The Helgason Conjecture for non-symmetric domains

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1 Introduction

Let $X = G/K$ be a Riemannian symmetric space and let $D_G(X)$ be the algebra of all G-invariant differential operators on X. Let $\mathcal{I} \subset D_G(X)$ be a co-finite ideal. A C^∞ function F on X is $\mathcal{I}$-*harmonic* if it is annihilated by every element of $\mathcal{I}$. For example, if χ is a character of $D_G(X)$ and F satisfies

$$XF = \chi(X)F \tag{1}$$

for all $X \in D_G(X)$, then F is $\mathcal{I}$-harmonic where $\mathcal{I}$ is the kernel of χ.

One of the most beautiful results in the harmonic analysis of symmetric spaces is the "Helgason Conjecture", which states that on a Riemannian symmetric space of non-compact type, a function satisfies 1 if and only if it is the Poisson integral of a hyperfunction over the Furstenberg boundary. A companion result, due to Oshima and Sekiguci, [13] says that the boundary hyperfunction is a distribution if and only if there are positive constants A and r_o (depending on F) such that

$$|F(x)| \leq Ae^{r_o \tau(x)} \tag{2}$$

for all $x \in X$ where $\tau(x)$ is the Riemannian distance in X from x to the base point $x_o = eK$.

In this work, we begin work on generalizing these results to general connected, homogeneous, Käher manifolds X. Specifically, we assume that $X = G/K$ where G is the connected component of the holomorphic isometry group of X and K is the isotropy subgroup of a point in X. In this context, we hope to

1. Define a collection G-invariant differential operators $\mathcal{F}$ on $C^\infty(X)$ to play the role of $D_G(X)$.
2. Define an appropriate boundary for X.
3. Define a distributional "boundary value" for any $\mathcal{I}$-harmonic function F satisfying 2.
4. Define a "Poisson" transform which reconstructs F from its boundary distribution.

A result of Dorfmeister and Nakajima [6] (generalizing earlier work of Gindikin and Vinberg [8]) states that the general homogeneous Kähler manifold is a holomorphic fiber bundle whose base is a bounded homogeneous domain in $\mathbb{C}^n$ and whose fiber is the product of $\mathbb{C}^k$ with a compact, complex, homogeneous Kähler manifold. Thus, *we assume that X is a bounded homogeneous domain in $\mathbb{C}^n$*. In this work, we solve (a)-(d).

Concerning (a), in the non-symmetric case, the group of bi-holomorphisms can be quite small, in which case the algebra $D_G(X)$ can be so large that the space of harmonic functions can consist of little more than the constant functions. In particular, holomorphic functions need not be harmonic. Hence, to produce an interesting theory we need a smaller algebra.

In place of $D_G(X)$ we use an algebra of "geometrically" defined invariant differential operators. Specifically, let $T(X)$ be the tangent bundle for X and let g be the Riemannian form on $T(X) \times T(X)$. Let

$$g_{her}(Z, W) = g(Z, \overline{W})$$

be the corresponding Hermitian form on $T_c(X)$ where g is extended to $T_c(X)$ by bi-linearity.

Let $\Delta_{(\cdot)}(\cdot)$ be the torsion free Riemannian connection defined by g and let

$$R(U, V) = \nabla_U \nabla_V - \nabla_V \nabla_U - \nabla_{[U,V]}$$

be the curvature operator. Then for each $k \in \mathbb{N}$, we define sections ω^k of $(T^*)^{2k}(X)$ by

$$\omega^k(X_1, Y_1, X_2, Y_2, \ldots, X_k, Y_k) = (-1)^k \text{ Tr } (\prod_{j=1}^{k} R(X_j, Y_j)) \quad (3)$$

It is clear that ω^k is invariant under any isometry of X. Let $T_{geo}(X)$ be the subalgebra of the full tensor algebra $T^*(X)$ generated by the ω^k, $k \geq 1$.

Let $T^{01}(X)$ denote the bundle of complex tangent vectors of type $(0, 1)$ and let $\{Z_j\}_{j=1\ldots n}$ be a (local) frame field for $T^{01}(X)$ which is orthonormal with respect to g_{her}.

For $f \in C^\infty(X)$ and $\omega \in T_{geo}(X)$ of degree $2k$, we define

$$D^\omega f = \sum_{i,j} \omega(Z_{i_1}, \overline{Z}_{j_1}, \ldots, Z_{i_k}, \overline{Z}_{j_k}) \nabla^{2k} f(\overline{Z}_{i_1}, Z_{j_1}, \ldots, \overline{Z}_{i_k}, Z_{j_k}) \quad (4)$$

where ∇^k denotes the k-fold covariant derivative of f and i and j range over the set of multi-indices of length k with entries between 1 and n. It is easily seen that these are real differential operators which are independent of the orthonormal frames and thus define canonical differential operators which commute with all holomorphic isometries of the domain; hence, they belong to $D_G(X)$. We extend this definition to all of $T_{geo}(X)$ by linearity in ω.

