \leq u < v \leq v_o$,

$$\rho(n, u, r + nb) \leq K \left( \frac{dC(u_o, v_o)}{b|u - v|^d} \right)^n \rho(0, v, r)$$

where

$$C(u_o, v_o) = \sum_i C_{N_i}(u_o, v_o).$$

and $K$ is independent of $u, v,$ and $n$.

**Proof**

For simplicity of notation, we let $C(u_o, v_o) = C$. It is easily seen that for $\text{re} \alpha = a$ and $r > a$,

$$\| \Lambda_{\alpha} F \|_{u,r} \leq (r - a)^{-1} \| F \|_{u,r}.$$  

Let $m = \text{re} \tilde{\alpha}_l$. Then for $r > m$,

$$\| \Lambda^{(d)} F \|_{u,r} \leq (r - m)^{-d} \| F \|_{u,r}.$$  

Let $0 < u_o \leq u < v \leq v_o$ be given. It follows from the preceding equality that

$$\| (\Lambda^{(d)} N_i e^{\beta_i t}) F \|_{u,r+b} \leq \frac{C_{N_i}(u_o, v_o)}{(r - m + b)^d |u - v|^d} \| F \|_{v,r} \quad (37)$$

We apply this inequality to 36 with $v$ replaced by $u + \epsilon$ where $\epsilon = (v - u)/n$, and $r$ replaced by $r + (n - 1)b$, finding

$$\rho(n, u, r + nb) \leq \frac{Cn^d}{b^d(n + (r - m)/b)^d |u - v|^d} \rho(n-1, u + \epsilon, r + (n - 1)b)$$
We repeat $n - 1$ more times, with $(u, v)$ replaced by $(u + k\epsilon, u + (k + 1)\epsilon)$, $k = 1, 2, \ldots, n - 1$, finding
\[
\rho(n, u, r + nb) \leq \left( \frac{C}{b^d|u - v|^d} \right)^n \frac{n^{nd} \Gamma((r - m)/b)^d}{\Gamma(n + (r - m)/b + 1)^d} \rho(0, v, r)
\]
Our lemma follows from Stirling’s formula.

The convergence of the series $35$ follows since for all $t \leq 0$,
\[
\|e^{r(n)t}H_n(t)\|_u \leq e^{(r + |n|)b(t)} \rho(|n|, u, r + |n|b)
\]
That $\pi^j(H)$ is a solution to $18$ follows since a series such as $17$ satisfies $18$ if and only if $33$ holds.

**Remark:** The $G = 0$ assumption is made only for convenience. In fact, suppose that $F$ satisfies $18$ with $G$ a non-zero exponential polynomial. Then there is a polynomial $Q$ such that
\[
Q(D)P(D)F = Q(D)NF.
\]
The reasoning from the remark following the proof of Theorem 3 allows us to reduce to the case of Theorem 3. Again, we leave the details to the reader as we don’t require this generality.

The following theorem, together with the uniqueness of the asymptotic expansions, implies the convergence of the asymptotic expansions in the topology of $V(s)$ for all $s > 0$. In the case of direct limits this implies convergence in $V$ since the injection of $V(s)$ into $V$ is continuous.

**Theorem 4.** Let $F \in \mathcal{C}$, satisfy $18$ with $G = 0$. Then for $1 \leq i \leq l$ there exist unique $V$-valued polynomials $H_j$ satisfying $P(D)(e^{\alpha_j t}H_j)(t) = 0$ such that for all $s > 0$ there is a $t_o$ (depending on $s$) such that in the $V(s)$ topology
\[
F(t) = \sum_{j=1}^l \pi^j(H_j)(t)
\]
for all $t \geq t_o$. Furthermore, $\deg H_j < n_j$, where $n_j$ is the multiplicity of $\alpha_j$ as a root of $P$. 
Proof
Let re α be minimal with respect to $F_\alpha \neq 0$. Then, as noted previously, $\alpha = \tilde{\alpha}_j$ is a root of $P$ and $P(D)(e^{\alpha t}F_\alpha)(t) = 0$. Let $H_1 = F_\alpha$. Since $H_1 \in \mathcal{V}$, $H_1 \in \mathcal{V}(s)$ for all $s > 0$. Fix $s > 0$ and set

$$F_1(t) = F(t) - \pi^j(H_1)(t)$$

Then, for sufficiently large $t$, $F_1$ is a $\mathcal{V}(s)$-valued solution to 18 such that $(F_1)\tilde{\alpha}_j = 0$. We repeat this argument $l$ times, producing $H_j$ such that

$$F_l = F - \sum_{j=1}^{l} \pi^j(H_j)$$

is a $\mathcal{V}(s)$-valued solution to 18 with all of its boundary functions zero. The following lemma shows that then $F_l = 0$, proving our theorem.

**Lemma 3.** Suppose that $F \in C_{r_\alpha}$ satisfies 18 with $G = 0$. If $F_\alpha = 0$ for all $\alpha \in E_\alpha$, then there is a $t_o$ such that $F(t) = 0$ for all $t \leq t_o$.

**Proof** It follows by induction from 33 that $F_\alpha = 0$ for all $\alpha$. Then Theorem 3 implies that $F \in C_r$ for all $r \in \mathbb{R}$. Since $\Lambda^{(d)}$ is a left inverse for $P(D)$ on $C_r$ for sufficiently large $r$, we have

$$F = \Lambda^{(d)}NF.$$  

Thus

$$F = (\Lambda^{(d)}N)^nF$$

for all $n \in \mathbb{N}$. Reasoning as in the proof of Lemma 2 using 37, we see that for all $0 < u_o \leq u < v \leq v_o$,

$$\|F\|_{u,r+nb} = \|(\Lambda^{(d)}N)^nF\|_{u,r+nb} \leq K\left(\frac{e^{dC(u_o,v_o)}}{b^d|u-v|^d}\right)^n\|F(t)\|_{v,r}$$

Our lemma follows by letting $n$ tend to infinity in

$$\|F(t)\|_u \leq e^{(r+nb)t}\|F\|_{u,r+nb}$$

The following result follows from Theorem 3 and the uniqueness of the asymptotic expansions.
Theorem 5. Assume that \( F \in \mathcal{C}_r \) satisfies 18 where the \( N_i \) in 19 have \( \deg N_i \leq \deg P \). Then for all \( s > 0 \) there is a \( t_o \leq 0 \) such that the expansion from Definition 4 converges pointwise in the \( \mathcal{V}(s) \) topology for all \( t \leq t_o \).