Definition 1. *The operator algebra generated over $\mathbb{C}$ by the D^ω for $w \in T_{geo}(X)$ is denoted $D_{geo}(X)$.*

It should be remarked that a given complex manifold may carry many non-isometric Kähler structures for which the corresponding group of biholomorphic isometries acts transitively. This is true even if the underlying manifold is bi-holomorphic with a symmetric space. On a bounded homogeneous domain, the Bergman metric yields the largest isometry group since all bi-holomorphisms are automatically isometries. However, the Dorfmeister, Nakajima, Gindikin, Vinberg Theorem does not imply that the induced metric on the base is the Bergman metric. Hence, we are forced to consider more general metrics, even in the symmetric case. The spaces of operators defined above are only guaranteed invariant under the holomorphic isometry group of X which will not typically be the full bi-holomorphism group unless we are actually using the Bergman metric. Fortunately, this all causes only minor complications.

Definition 2. *By a "co-finite ideal $\mathcal{I}$", we mean a co-finite ideal of $D_{geo}(X)$. In this case we say that $F \in C^\infty(X)$ is $\mathcal{I}$-harmonic if it is annihilated by every element of $\mathcal{I}$.*

We use the concept of $\mathcal{I}$-harmonic as a replacement for the harmonicity studied in the semi-simple case.

The next question is, "What should play the role of the Furstenberg boundary in the non-symmetric case?" There seems, in general, to be no way of constructing an analogue of the Furstenberg boundary. We can, however construct what, in the symmetric case, is an open subset of the Furstenberg boundary. Specifically, in general, G is algebraic and has an "Iwasawa" decomposition

$$G = AN_SK$$

where A is an $\mathbb{R}$ split algebraic torus, N_S is a unipotent subgroup normalized by A, and K is a maximal compact subgroup. Then $S = AN_S$ acts simply-transitively on X.

We identify X with S.

As an algebraic variety,

$$S = N_S \times (\mathbb{R}^+)^d \subset N_S \times \mathbb{R}^d$$

where d is the rank of X. Under this identification, N_S is contained in the topological boundary of AN_S. We use N_S as a substitute for the Furstenberg boundary. In the semi-simple case this amounts to restricting to a dense, open, subset of the Furstenberg boundary. We refer to N_S as the *naive* boundary.

We prove the following result in the Hermitian-symmetric case. Our proof carries over to the non-symmetric case. However, in the non-symmetric case, our operator algebras are non-abelian so the concept of regular singularities requires some conditions on the commutators of the operators which we have not been able to verify. (See [10] and [12].) Our techniques do, however, provide an especially simple way of proving the regular singularity property for Hermitian symmetric spaces.

Theorem 1. *Let X be a Hermitian symmetric space and let $\mathcal{I}$ be a co-finite ideal. There are elements $D_i \in \mathcal{I}$ and elements $Q_i \in (S)$, $1 \leq i \leq d$, such that the system $R(Q_i)D_i$ has regular singularities in the week sense along the walls $t_i = 0$ with edge N_S where R is the right action of S on $C^\infty(X) = C^\infty(S)$.*

Without the regular singularity property, we cannot appeal to the general theory of hyperfunctions to define the boundary values. Instead, we use ideas due to Wallach [16] as extended by van den Ban and Schlichtkrull [1] to construct a family of boundary values on the naive boundary for F. To describe these ideas we require some notation. Our basic references for the structure of bounded homogeneous domains are [7] and [15], although we will at times refer the reader to some of our papers where the results are presented in similar notation to our current needs. In particular, the summary given on p. 86-91 and p. 94-97 of [4] covers many of the essentials.

Throughout this work, we will usually denote Lie groups by upper case Roman letters, in which case the corresponding Lie algebra will automatically be denoted by the corresponding upper case script letter.

Since the elements of $D_{geo}(X)$ commute with the left action of S on $X = S$, we may consider $D_{geo}(X) \subset (\mathcal{S})$ where the universal enveloping algebra is identified with the left invariant differential operators, in which case we will usually set $D_{geo}(X) = \mathcal{I}_{geo}$.

Let $\mathcal{I} \subset \mathcal{I}_{geo}$ be a co-finite ideal. Let

$$\mathcal{J} = (\mathcal{S})\mathcal{I} \subset (\mathcal{S}) \tag{5}$$

be the left ideal generated by $\mathcal{I}$ and

$$\mathcal{P} = (\mathcal{S})/\left(\mathcal{J} + (\mathcal{S})\mathcal{N}_S\right)$$

Since

$$(\mathcal{S}) = (\mathcal{A}) + (\mathcal{S})\mathcal{N}_S, \tag{6}$$

it follows that

$$\mathcal{P} = (\mathcal{A})/(\mathcal{A}) \cap (\mathcal{J} + (\mathcal{S})\mathcal{N}_S) \tag{7}$$

In particular, $\mathcal{P}$ is an abelian algebra over $\mathbb{R}$ which is also an $(\mathcal{S})$-module. The following result, which is proved in Section 2, is central:

Proposition 1. *$\mathcal{P}$ is finite dimensional.*

An element $\tilde{\lambda} \in \mathcal{P}_c^*$ is a root of $\mathcal{P}$ if there is a non-zero $X \in \mathcal{P}$ such that

$$AX = \tilde{\lambda}(A)X$$

for all $A \in \mathcal{P}$. The roots are characters on $\mathcal{P}$. In particular, $\tilde{\lambda}$ is determined by its lift $\lambda \in \mathcal{A}_c^*$. The set of such functionals in $\mathcal{A}_c^*$ is denoted $\mathcal{E}_o$ and is referred to as the set of *characteristic exponents.*

Since $\mathcal{P}$ is abelian, there is a direct sum decomposition

$$\mathcal{P}_c = \sum_{\alpha \in \mathcal{E}_o} \mathcal{P}_\alpha \tag{8}$$

where each $\mathcal{P}_\alpha$ is an ideal in $\mathcal{P}_c$ and for all $A \in \mathcal{A}$

$$(L_A - < A, \alpha >)^{n_\alpha}\big|_{\mathcal{P}_\alpha} = 0 \tag{9}$$

where $n_\alpha = \dim \mathcal{P}_\alpha$ and L_A denotes the action of A on $\mathcal{P}$. The of α is, by definition, n_α. Let $\Sigma \subset \mathcal{A}^*$ be be the set of roots of $\mathcal{A}$ on $\mathcal{N}_S$–i.e. $\lambda \in \Sigma$ if and only if there is a non-zero vector $X \in \mathcal{N}_S$ such that

$$[A, X] =< A, \lambda > X$$

There is an ordered basis $\lambda_1, \lambda_2, \ldots, \lambda_d$ for $\mathcal{A}^*$ consisting of roots for which the root space of λ_i is a one dimensional subspace $\mathcal{M}_{ii}$ of $\mathcal{N}_S$. All of the other roots are one of the following types

1. $\beta_{ij} = (\lambda_i - \lambda_j)/2$ where $i < j$,
2. $\tilde{\beta}_{ij} = (\lambda_i + \lambda_j)/2$ where $i \leq j$,
3. $\lambda_i/2$.

The root spaces are denoted, respectively, (a): $\mathcal{S}_{ij}$, (b): $\mathcal{M}_{ij}$, and (c): $\mathcal{Z}_i$. We let $\nu_{ij} = \dim \mathcal{S}_{ij} = \dim \mathcal{M}_{ij}$ and $\nu_i = \dim \mathcal{Z}_i$. Note that some of these dimensions may be 0.

The ordered basis of $\mathcal{A}$ that is dual to the basis formed by $\{\lambda_i\}$ is denoted $\{A_i\}$. Let

$$\mathcal{W}^+ = \text{span}_{\mathbb{R}^+}\Sigma.$$

Then, $\mathcal{W}^+$ is an open cone in $\mathcal{A}^*$ which plays the role of a positive Weyl chamber. Let

$$\begin{aligned}\mathcal{A}^+ &= \{A \in \mathcal{A} \mid < A, \lambda > > 0, \quad \lambda \in \Sigma\} \\ \mathcal{A}^- &= -\mathcal{A}^+\end{aligned}$$

Finally, let

$$\mathcal{E} = \mathcal{E}_o + \text{span}_{\mathbb{N}_0}(\Sigma)$$

where

$$\mathbb{N}_0 = \mathbb{N} \cup \{0\}$$

Now for $r \in \mathbb{R}$, let

$$L^1_r(S) = L^1(S, e^{r\tau(x)}\, dx)$$

where dx is a choice right-invariant of Haar measure on S and $\tau(x)$ is the Riemannian distance from x to e in $S = X$. Since S acts on X by isometries, it is easily seen that

$$\tau(xy) \leq \tau(x) + \tau(y) \tag{10}$$

for all $x, y \in S$. It follows that $L^1_r(S)$ is invariant under right translation by elements of S. Let π_r be the right-regular representation of S in $L^1_r(S)$. Let $\mathcal{H}_\omega(\pi_r)$ (resp. $\mathcal{H}^\infty(\pi_r)$) be the space of analytic vectors (resp. C^∞ vectors) for π_r–i.e. the space of functions $f \in L^1_r(S)$ for which $g \to \pi_r(g)f$ extends holomorphically to a neighborhood of e in the complexification S_c of S (resp. is C^∞ on a neighborhood of e in S). It follows from Theorem 4 of [11] that $\mathcal{H}_\omega(\pi_r)$ is dense in $L^1_r(S)$.