Next we consider multi-variable expansions. By a \( \mathcal{V} \)-valued exponential series on \( \mathbb{R}^n \) with exponents from \( \mathcal{E} \in \mathbb{C}^n \) we mean a formal sum of the form

\[
\tilde{F}(t) = \sum_{\gamma \in \mathcal{E}} e^{t\gamma} F_\gamma(t)
\]

where the \( F_\gamma \) are \( \mathcal{V} \)-valued polynomials on \( \mathbb{R}^n \). If \( \mathcal{E} \) is finite, then \( \tilde{F}(t) \) is an exponential polynomial.

Let \( \beta \in \mathbb{R}^n \). We say that \( F \in \mathcal{C}_\beta \) if

\[
\{e^{-\beta t} F(t) \mid t \in (-\infty, 0]^n\}
\]

is bounded in \( \mathcal{V} \). We say that \( F \in \mathcal{C}_\beta^\infty \) if for all multi-indecies \( j \) of length \( n \),

\[
D^j F = D_1^{j_1} D_2^{j_2} \ldots D_n^{j_n} F \in \mathcal{C}_\beta.
\]

For \( \beta \) and \( \gamma \) in \( \mathbb{R}^n \), we say

\[
\beta \succ \gamma
\]

if, for all \( i, \beta_i > \gamma_i \). If \( B \subset \mathbb{R}^n \) is finite, we define \( \inf B \), to be the vector \( c \) where \( c_i = \min \{b_i \mid (b_1, \ldots, b_n) \in B\} \).

Suppose that \( F \in \mathcal{C}_\beta \). We say that a series as in 38 is an asymptotic expansion for \( F \) if for all \( \gamma \in \mathbb{R}^n \), there is a finite subset \( \mathcal{E}(\gamma) \subset \mathcal{E} \) such that

\[
F - F^\gamma \in \mathcal{C}_\gamma
\]

where \( F^\gamma \) is the sum in 38 with \( \mathcal{E} \) replaced by \( \mathcal{E}(\gamma) \).

Proposition 4. Suppose that \( F \in \mathcal{C}_\beta \) has an asymptotic expansion as in 38. Then both the subset \( \mathcal{E} \) and the polynomials \( F_\gamma \) are uniquely determined.

Proof

Our Proposition clearly follows from the following lemma.

Lemma 4. Suppose that \( \tilde{F} \) is an exponential polynomial as in 38 where \( \tilde{F} \in \mathcal{C}_\gamma \). Then \( F_\alpha = 0 \) for all \( \gamma \succ \re \alpha \).
Proof Let \( U \) be the set of \( X \in (-\infty,0)^n \) such that \( X \cdot \alpha \neq X \cdot \beta \) for all \( \alpha, \beta \in E \). Since \( E \) is finite, \( U \) is an open, not necessarily convex, cone. For \( X \in U \),

\[
G(t) = \tilde{F}(-tX)
\]

is an exponential polynomial on \((-\infty,0]\) with exponents from \(-X \cdot E\) which belongs to \( C_{-X^{-}} \). Since \( X \in U \),

\[
G^{-X \cdot \alpha}(t) = F_{\alpha}(-tX)
\]

for all \( \alpha \in E \). For \( \gamma > \text{re} \alpha \) and \( X \in (-\infty,0)^n \), \( -X \cdot \gamma > -X \cdot \text{re} \alpha \). Hence, \( G_{-X^{-} \alpha}(t) \equiv 0 \). Our lemma follows since a polynomial which is zero on an open subset is zero.

Let the general variable \( t \in \mathbb{R}^n \) be \( t = (t_1, \ldots, t_n) \) and let \( D_j = \frac{d}{dt_j} \). We assume that \( F : (-\infty,0]^n \to \mathcal{V} \) satisfies a system of differential equations of the form

\[
P_j(D_j)F(t) = N_jF(t) + G_j(t) \quad (41)
\]

for \( 1 \leq j \leq n \) where:

1. The \( P_j \) are polynomials of degree \( d_j > 0 \).

2. For \( 1 \leq j \leq n \),

\[
N_j = \sum_{i=1}^{k} e^{t_{i} \beta_{i,j}} N_{ij} \quad (42)
\]

where the \( N_{ij} \) are continuous linear operators on \( \mathcal{V} \) and \( \beta_{i,j} = (\beta_{i,j}^1, \ldots, \beta_{i,j}^n) \in \mathbb{C}^n \) satisfies

\[
\text{re} \beta_{i,j}^k > b > 0
\]

where \( b \) is independent of \( i, j, k \).

3. The \( G_j \) are exponential polynomials in \( t \) with exponents from \( \mathcal{E}_1 \in \mathbb{C}^n \).

We factor \( P_j \) as in 21 where the \( \alpha_i \) corresponding to \( P_j \) are denoted \( \alpha_i^j \). We assume that for each \( j \), the \( \alpha_i^j \) are ordered so that \( \text{re} \alpha_i^j \leq \text{re} \alpha_{i+1}^j \) for all \( i \). Let

\[
a_i^j = \begin{cases} 
\text{re} \alpha_i^j & 1 \leq i \leq d_j \\
\infty & i > d_j 
\end{cases} \quad (43)
\]
We also let
\[ a^i = (a^i_1, a^i_2, \ldots, a^i_n). \]

Let \( E^j_0 \) be the set of roots of \( P_j \) and let
\[ E_0 = E^1_0 \times \ldots \times E^n_0 \subset \mathbb{C}^n \]
\[ I = \{ \alpha + \sum_{j} \beta_j k_{ij} \mid \alpha \in E_0 \cup E_1, k_{ij} \in \mathbb{N}_0 \}. \]

Let \( \Lambda^\alpha_0 \) and \( \Lambda^\alpha_1 \) be, respectively, the analogues of the operators \( \Lambda^\alpha_0 \) and \( \Lambda^\alpha_1 \) from 22 and 28 defined by integration in the \( j \)th variable. The analogs of the \( \Lambda^{(i)} \) are the operators defined by
\[ \Lambda^{(i)}_j = \prod_{k=1}^{i} (\Lambda^{(i)}_{\alpha_k^j}) \prod_{k=i+1}^{d_j} (\Lambda^{j,\alpha}_0). \]
(44)

where \( 1 \leq i \leq d_j \). For \( i > d_j \), we set \( \Lambda^{(i)}_j = \Lambda^{(d_j)}_j \).

\( \Lambda^{j,\alpha}_0 \) is defined on \( C_\beta^\infty \) for all \( \beta \) while \( \Lambda^{j,\alpha}_1 \) is defined on \( C_\beta^\infty \) for \( \beta_j > \text{re} \alpha_k^j \). Hence \( \Lambda^{(i)}_j \) is defined on \( C_\beta^\infty \) as long as
\[ \beta_j > \text{re} \alpha_k^j, \quad 1 \leq k \leq i. \]
(45)

We also let
\[ \Delta^i_j = I - \Lambda^{(i)}_j P(D_j) \]
(46)
and
\[ N^i = \Lambda^{(i)}_1 N_1 - \Lambda^{(i)}_2 \Delta^1_1 N_2 - \cdots - \Lambda^{(i)}_n \Delta^1_1 \Delta^2_2 \cdots \Delta^i_{n-1} N_n. \]

Note that for \( i \geq d_j \), \( \Delta^i_j = 0 \).

Since on its domain, \( \Lambda^\alpha_0 \) is a \textit{two-sided} inverse for \( (D_j - \alpha I) \), for \( i < d_j \),
\[ \Delta^i_j = I - \tilde{\Lambda}^{(i)}_j \tilde{P}^i_j(D_j) \]
where
\[ \tilde{\Lambda}^{(i)}_j = \prod_{k=i+1}^{d_j} (\Lambda^{j,\alpha}_0 \alpha_k^j) \]
and
\[ \tilde{P}^i_j(x) = \prod_{k=i+1}^{d_j} (x - \alpha_k^j) \]
Thus $\tilde{P}_j(D_j)\Delta^j = 0$. Hence, if $H \in C^\infty_\beta$ where $\beta$ satisfies 45, there are constants $c_{kl}$ (independent of $H$) such that

$$\Delta^j H(t) = \sum_{k=i+1}^{d_j} \sum_{m=0}^{d_j-i-1} c_{km} D^m_j H(t^j) e^{\alpha_k t_j}.$$  \hfill (47)

where $t^j = (t_1, \ldots, t_{j-1}, 0, t_{j+1}, \ldots, t_n)$. It also follows that $\Delta^j$ is defined on $C^\infty_\beta$ for all $\beta$ and for $i < d_j$, $\Delta^j : C^\infty_\beta \to C^\infty_\gamma$ where

$$\gamma = (\beta_1, \ldots, \beta_{j-1}, a^{i+1}_j, \beta_{j-1}, \ldots, \beta_n)$$ \hfill (48)

For the next lemma, let $1 = (1, 1, \ldots, 1) \in \mathbb{R}^n$. We remind the reader that for $i > d_j$, $a^i_j = \infty$. (See 43.)

**Lemma 5.** Let $\beta \in \mathbb{R}^n$ and $i \in \mathbb{N}$ be such that $\beta_j > \Re \alpha_k^j$ for $1 \leq k \leq i$. Then $N^i : C^\infty_\beta \to C^\infty$ where $\gamma = \inf \{\beta + b 1, a^{i+1}\}$.

**Proof** This all follows easily from the observation that $N^i : C^\infty_\beta \to C^\infty_{\beta + b 1}$ along with 45, 31, and the comments following 47. \hfill \Box

**Theorem 6.** Let $F \in C^\infty_\beta$ satisfy 41. Then $F$ has an asymptotic expansion with exponents from $I$.

**Lemma 6.** Let $E_j = P_j(D_j)$ and let $\beta$ satisfy the hypothesis of Lemma 5 with respect to $i$. Then on $C^\infty_\beta$,

$$E_j N^i = N^i_1 + \Lambda_1^{(i)} (E_j N_1 - E_1 N_j) + \Delta^i_1 \Lambda_2^{(i)} (E_j N_2 - E_2 N_j) + \ldots + \Delta^i_1 \ldots \Delta^i_{j-1} \Lambda_j^{(i)} (E_j N_j - E_{j-1} N_j)$$

**Proof** For $j = 1$, our lemma claims that $E_1 N^1 = N_1$ which is clear from $E_1 \Delta^1_1 = 0$ and $E_1 \Lambda_1^{(i)} = I$. Thus we assume by induction that our result is known for all $k < j$.

Using $E_j \Lambda^j_j = 0$ and $E_j \Lambda_j^{(i)} = I$, we find

$$E_j N^i = \Lambda_1^{(i)} E_j N_1 + \Lambda_2^{(i)} \Delta^1_1 E_j N_2 + \cdots + \Lambda_j^{(i)} \Delta^1_1 \Delta^2_1 \ldots \Delta^i_{j-2} E_j N_{j-1} + \Delta^i_1 \Delta^2_1 \cdots \Delta^i_{j-1} N_j$$
If we replace $\Delta_{j-1}^i$ with $I - \Lambda_{j-1}^{(i)}E_{j-1}$ and combine the last two terms, we obtain

$$E_jN^i = \Lambda_1^{(i)}E_jN_1 + \Lambda_2^{(i)}\Delta_1^iE_jN_2 + \cdots + \Lambda_{j-2}^{(i)}\Delta_1^i\Delta_2^i\ldots\Delta_{j-3}^iE_jN_{j-2} + \Delta_1^i\Delta_2^i\ldots\Delta_{j-2}^iN_j + \Delta_1^i\Delta_2^i\ldots\Delta_{j-2}^i\Lambda_{j-1}^{(i)}(E_jN_{j-1} - E_{j-1}N_j)$$

The last term on the right is as required for the lemma. The sum of the other terms equals

$$E_j\left(\Lambda_1^{(i)}N_1 + \Lambda_2^{(i)}\Delta_1^iN_2 + \cdots + \Lambda_{j-2}^{(i)}\Delta_1^i\Delta_2^i\ldots\Delta_{j-3}^iN_{j-2} + \Lambda_j^{(i)}\Delta_1^i\Delta_2^i\ldots\Delta_{j-2}^iN_j\right)$$

This has the same form as that described in our lemma except that the terms corresponding to the $j - 1$st variable are omitted. We use the inductive hypothesis to simplify this expression, proving the lemma.

**Corollary 1.** Let $F \in C_{\beta}^\infty$ satisfy 41 where $\beta$ satisfies the hypothesis of Lemma 5 with respect to $i$. Then $G = (I - N^i)F$ is an exponential polynomial.

**Proof** Since $N_jF = E_jF - G_j$, it follows that $(E_jN_i - E_iN_j)F$ is an exponential polynomial. Hence, form Lemma 6, for each $j$, $H_j = P_j(D_j)N^iF$ is an exponential polynomial. Let

$$\tilde{G} = G - \Lambda_1^{(0)}H_1 - \Lambda_2^{(0)}\Delta_1^0H_2 - \cdots - \Lambda_n^{(0)}\Delta_1^0\Delta_2^0\ldots\Delta_{n-1}^0H_n.$$

it follows easily from the following lemma that $\tilde{G}$ is an exponential polynomial, proving our corollary.