The topology on $\mathcal{H}_\omega(\pi_r)$ is of particular importance to us. Let $\rho(\cdot,\cdot)$ be some metric on the complexification S_c of S which defines the topology of S_c and, for $s > 0$, let $B_s \subset S_c$ be the closed ρ-ball of radius s centered at e. For each $s > 0$, let $\mathcal{H}^s_\omega(\pi_r)$ be the set of $v \in \mathcal{H}_\omega(\pi_r)$ such that $g \to \pi_r(g)v$ extends continuously to B_s and holomorphically to the interior of this set. This space is non-zero for all sufficiently small s. For $v \in \mathcal{H}^s_\omega(\pi_r)$ let

$$\|v\|_s = \sup_{g \in B_s} \|\pi_r(g)v\|$$

where we use the $L^1_r(S)$ norm on the right. Then $\mathcal{H}^s_\omega(\pi_r)$ is a Banach space in this norm. Furthermore, for $s < t$, there is an obvious injection of $\mathcal{H}^t_\omega(\pi_r)$ into $\mathcal{H}^s_\omega(\pi_r)$ where the norm of the injection mapping is ≤ 1. The $\mathcal{H}_\omega(\pi_r)$ topology is defined by the equality

$$\mathcal{H}_\omega(\pi_r) = \mathrm{Dir}\lim \mathcal{H}^s_\omega(\pi_r)$$

The dual topology is defined by

$$\mathcal{H}_{-\omega}(\pi_r) = \mathrm{Inv}\lim(\mathcal{H}^s_\omega(\pi_r))^*$$

(See p. 155 and p. 174 of [9] for notation.)

Now let F be $\mathcal{I}$-harmonic on $X = S$ and satisfy 2. For each $A \in \mathcal{A}$, let $F_A \in L^1_r(S)^*$ be defined by

$$< \phi, F_A >= \int_S \phi(x) F(x \exp A)\, dx. \tag{11}$$

By restriction, we may also consider F_A as an element of either $\mathcal{H}^{-\infty}(\pi^r)$ or $\mathcal{H}_{-\omega}(\pi^r)$.

The following result is a strengthening of Theorem 3.5 of [1]. The convergence of this expansion, which seems to be new even in the symmetric case, is one of or main results. Our arguments are based techniques of Baouendi and Goulaouic [3]. (In both [1] and [16], only *aysmptotic* convergence, similar to (a) below, was proven.)

Theorem 2. *Assume that $F \in C^\infty(X)$ satisfies 2 and is $\mathcal{I}$-harmonic where $\mathcal{I}$ is either a co-finite ideal in $D_{geo}(X)$ or X is a Riemannian symmetric space and $\mathcal{I}$ is a co-finite ideal of $D_G(X)$. Let $s > 0$. Then for each $\alpha \in \mathcal{E}$, there exists a unique $\mathcal{H}_{-\omega}(\pi_r)$-valued polynomial F_α on $\mathcal{A}$ (independent of s) and a $t_o > 0$ (which may depend on s) such that*

$$F_A = \sum_{\beta \in <A,\mathcal{E}>} \left(\sum_{\alpha \in \mathcal{E}, <A,\alpha> = \beta} F_\alpha(A) e^{<A,\alpha>} \right) \tag{12}$$

for all $A \in \mathcal{A}^-_K$, $|A| > t_o$, where the convergence is in $(\mathcal{H}^s_\omega(\pi_r))^*$. (The inner sum is finite and the outer is countable.) Furthermore

1. *For all A and α, $F_\alpha(A) \in \mathcal{H}^{-\infty}(\pi^r)$. Hence the $F_\alpha(A)$ define distributions on S. Also, for all $s \in \mathbb{R}$ and $A \in \mathcal{A}^-$ there is a finite set $J_s \subset \mathcal{E}$ such that the following set is bounded in $\mathcal{H}^{-\infty}(\pi^r)$:*

$$\{e^{-st}(F_{tA} - \sum_{\alpha \in J_s} F_\alpha(A) e^{<A,\alpha>t}) \mid t \in \mathbb{R}^+\}$$

2. *The F_α have bounded homogeneous degree. Specifically, for $\alpha \in \mathcal{E}_o$, $\deg F_\alpha < n_\alpha$ where n_α is the multiplicity of α.*

3. *For all $A, B \in \mathcal{A}^-$*

$$\pi_r(\exp B) F_\alpha(A) e^{-<A,\alpha>} = F_\alpha(A+B).$$

Remark: Since $C_c^\infty(S) \subset \mathcal{H}^{-\infty}(\pi^r)$, part (a) of Theorem 2 implies that the expansion 12 converges in the sense of distribution valued asymptotic expansions. Hence, our result implies Theorem 3.5, part (i), of [1].