**Lemma 7.** For all $1 \leq i \leq n$, $P(D_i)\tilde{G} = 0$.

**Proof** Using the observations

$$P(D_i)\Lambda_i = I$$

$$\Lambda_iP(D_i) = I - \Delta_i$$

$$P(D_i)H_j = P(D_j)H_i$$

$$P(D_i)\Delta_i = 0$$
we see
\[ P(D_i)G_0 = H_i - (I - \Delta^0_1)H_i - (I - \Delta^0_2)\Delta^0_1H_i - \ldots \]
\[ - (I - \Delta^0_{i-1})\Delta^0_1 \ldots \Delta^0_{i-2}H_i - \Delta^0_1 \Delta^0_2 \ldots \Delta^0_{i-1}H_i = 0 \]
proving the lemma.

We set
\[ F_n = \sum_{k=0}^{n}(N^0)^kG \] (49)
so that
\[ F - F_n = (N^0)^{n+1}F. \]

Note that \( F_n \) is an exponential polynomial. It follows from Lemma 5 that for sufficiently large \( n \), \( F - F_n \in C_{a_1}^\infty \). Then \( F - F_n \) satisfies a similar system of differential equations as \( F \), allowing us to assume that \( F \in C_{a_1}^\infty \). In this case \( G_i = F - N_iF \in C_{a_2}^\infty \).

Now we repeat the preceding argument using \( N^1 \) instead of \( N^0 \). This is allowed since now \( F, P(D_i)F, N_iF \) and \( G_i \) all belong to the domain of \( \Lambda_i^{(1)} \). We conclude that there is an exponential polynomial \( F_n \) such that \( F - F_n \in C_{a_1}^\infty \). We replace \( F \) by \( F - F_n \), and continue the argument.

Once we reach the point where \( F \in C_{a_1}^\infty \), then the proof is finished just as in the one variable case. \qed

3 Invariant Operators

We begin with a few observations concerning the structure of homogeneous domains. We assume the notation from the introduction is still in force. Let \( \mathcal{Q} \subset \mathcal{S} \) be the set of complex tangent vectors at \( e \) of type \((1,0)\) i.e. \( \mathcal{Q} \) is the Lie algebra of left invariant vector fields which annihilate holomorphic functions. Let \( J : \mathcal{S} \to \mathcal{S} \) be the operator whose \( \pm i \) eigenspaces are \( \mathcal{Q} \) and \( \overline{\mathcal{Q}} \) respectively. Thus, \( J \) is the complex structure on \( \mathcal{S} \) corresponding with the identification of \( \mathcal{S} \) with \( \mathcal{D} \).
It is known that
\[ J : S_{ij} \rightarrow \mathcal{M}_{ij} \]
\[ J : Z_i \rightarrow Z_i \]
\[ J : A \rightarrow \sum_i \mathcal{M}_{ii} \]
where the notation is explained below Proposition 1. Let \( S, \mathcal{M}, \) and \( Z \) be, respectively, the spans of the \( S_{ij}, \mathcal{M}_{ij}, \) and \( Z_{ij}. \)

Since the Hermitian structure on \( X \) is invariant, it is determined by a Hermitian product \( H \) on \( S. \) Let \( g = \text{re} \ H. \) Then \( g \) defines a real scalar product on \( S \) which defines the Riemannian structure on \( X. \) The Kähler form on \( X \) is then defined by
\[ \phi(X, Y) = g(X, JY). \]

The Kähler assumption is equivalent with the statements that \( \phi \) is \( J \)-invariant, skew-symmetric, and closed–i.e.
\[ \phi([X, Y], Z) = \phi([X, Z], Y) + \phi(X, [Y, Z]). \]

For \( X \in S, \) we define
\[ |X| = \sqrt{g(X, X)} \]

Many of the results in [4] were based on the assumption that there is a linear functional \( \nu \) such that for all \( X \) and \( Y \) in \( S, \)
\[ \phi(X, Y) = \langle [X, Y], \nu \rangle \quad (50) \]

The following lemma is certainly known, although we lack a reference.

**Lemma 8.** The functional \( \nu \) described above exists.

**Proof** For each \( i \) we let \( E_i = -JA_i \in \mathcal{M}_{ii}. \) Then
\[ [A_i, E_i] = E_i. \quad (51) \]

Let
\[ E = \sum_i E_i \]
Then
\[ J E = \sum_{i=1}^{d} A_i. \]
It follows that
\[ \text{ad } J E \big|_{S} = 0 \]
\[ \text{ad } J E \big|_{M} = I, \]
\[ \text{ad } J E \big|_{Z} = I/2. \] (52)

We define \( \nu \) to be zero on \( Z \) and \( S \) and,
\[ < M, \nu > = g(M, E) \]
for \( M \in M \). We claim that formula 50 holds. To see this, consider first the case where \( X \in S \) and \( Y \in M \). Then \( [X, Y] \in M \) so
\[ < [X, Y], \nu > = g([X, Y], E) \]
\[ = -\phi([X, Y], J E) \]
\[ = -\phi([X, J E], Y) - \phi(X, [Y, J E]) = \phi(X, Y) \]
as desired.

The equality for \( X \) and \( Y \) in \( Z \) is similar.

For \( X \) and \( Y \) in \( M \), we must show that \( \phi(X, Y) = 0 \). However,
\[ \phi(X, Y) = \phi([J E, X], Y) \]
\[ = \phi([J E, Y], X) + \phi(J E, [X, Y]) \]
\[ = \phi(Y, X) \]
which must be zero due to the skew-symmetry of \( \phi \).

It follows from the \( J \)-invariance of \( \phi \) that \( \phi \) is also zero on \( S \times S \) which is consistent with our definition of \( \nu \). \( \square \)

Let \( \pi_Q \) be the projection to \( Q \) along \( \overline{Q} \). For each \( Z \in \overline{Q} \), we define an operator \( M(Z) : Q \to Q \) by
\[ M(Z)(X) = \pi_Q([Z, X]). \]

Then
\[ \nabla_Z X = M(Z)(X). \]
(See the discussion following formula (1.7) in [4].)

Since the connection is real, it follows that

$$\nabla_X Z = M(X)(Z),$$

where

$$M(X)Z = \bar{M(\bar{X} \bar{Z})}.$$  

From Theorem (1.9) of [4], on $Q$, for $Z$ and $W$ in $Q$,

$$R(Z, \overline{W}) = -M^*(Z)M(\overline{W}) + M(\overline{W})M^*(Z)$$

$$- M^*(M(\overline{W})Z) - M(M(Z)\overline{W})$$

where $M^*(Z)$ is the adjoint of $M(Z)$ on $Q$ with respect to the Hermitian form.

For $X \in S$, let $X^Q = X - iJX \in Q$ and $X^{\overline{Q}} = \overline{X^{Q}} \in \overline{Q}$. Then $Q = S^Q$. Let $A = a_1A_1 + \cdots + a_dA_d$.

**Lemma 9.** For $A \in A$,

$$M(A^{\overline{Q}})X^Q = \frac{1}{2}(a_j + a_k)X^Q \quad X \in S_{jk} + \mathcal{M}_{jk}$$

$$M(A^{\overline{Q}})X^Q = \frac{1}{2}a_jX^Q \quad X \in Z_j$$

**Proof** Let $JA = M \in \sum_{i} \mathcal{M}_{ii}$. Let $X \in S_{jk}$ and let $Y = JX \in \mathcal{M}_{jk}$. Then, since the span of the $\mathcal{M}_{jk}$ is abelian, we have (mod $\overline{Q}$)

$$= [A + iM, X + iY] - 2i[A + iM, Y]$$

$$\equiv -2i[A, Y] = -i(a_j + a_k)Y$$

$$\equiv \frac{1}{2}(a_j + a_k)(X - iY)$$

proving the first equality for $X \in S_{ij}$. The equality for $X \in \mathcal{M}_{ij}$ follows from the complex linearity of $M(Z)$. The second equality is a similar argument. \qed

**Proposition 5.** For $A \in A$

$$R(A^Q, A^{\overline{Q}})X^Q = -(a_i^2 + a_j^2)X^Q \quad X \in S_{ij} + \mathcal{M}_{ij}$$

$$R(A^Q, A^{\overline{Q}})X^Q = -a_i^2X^Q \quad X \in Z_i$$
Proof From Lemma 9, \( M^*(A^\varnothing) = M(A^\overline{\varnothing}) \). Hence, our proposition follows from formula 53 and Lemma 9.

For \( X \in (S) \), we let \( \delta(X) \in (A) \) be the \((A)\) component in the decomposition 6. Then

\[
\delta(X) = \sum_{|k| \leq l} C_k A_1^{k_1} A_2^{k_2} \cdots A_d^{k_d}
\]

where \( k = (k_1, k_2, \ldots, k_d) \) is a multi-index of length \( d \) and \( |k| = k_1 + \cdots + k_d \). The minimum value of \( l \) for which such an inequality holds is referred to as the \( A\)-degree of \( X \) and is denoted \( \deg_A(X) \).

Let

\[
E_i = \frac{A_i}{|A_i|}
\]

We identify \( A \) with \( \mathbb{R}^d \) via the orthogonal mapping

\[(x_1, \ldots, x_d) \rightarrow x_1 E_1 + \cdots + x_d E_d.\]

For \( l \geq \deg_A(X) \) we define the symbol \( \sigma_l(X) \) to be the polynomial on \( A \)

\[
\sigma_l(X)(A) = \sum_{|k|=l} C_k x_1^{k_1} x_2^{k_2} \cdots x_d^{k_d}.
\]

(If \( l > \deg_A(X) \), \( \sigma_l(X) = 0 \), while \( \sigma_l(X) \) is undefined if \( l < \deg_A(X) \).)

Let \( \omega \in T_{\text{geo}}(X) \) have degree \( 2k \). We identify \( D^\omega \) from formula 4 with an element of \((S)\). Our first goal is to compute \( \sigma(D^\omega) \).

**Proposition 6.** Let \( \omega \in T_{\text{geo}}(X) \) have degree \( 2k \). Then

\[
\sigma_{2k}(D^\omega)(A) = 2^{-k} \omega(A^\overline{\varnothing}, A^\varnothing, \ldots, A^\overline{\varnothing}, A^\varnothing)
\]

**Proof** The spaces \( S_{ij} \), \( M_{ij} \), and \( Z_j \) are all mutually orthogonal. (See [4].) Furthermore,

\[
H(E_i^\overline{\varnothing}, E_i^\varnothing) = g(E_i, E_i) + g(J E_i, J E_i) = 2 g(E_i, E_i) = 2
\]

We may choose an orthonormal basis \( Z_i \) for \( P \) such that

1. \( Z_i = 2^{-1/2} E_i^\overline{\varnothing} \) for \( 1 \leq i \leq d \).
2. \( Z_i \in (\mathcal{N}_S)_c \) for \( i > d \).

Let \( W_1, W_2, \ldots, W_{2k} \in \mathcal{P} \). The differential operator
\[
f \rightarrow \nabla^2 f(W_1, W_2, \ldots, W_{2k-1}, W_{2k})
\]
is degree \( 2k \) with leading term
\[
L = W_1 W_2 \ldots W_{2k-1} W_{2k}.
\]

If \( W_i \in (\mathcal{N}_S)_c \) for any \( i \), then \( L \in (\mathcal{S}_c)(\mathcal{N}_S)_c \) and \( \sigma_{2k}(L) = 0 \). Hence, only those terms in 4 where all of the \( Z_{ij} \) equal \( E_{ij} \) can contribute to \( \sigma_{2k}(D^\omega) \).

Thus, assume that the operator \( L \) above is such that \( W_j = 2^{-1/2} E_j^Q \) for all \( j \). Since \( JE_i \in \mathcal{N}_S \), the leading part of \( \delta(L) \) is \( 2^{-k} E_1 E_2 \ldots E_{2k} \).

Finally, from formula 4
\[
\sigma_{2k}(D^\omega)(A) = 2^{-k} \sum_{i,j} \omega(E_{i1}^Q, E_{j1}^Q, \ldots, E_{ik}^Q, \ldots, E_{jk}^Q) x_i x_j \ldots x_k x_{jk}
\]
which is equivalent with the stated formula.

For the sake of the next proposition, we remind the reader that \( \nu_{ij} = \dim \mathcal{M}_{ij} = \dim \mathcal{S}_{ij} \) and \( \nu_i = \dim \mathcal{Z}_i \).

**Proposition 7.** Let \( D^k = D^{\omega^k} \) where \( \omega^k \) is as in 3 and let \( A = a_1 A_1 + \cdots + a_d A_d \). Then
\[
\sigma_{2k}(D^k)(A) = 2^{-k} \left( \sum_{1 \leq i \leq j \leq d} \nu_{ij} (a_i^2 + a_j^2)^k + \sum_{1 \leq i \leq d} \nu_i a_i^{2k} \right).
\]

**Proof** This follows immediately from Propositions 5 and 6.

Next we consider a general co-finite ideal \( \mathcal{I} \).