In [1], the asymptotic expansions are over $\mathcal{A}^+$ as $A \to \infty$. This is because they use the parabolic opposite to ours–i.e., they use $\overline{\mathcal{N}}_S$ rather than $\mathcal{N}_S$. The difference is really just a matter of notation. If we think of $\mathcal{N}_S$ as being in the opposite parabolic, then we should call the roots $-\lambda_i$ rather than λ_i, in which case our $\mathcal{A}^-$ becomes their $\mathcal{A}^+$.

Definition 3. The boundary values of F are the set of polynomials $F_\alpha(A)$ for $\alpha \in \mathcal{E}_o$.

According to the preceding definition, the boundary values are distributions on $S \times \mathcal{A}$. It appears that we have made describing the harmonic functions *more* difficult in that we have replaced functions on S with distributions on $S \times \mathcal{A}$. It turns out, however, that each boundary function is uniquely determined by a distribution on the C_c^∞ sections of a finite dimensional line bundle over N_S.

To describe this, let F satisfy the same conditions as F_α in conditions (b) and (c) in Theorem 2. For $n = n_\alpha$, let $\mathcal{W}_n$ be the space of polynomial functions on $\mathcal{A}$ of total degree $\leq n$ and let ρ_n be the representation of A in $\mathcal{W}_n$ defined by right translation. F defines an element of $\mathcal{H}^{-\infty}(\pi^r) \otimes \mathcal{W}_n$, which is the dual space of $\mathcal{H}^{\infty}(\pi^r) \otimes \mathcal{W}_n^*$. The covariance condition becomes

$$\pi_r(a)F = \rho_n(a^{-1})F \tag{13}$$

for all $a \in A$. (With obvious abuse of notation, we denote $\pi_r \otimes I$ and $I \otimes \rho_n$ by π_r and ρ_n respectively.)

Formally, for $\phi \in C_c^{\infty}(S) \otimes \mathcal{W}_n^*$,

$$\begin{aligned} < \phi, F > &= \int_S < \phi(x), F(x) > \, dx \\ &= \int_{N_S} \int_A < \phi(na), F(na) > \, dadn \\ &= \int_{N_S} < T\phi(n), F(n) > \, dn \end{aligned} \tag{14}$$

where

$$T\phi(x) = \int_A \rho_n^*(a)\phi(xa)\, da \tag{15}$$

and ρ_n^* is the contragrediant representation to ρ_n in $\mathcal{W}_n^*$. Then

$$T\phi(xa) = \rho_n^*(a^{-1})T\phi(a).$$

Hence, $T\phi$ is a section of the homogeneous line bundle L_n over N_S defined by

$$L_n = (S \times \mathcal{W}_n^*)/A$$

where the A-action is defined by

$$(x, p)a = (xa, \rho_n^*(a^{-1})p)$$

As is well known, and easily shown, T maps $C_c^{\infty}(S, \mathcal{W}_n)$ onto the space $\Gamma_c^{\infty}(L_n)$ of C^{∞}, compactly supported sections of L_n. These calculations suggest the following proposition.

Proposition 2. *For any $F \in \mathcal{H}^{-\infty}(\pi^r) \otimes \mathcal{W}_n$ which satisfies 13, there is a unique element $\tilde{T}F \in \Gamma_c^{\infty}(L_n)^*$ such that for all $\phi \in C_c^{\infty}(S) \otimes \mathcal{W}_n^*$,*

$$< \phi, F > = < T\phi, \tilde{T}F >$$

Proof For the functional $\tilde{T}F$ to be well defined it suffices to show that $T\phi = 0$ implies $< \phi, F >= 0$. If F is a function, this follows from 14. The general case follows by convolving F on the left with a C_c^∞ approximate identity. The continuity of $\tilde{T}F$ is due to the observation that kernel of T is a closed subspace of $C_c^8(S, \mathcal{W}_n)$. $\square$

It is clear that $\Gamma_c^\infty(L_n) = C_c^\infty(N_S) \otimes \mathcal{W}_n^*$. Hence, $\Gamma_c^\infty(L_n)^* = \mathcal{D}'(N_S) \otimes \mathcal{W}_n$, implying that each boundary function is uniquely determined by a finite family of distributions on N_S.