If \( \mathcal{I} \) is an ideal in \( \mathcal{I}_{geo} \), then we will (without comment) set
\[
\mathcal{J} = (S)\mathcal{I}
\]
\[
\mathcal{K} = \mathcal{J} + (S)\mathcal{N}_S
\]

Note that \( \mathcal{K} \) is an algebra since \( (S)\mathcal{N}_S \) is an ideal in \( (S) \). In the case of \( \mathcal{I}_{geo} \), we will denote \( \mathcal{K} \) by \( \mathcal{K}_{geo} \) and \( \mathcal{J} \) by \( \mathcal{J}_{geo} \).
Now let \( \text{Pol}(\mathcal{A}) \) be the space of polynomial functions on \( \mathcal{A} \). For any subset \( \mathcal{V} \subset (\mathcal{S}) \), let 

\[ \tilde{\mathcal{V}} = \text{span} \{ \sigma_k(X) \mid X \in \mathcal{V}, \sigma_k(X) \text{ defined} \}. \]

Then \( \tilde{\mathcal{K}} \) is an ideal in \( \text{Pol}(\mathcal{A}) \) since, if \( X \) and \( Y \) are elements of \((\mathcal{S})\) of \( A \)-degrees \( k \) and \( l \) respectively, then 

\[ \sigma_{k+l}(XY) = \sigma_k(X)\sigma_l(Y). \]

Since \( \sigma \) factors through \( \delta \), it is clear that \( \tilde{\mathcal{I}} \) is co-finite in \( \tilde{\mathcal{I}}_{geo} \).

**Proposition 8.** \( \tilde{\mathcal{J}} \) is co-finite in \( \text{Pol}(\mathcal{A}) \).

**Proof** We consider first the case where \( \mathcal{I} = \mathcal{I}_{geo} \) so that \( \tilde{\mathcal{I}} \) contains all of the elements \( \sigma_{2k}(D^k) \) from Proposition 7. Let 

\[ m = \sum_{i \leq j} \nu_{ij} = \dim \mathcal{M} \]

\[ f = \sum_{i=1}^{d} \nu_i = \dim \mathcal{Z} \]

Let \( A_{ij} = a_i^2 + a_j^2 \). We embed \( \mathcal{A} \) into \( \mathbb{R}^{f+m} \) using the mapping \( \phi \) where

\[ \phi(A) = (a_1^2, \ldots, a_1^2, a_2^2, \ldots, a_2^2, \ldots, a_d^2, \ldots, a_d^2, A_{11}, \ldots, A_{11}, A_{12}, \ldots, A_{12}, \ldots, A_{dd}, \ldots, A_{dd}) \]

(54)

where we only use the \( A_{ij} \) for \( i \leq j \), \( a_i^2 \) is repeated \( \nu_i \) times, and and \( A_{ij} \) is repeated \( \nu_{ij} \) times. Then for \( t = \phi(A) \)

\[ \sigma_{2k}(D^k)(A) = t_1^k + t_2^k + \cdots + t_{f+m}^k \]

Let \( Q_k(t) \) be the polynomial on the right side of the above equality. The \( Q_k, \ 0 \leq k \leq f + m, \) generate the algebra of all symmetric polynomials on \( \mathbb{R}^{f+m} \). (See [5], pp. 2-4.) Hence \( \tilde{\mathcal{I}}_{geo} \) contains all polynomials \( p \circ \phi \) where \( p \) is an arbitrary, non-constant, symmetric polynomial.

The elementary symmetric polynomials \( S_j(t) \) in \( f + m \) variables are defined by the equality

\[ \prod_{i=1}^{f+m} (x + t_i) = \sum_{k=0}^{f+m} S_{f+m-k}(t)x^k \]

(55)
Letting \( x = -t_i \) in 55 shows that
\[
(-1)^{f+m} t_i^{f+m} = \sum_{k=0}^{f+m-1} (-1)^k S_{f+m-k}(t) t_i^k
\] (56)

Composing with \( \phi \) and choosing \( i \) so that \( t_i = A_{jj} = 2a_{jj}^2 \) shows that \( a_j^{2(f+m)} \in \tilde{J}_{geo} \). Hence, the monomials \( a_1^{n_1} a_2^{n_2} \ldots a_{d}^{n_d} \), \( n_i < 2(f + m) \), span \( \text{Pol}(\mathcal{A})/\tilde{J}_{geo} \), proving our proposition in this case.

To prove the general case, suppose that \( \mathcal{I} \) is a co-finite ideal of \( \mathcal{I}_{geo} \). From 56, for all \( l \geq 2(f + m) \), there are polynomials \( P_{ij}^l \) such that
\[
da_j^l = \sum_{k=0}^{f+m-1} P_{ij}^l(S_1 \circ \phi, \ldots, S_{f+m} \circ \phi)a_j^{2k}
\] (57)

Since \( \tilde{I}_{geo}/\tilde{I} \) is finite dimensional, the \( P_{ij}^l(S_1 \circ \phi, \ldots, S_{f+m} \circ \phi) \) span a finite dimensional subspace of \( \tilde{I}_{geo}/\tilde{I} \). Hence, \( \{a_j^l \mid 1 \leq j \leq d, l \geq 0 \} \) spans a finite dimensional subset mod \( \tilde{J} \). In particular, for each \( j \) there is an \( l \) such that
\[
a_j^l \equiv \sum_{k=1}^{l-1} C_k a_j^k \mod \tilde{J}
\]
for some scalars \( C_k \). Our proposition follows as before. \( \square \)

**Corollary 2.** For any homogeneous, symmetric polynomial \( p \) in \( f + m \) variables, there is a polynomial \( P \) and a \( k \in \mathbb{N} \) such that
\[
p \circ \phi = \sigma_k(P(D_1, \ldots, D^{f+m})).
\]

**Proof** Let \( p \) be homogeneous of degree \( q \). Choose \( P \) so that
\[
p = P(Q_1, \ldots, Q_{f+m}).
\]
Since the \( Q_i \) are homogeneous of degree \( i \),
\[
P(t) = \sum C_i t^i
\]
where \( i = (i_1, \ldots, i_{f+m}) \) ranges over a set of multi-indices such that
\[
i_1 + 2i_2 + \cdots + (f + m)i_{f+m} = q.
\]
Then,

$$\sigma_{2q}(P(D^1, \ldots, D^{f+m})) = \sigma_{2q} \left( \sum C_i(D^1i \ldots (D^{f+m})_{i+1} \right)$$

$$= P(\sigma_2(D^1), \ldots, \sigma_{2(f+m)}(D^{f+m}))$$

$$= P(Q_1 \circ \phi, \ldots, Q_{f+m} \circ \phi)$$

$$= p \circ \phi$$

as desired. \hfill \Box

We grade \((S)\) by degree. For any \(V \subset (S)\) we let \(V_l\) be the set of \(X \in V\) with \(\deg(X) \leq l\). We say that an element \(X \in (S)\) is \textit{non-expansive} if \(\deg(X) = \deg(\delta(X))\). Note that the product of two non-expansive elements is non-expansive. We say that a subspace \(V \subset (S)\) is \textit{non-expansive} if it is spanned by a (possibly infinite) set of non-expansive elements. It is clear from Proposition 7 that the \(D^k\) are non-expansive, implying that \(I_{\text{geo}}\) is non-expansive.