In the Hermitian symmetric case, our boundary values are the restrictions of those of [1] to the naive boundary. In [2] it was shown that in the Hermitian symmetric case, the function F is uniquely determined by the restrictions of its boundary values to any open subset of the Furstenberg boundary. *It is a consequence of our convergence result mentioned above that the same holds in the general case for restrictions to the naive boundary.* As in the symmetric, case we require all of the boundary values. (For the Furstenberg boundary, there is a distinguished boundary distribution that uniquely determines the solution. This, however, is not true for restrictions to open subsets of the boundary.) We also describe an algorithm for reconstructing all of the F_α from the boundary distributions. From the convergence result mentioned above, this then reconstructs F, producing a kind of Poisson transformation.

2 Abstract Asymptotic Expansions

Here we prove the existence and convergence of general asymptotic expansions. The existence, but not the convergence nor the boundedness of the degrees, was already proven in [14].

Let $\mathcal{V}$ be a locally convex, topological vector space space over $\mathbb{C}$. For $r \in \mathbb{R}$, let $\mathcal{C}_r$ be the set of $F : (-\infty, 0] \to \mathbb{R}$ such that

$$\{e^{-rt}F(t) \,|\, t \in (-\infty, 0]\} \tag{16}$$

is bounded in $\mathcal{V}$. Let

$$\mathcal{C} = \cup_r \mathcal{C}_r$$

Let $I \subset \mathbb{C}$ be countable. An *exponential series with exponents from* I is a formal sum

$$\tilde{F}(t) = \sum_{\gamma \in I} e^{\gamma t} F_\gamma(t) \tag{17}$$

where F_γ is an $\mathcal{V}$-valued polynomial. If I is finite the above sum (which is now an element of $\mathcal{C}$) is referred to as an *exponential polynomial.*

Let $\mathcal{F}$ be the family of finite subsets of I, directed by inclusion.

Definition 4. *Let $F \in \mathcal{C}$. Given a topology $\mathcal{T}$ on $\mathcal{C}$, we say that the exponential series 17 equals $F(t)$ in $\mathcal{T}$ if*

$$F(t) = \lim_{J\in\mathcal{F}} \sum_{\gamma\in J} e^{\gamma t} F_\gamma(t)$$

where the limit is in the sense of nets.

The two topologies of interest are

1. The topology of point-wise convergence.
2. The locally convex TVS-topology for which the spaces $\mathcal{C}_r$ form a base of neighborhoods of 0. We refer to this as the *asymptotic* topology. Convergence in this topology is called *asymptotic* convergence. It is a T_1 topology.

Let $D = \dfrac{d}{dt}$. We consider a differential equation on $\mathcal{C}$ of the form

$$P(D)F(t) = NF(t) + G(t) \tag{18}$$

where:

1. P is a polynomial of degree d.
2. $$N = \sum_{i=1}^{k} e^{\beta_i t} N_i \tag{19}$$
   where the N_i are continuous linear operators on $\mathcal{V}$ and
   $$\text{re } \beta_i > b > 0$$
   for all $1 \le i \le k$.
3. G is a exponential polynomial with exponents from $\mathcal{E}_1 \subset \mathbb{C}$.

Note that under these assumptions,

$$N : \mathcal{C}_r \to \mathcal{C}_{r+b} \tag{20}$$

We factor $P(D)$ as

$$P(D) = (D - \alpha_1)(D - \alpha_2) \dots (D - \alpha_d) \tag{21}$$

where some of the roots may be repeated. Let $a_i = \text{ re } \alpha_i$. We assume that the α_i are ordered so that

$$a_i \leq a_{i+1}.$$

Let

$$I = \{\alpha + \sum_j \beta_j k_j \mid \alpha \in \mathcal{E}_o \cup \mathcal{E}_1, k_j \in \mathbb{N}_0\}$$

where

$$\mathcal{E}_o = \{\alpha_1, \dots, \alpha_d\}.$$

Theorem 3. *Let $F \in \mathcal{C}$ satisfy 18. Then F has an asymptotically convergent expansion with exponents from I. Furthermore, the F_α have degrees bounded independently of α.*

Proof From Corollary 1.7 of [14], it suffices to show that for all r there is an exponential polynomial F_r with exponents from I such that $F - F_r \in \mathcal{C}_r$.