We say that a not necessarily co-finite ideal \(I \subset I_{\text{geo}}\) is \textit{\(N\)-co-finite} if it satisfies the conclusion of Proposition 8. It turns out that the general theory we will develop requires only that \(I\) be \(N\)-co-finite and non-expansive. The ability to work in this generality is important due to the following lemma, which allows us to replace a co-finite ideal by a non-expansive, \(N\)-co-finite one.

**Lemma 10.** Let \(I \subset I_{\text{geo}}\) be a co-finite ideal. Then there is a \(N\)-co-finite, non-expansive, ideal \(I_1 \subset I\).

\textit{Proof} From Lemma 2, there are non-expansive elements \(E_k \in I_{\text{geo}}\) such that

$$\sigma_{2k}(E_k) = S_k \circ \phi$$

for \(1 \leq k \leq f + m\). From the co-finite condition, for each \(k\) there is a non-zero monic polynomial \(P_k \in \mathbb{R}[x]\) such that \(P_k(E_k) \in I\). Let \(I_1\) be the ideal in \(I_{\text{geo}}\) generated (as an ideal) by the elements \(P_k(E_k)\), \(1 \leq k \leq f + m\). \(I_1\) is non-expansive since it is spanned by products of the \(P_k(E_k)\) and \(D^j\), both of which are non-expansive.

To see that \(I_1\) is \(N\)-co-finite, let \(d_k = \deg P_k\). Note that

$$(S_k \circ \phi)^{d_k} = \sigma_{2kd_k}(P_k(E_k)) \in \tilde{I}_1.$$
Hence, any polynomial in the $S_k \circ \phi$ is equivalent to one of degree less than $\sum d_j$ in the $S_k \circ \phi$, mod $\tilde{I}_1$. Our lemma follows from the reasoning following 57. \hfill \Box

Let
\[ P_l = (A)_l / (A)_l \cap (J_l + (S)_{l-1}N_S) \]

**Proposition 9.** Let \( \{X_1, \ldots, X_k\} \subset (S) \) be a set of non-expansive elements such that there are \( i \) such that \( \{\sigma_i(X_i)\} \) projects to a basis for \( \text{Pol}(A)/\tilde{J} \) where \( I \) is \( N \)-co-finite and non-expansive. (It is clear that such \( X_i \) exist.) Then \( B_l \) projects to a basis of \( P_l \) for all \( l \). Hence \( B \) projects to a basis of \( P \). Furthermore
\[ (A)_l \cap (J + (S)_{N_S}) = J_l + (S)_{l-1}N_S \]

**Proof** Let \( X \in (S)_l \). Let \( l_o = \deg(\delta(X)) \leq l \). Then there are scalars \( c_i \), elements \( B_j \in (S) \), and non-expansive elements \( I_j \in I \), with \( \deg(\delta(B_jI_j)) = l_o \), such that
\[ \sigma_{l_o}(X) - \sum c_i\sigma_{l_o}(X_i) = \sum \sigma_{l_o}(B_jI_j). \]

We may in fact choose \( B_j \in (A) \) since \( \delta \) is zero on \( (S)N_S \).

Let \( J_j = B_jI_j \), a non-expansive element. From the non-expansive property, \( X_i \) and \( J_j \) belong to \( (S)_l \).

Let
\[ X_1 = X - \sum c_iX_i - \sum J_j. \]

Then \( X_1 \in (S)_l \) and \( \delta(X_1) \) has lower degree than \( \delta(X) \).

We may repeat this argument with \( X_1 \) in place of \( X \). It follows by induction that there are elements \( J_j \) in \( \tilde{J}_l \) and constants \( c_i \) as above such that \( \delta(X_1) = 0 \). Then \( X_1 \in ((S)_{N_S})_l \), proving the first part of our proposition.

To prove the last statement, let \( A \in (J + (S)_{N_S})_l \). From the preceding argument
\[ A = \sum c_iX_i + J + Y \]
where the $c_i$ are scalars, $J \in \mathcal{J}$, and $Y \in (\mathcal{S})\mathcal{N}_S$. $A \in \mathcal{J} + (\mathcal{S})\mathcal{N}_S$ implies that the $c_i$ are all 0. Hence $Y = A - J$ belongs to $((\mathcal{S})\mathcal{N}_S)_t = (\mathcal{S})_{t-1}\mathcal{N}_S$, proving our result.

\[ \square \]

**Remark:** If $X$ is a Riemannian symmetric space and $\mathcal{I}$ is $D_G(X)$, then the finite dimensionality of $\mathcal{P}$ is known. (See the discussion in Section 5 of [1].) The existence of a non-expansive spanning set for $\mathcal{I}$ follows from the observation that the Harish Chandra homomorphism preserves degree. The analogue of Lemma 10 follows from a similar argument. These comments are used to prove Theorem 2 in the Riemannian symmetric case.

## 4 Explicit Expansions

In this section we prove the existence and convergence of the asymptotic expansions. We refer the reader to §1 for our notation.

**Proposition 10.** Let $F$ be an $\mathcal{I}$-harmonic function on $S$ which satisfies 2 and let $F_A \in \mathcal{H}_\omega(\pi_r)$ be defined by formula 11 where $A \in \mathcal{A}^+$. Let $s > 0$ be given. Then for each $A \in \mathcal{A}^+$ and each $\gamma \in \langle A, \mathcal{E} \rangle$, there exists an unique $\mathcal{H}_\omega(\pi_r)$-valued function $F_\gamma(A, t)$ (not depending on $s$) which is polynomial of bounded degree in $t$ and a $t_o \geq 0$ (which may depend on $s$) such that for all $t \geq t_o,$

\[ F_A = \sum_{\gamma \in \langle A, \mathcal{E} \rangle} F_\gamma(A, t)e^{\gamma t} \quad (58) \]

where the convergence is in $(\mathcal{H}_\omega(\pi_r))^*$. Furthermore for all $t \in \mathbb{R}^+$, $F_\gamma(A, t) \in C^{-\infty}(\pi_r)$ and the above equality is valid in the asymptotic topology on $C^{-\infty}(\pi_r)$-valued functions.

**Proof** From Lemma 10 we may assume that $\mathcal{I}$ is non-expansive and $\mathcal{N}$-co-finite. Then Proposition 9 shows that $\mathcal{P}$ is finite dimensional. Let $\mathcal{P}$ be decomposed as in 8.

For $A \in \mathcal{A}$ and $t \in \mathbb{R}$, let

\[ P^A(t) = \prod_{\alpha \in \mathcal{E}_a} (t - \langle A, \alpha \rangle)^{n_{\alpha}}. \]
$P^A$ is a real polynomial since the roots of $P$ occur in conjugate pairs with equal multiplicity. The following lemma is clear from 9 since $P$ has a unit.

**Lemma 11.** For $A \in A$, let $\tilde{A} \in P$ be the projection of $A$. Then $P^A(\tilde{A}) = 0$.

It follows from the preceding lemma that

$$P^A(A) = X^A + J^A$$

where $X^A \in (S) N_S$ and $J^A \in J$. From Proposition 9, we may take $\deg X^A \leq \deg P^A$. We may also assume that $X^A$ depends linearly on $P^A(A)$ and thus polynomially on $A$. Thus $X^A$ is a sum of terms of the form

$$p(A)X_1X_2 \ldots X_l$$

where the $p$ are polynomials in $A$ and each $X_k$ belongs to either $S_{ij}$ or $M_{ij}$ for some $i \leq j$ depending on $k$. Furthermore, since $X^A \in (S) N_S$, there is at least one $X_k$ in each term for which $i < j$.

Since $F$ is annihilated by $J$ it follows that $F$ satisfies

$$P^A(A)F(x) = X^A F(x)$$

where elements of the enveloping algebra are identified with left invariant differential operators on $S$.

For $X \in (S)$, let $X(t) = \text{Ad} (\exp(tA))X$. We replace $x$ by $x(\exp tA)$ in 60 discovering that

$$P^A(D)F_{tA} = X^A(t)F_{tA}$$

where $D = \frac{d}{dt}$. Expanding $X^A$ in a sum of terms of the form of 59, we find that $F_{tA}$ satisfies an equation of the form

$$P^A(D)F_{tA} = \sum_k e^{<A,\beta_k>t}X_i^A F_{tA}$$

where $\beta_k \in \text{span} N_S$ and $X_i^A \in (S)$ depends polynomially on $A$. In particular, $A \in A^+$ implies that $<A, \beta_k> > 0$. 

We interpret 62 as an $\mathcal{H}_-\omega(\pi_r)$ valued differential equation. We claim that in this case, the hypotheses of Theorem 5 are satisfied. We note first that for all $\phi \in \mathcal{H}_-\omega(\pi_r)$, $F \in \mathcal{H}_-\omega(\pi_r)$, and $X \in (S)$,

$$< \pi_r(g^{-1})\phi, XF > = < \pi_r(g^{-1})\phi, \pi^*_r(X)F > = < \pi_r(X^*)\pi_r(g^{-1})\phi, F > = X(< \pi_r(g^{-1})\phi, F >).$$

It follows from Lemma 1, p. 459 of [3] and the definition of the $\mathcal{H}_-\omega(\pi_r)$ topology, that if $X$ has degree $d$ as an element of $(S)$ then it has degree $\leq d$ as an operator on $\mathcal{H}_-\omega(\pi_r)$.

From the example on p. 282 of [17], there are positive constants $C$ and $r'$ such that

$$e^{\tau(x)} \leq C\|\text{Ad}(x)\|^r$$

where $\|\cdot\|$ denotes the operator norm with respect to any conveniently chosen norm on $\mathcal{L}$. In particular, if $x = \exp tA$

$$e^{\tau(\exp tA)} \leq C e^{r''t}$$

for some constant $r''$. Hence, from inequalities 2 and 10,

$$\{e^{-r''t}F_{A} \mid t \in (-\infty, 0]\}$$

is bounded in $L^1_t(S)^*$ and, thus, in $\mathcal{H}_-\omega(\pi_r)$.

Formula 58 now follows immediately from Theorem 5. The statement about $C^{-\infty}(\pi_r)$ follows from Theorem 3 together with the uniqueness of the coefficients.  

Next we prove Theorem 2:

Proof (Of Theorem 2)

Let $F$ be $I$-harmonic and let $B_1, B_2, \ldots, B_r$ be a basis for $\mathcal{A}$ contained in $\mathcal{A}^+$. For $t \in (-\infty, 0)^n$, let

$$G_F(t) = F_{B(t)}$$

where $B(t) = t_1B_1 + t_2B_2 + \cdots + t_dB_d$. Then letting $D_i = \frac{d}{dt_i}$ and reasoning as in 62, we find

$$P^{B_j}(D_j)G_F = \sum_k e^{<B(t), \beta_k>} X_k^j G_F$$
where $\beta_k \in \Delta$ and $X^j \in (S)$.

We consider the preceding set of equations as a $C^{-\infty}(\pi_r)$ valued system. Theorem 6 shows that under asymptotic convergence

$$G_F(t) = \sum_{\delta \in \mathcal{E}_1} e^{t\delta} F_\delta(t)$$

where $\mathcal{E}_1 \subset \mathbb{R}^d$ and $F_\delta(t)$ depends polynomially on $t$.

Let $A = B(s)$ where the $s_i > 0$. Then, under asymptotic convergence,

$$F_{tA} = G_F(ts) = \sum_{\delta \in \mathcal{E}_1} e^{t(s,\delta)} F_\delta(ts). \quad (63)$$

The uniqueness of coefficients in asymptotic expansions shows that for the $F_\gamma$ as in 58,

$$F_\gamma(A, t) = \sum_{\gamma = s, \delta} F_\delta(ts)$$

It follows that the series 63 converges in $\mathcal{H}_{\omega}(\pi_r)$ for all $t > t_o$. From the proof of Lemma 2, $t_o$ depends continuously on $A$. In particular, we may choose a value of $t_o$ so that the series for $t \to F_{tA/|A|}$ converges for all for $t > t_o$ and all $A$ which are positive linear combinations of the $B_i$. Hence, the series in question converges at $t = 1$ if $|A| \geq t_o$. For such $A$, we define

$$F_\delta(A) = F_\delta(s).$$

This finishes the proof of Theorem 2 for homogeneous domains, with the exception of the covariance property (c). This, however, is a simple consequence of the equality

$$R(\exp B(s))F_{B(t)} = F_{B(s+t)}.$$

The proof of 2 in the case of Riemannian symmetric spaces is almost identical. See the remarks at the end of §2.

**Remark** We still need to discuss the Poisson transformation. This however, is more or less immediate from Theorems 3 and 4. Explicitly, let $A \in A^+$. Theorem 4 allows us to construct $F_{tA}$ for $t \geq t_o$ using the boundary values and the operators $\pi^j$. Hence, we can construct

$$F = \pi_r (\exp -t_o A) F_{t_o A}.$$
If $F$ is harmonic, then the result will of course be independent of the choice of $A$. Conversely, one might hope that if the result is independent of $A$ then $F$ would be harmonic. This, however, is the subject of future research.

References


REFERENCES