Let Λ^o_α be the integral operator on $\mathcal{C}$ defined by

$$\Lambda^o_\alpha(F)(t) = e^{\alpha t} \int_0^t e^{-\alpha x} F(x)\, dx$$

It is easily checked that Λ^o_α is a right inverse for $D - \alpha$ and

$$\Lambda^o_\alpha : \mathcal{C}_r \to \mathcal{C}_{r_o} \tag{22}$$

where $r_o = \min\{r, \text{ re } \alpha\}$. Let

$$\Lambda^{(0)} = \prod_{i=1}^{d} (\Lambda^o_{\alpha_i}) \tag{23}$$

Then, $\Lambda^{(0)}$ is a right inverse for $P(D)$ and

$$\Lambda^{(0)} : \mathcal{C}_r \to \mathcal{C}_{r_1} \tag{24}$$

where $r_1 = \min\{r, a_1\}$.

Let $H_0 = \Lambda^{(0)} NF$. Then from 18

$$P(D)(F - H_0) = NF - NF + G = G$$

Hence

$$F - H_0 = \Lambda^{(0)} G + F_0 \equiv H_1 \tag{25}$$

where $P(D)F_0 = 0$. Thus, F_0, $\Lambda^{(0)}G$, and, H_1 are exponential polynomials with exponents from I.

We write equation 25 as

$$(I - \Lambda^{(0)} N)F = H_1$$

Let

$$F_n = \sum_{k=0}^{n} (\Lambda^{(0)} N)^k H_1 = (I - (\Lambda^{(0)} N)^{n+1})F \tag{26}$$

so that

$$F - F_n = (\Lambda^{(0)} N)^{n+1} F.$$

F_n is an exponential polynomial with exponents from I. Also, from 24 and 20, for $r + nb \geq a_1$,

$$\tilde{F} = F - F_n \in \mathcal{C}_{a_1}.$$

Note that

$$\begin{aligned} P(D)\tilde{F} &= P(D)F - P(D)F_n \\ &= N\tilde{F} + \tilde{G} \end{aligned} \tag{27}$$

where

$$\tilde{G} = NF_n - P(D)F_n + G$$

is an exponential polynomial. Equation 27 has precisely the same form as 18. Hence, we are reduced to the case where $F \in \mathcal{C}_{a_1}$.

For $r >$ re α, the operator Λ_α defined by

$$\Lambda_\alpha(F)(t) = e^{\alpha t} \int_{-\infty}^{t} e^{-\alpha x} F(x)\, dx \tag{28}$$

maps $\mathcal{C}_r$ into itself and is the (two sided) inverse of $D-\alpha$ on $\mathcal{C}_r$. Also, from 20, for $r >$ re $\alpha - b$

$$\Lambda_\alpha N : \mathcal{C}_r \to \mathcal{C}_{r+b}. \tag{29}$$

We repeat the argument leading up to 27 with $\Lambda^{(0)}$ replaced by $\Lambda^{(1)}$ where

$$\Lambda^{(i)} = \prod_1^i (\Lambda_{\alpha_i}) \prod_{i+1}^d (\Lambda^o_{\alpha_i}) \tag{30}$$

Note that from 24 and 29, for $r > a_j$,

$$\Lambda^{(j)} : \mathcal{C}_r \to \mathcal{C}_{r_j} \tag{31}$$

where $r_j = \min\{r, a_{j+1}\}$. In particular, $\Lambda^{(1)} NF$ is defined since $F \in \mathcal{C}_{a_1}$. Precisely as before we are able to replace 18 with a similar differential equation where $F \in \mathcal{C}_{a_2}$.

We continue, using each of the operators $\Lambda^{(i)}$ in succession, eventually reducing to the case where $F \in \mathcal{C}_{a_l}$. In this case, the asymptotic series is produced by 26 with $\Lambda^{(0)}$ replaced by $\Lambda^{(d)}$. The boundedness of the degrees follows from the observation that Λ_α preserves degrees of exponential polynomials. □

For future reference, we note the following lemma:

Lemma 1. *If F satisfies 18 where F and G both belong to $\mathcal{C}_r$ then $F^{(n)} \in \mathcal{C}_{r_o}$ for all n where $r_o = \min\{r,\ re\ \alpha_1\}$.*

Proof

Suppose that $F \in \mathcal{C}_r$ is such that $P(D)F = H \in \mathcal{C}_r$. Then

$$F = \Lambda^{(0)} H + K$$

where $P(D)K = 0$. In particular, K is an exponential polynomial with exponents from the α_i. Hence

$$(D-\alpha_1)(D-\alpha_2)\dots(D-\alpha_k)F = \prod_{i=k+1}^d (\Lambda^o_{\alpha_i})H + K_k$$

where K_k is an exponential polynomial with exponents from the α_i. It follows easily by induction on k that $F^{(k)} \in \mathcal{C}_{r_o}$ for $0 \le k \le \deg P(D)$. Our result now follows by repeated differentiation of 18. □

Remark: We can actually generalize Theorem 3 to apply to equations 18 where

$$N = \sum_{i=1}^{k} e^{\beta_i t} P_i(D) N_i$$

where the P_i are polynomials. In fact, suppose that F satisfies such an equation where $F \in \mathcal{C}_r$. Let $\tilde{F} : (-\infty, 0] \to \mathcal{C}_r$ be defined by

$$\tilde{F}(t)(s) = F(t+s)$$

Then $\tilde{F}$ satisfies

$$P(D_t)\tilde{F}(t) = \sum_{i=1}^{k} e^{\beta_i t} e^{\beta_i s} P_i(D_s) N_i \tilde{F}(t) + \tilde{G}(s)$$

where $\tilde{G}$ is a $\mathcal{C}_r$ valued exponential polynomial. Hence, the more general result follows from Theorem 3. We leave the details to the reader as we don't require the more general result.

Next we define a "Poisson transformation" for 18 with $G = 0$. For the remainder of this section, we assume that there is an increasing family of Banach spaces $(\mathcal{V}(s), \|\cdot\|_s)$ for $s \in \mathbb{R}^+$, with continuous and dense, injections, such that

$$\mathcal{V} = \operatorname{Inv}\lim \mathcal{V}(s).$$

For example, if $\mathcal{V}(s) = \mathcal{H}^s_{-\omega}(\pi_r)$, then $\mathcal{V} = \mathcal{H}_{-\omega}(\pi_r)$. (See the discussion below 10 for the notation.) The theory also works, with only slight modifications, for inverse limits. We do not, however, require this case.

We say that an operator $N : \mathcal{V} \to \mathcal{V}$ has degree $\leq d$ if for all $0 < a \leq u < v \leq b$, $N : \mathcal{V}(u) \to \mathcal{V}(v)$, and there is a constant $C_N(a,b)$ such that

$$\|Nw\|_v \leq \frac{C_N(a,b)}{|v-u|^d} \|w\|_u \tag{32}$$

for all $w \in \mathcal{V}(u)$.

Assume that F satisfies 18 with $G = 0$. We assume also that

$$\text{degree} N_i \leq d$$

for all i where d is the degree of $P(D)$. By definition, the Poisson transformation maps the F_α, $\alpha \in \mathcal{E}_o$, into F. We showed in [14] that asymptotic expansions may be differentiated term-by-term. Substitution of 17 into 18 and equating terms with the same exponent shows that

$$P(D)(e^{\gamma t}F_\gamma(t)) = e^{\gamma t}\sum_{i=1}^{k} N_i F_{\gamma-\beta_i}(t). \qquad (33)$$

Let $\alpha \in \mathcal{E}$ have $\operatorname{re}\alpha$ minimal with respect to $F_\alpha \neq 0$. Equation 33 shows that

$$P(D)(e^{\alpha t}F_\alpha(t)) = 0.$$

Hence, α is a root of P. Let the distinct roots of P be $\tilde{\alpha}_1 \ldots, \tilde{\alpha}_l$ so $\alpha = \tilde{\alpha}_j$ for some j.

Then $\deg F_\alpha \leq n_j$ where n_j is the multiplicity of $\tilde{\alpha}_j$ as a root of P. Let the $\tilde{\alpha}_i$ be ordered so that $\operatorname{re}\tilde{\alpha}_i \leq \operatorname{re}\tilde{\alpha}_j$ for $i \leq j$. For each multi-index n of length k, let

$$\gamma(n) = \tilde{\alpha}_j + n_1\beta_1 + \cdots + n_k\beta_k.$$

Given a $\mathcal{V}(s)$-valued polynomial H, we inductively define for each $n \in \mathbb{Z}^k$ a polynomial H_n (which also depends on j through $\gamma(n)$) by the stipulations

1. $H_0(t) = H(t)$.

2. $H_n = 0$ if any of the components of n are negative.

3. $$P(D)(e^{\gamma(n)t}H_n)(t) = e^{\gamma(n)t}\sum_{i=1}^{k} N_i H_{n-e_i}(t) \qquad (34)$$

   where e_i is the ith standard basis element in $\mathbb{R}^k$.

4. If for some $n \neq 0$, $\gamma(n) = \tilde{\alpha}_k \in \mathcal{E}_o$, then

   $$D^m(e^{\gamma(n)t}H_n)(0) = 0 \quad 0 \leq m < n_k$$

   where n_k is the multiplicity of $\tilde{\alpha}_k$ as a root of P.

We remind the reader that $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Proposition 3. *Conditions (1)-(4) uniquely determine polynomials H_n which are valued in $\mathcal{V}(u)$ for all $u > s$. Furthermore, for all $u > s$, there is a $t_o \leq 0$ such that*

$$\pi^j(H)(t) = \sum_{n \in \mathbb{N}_0^k} e^{\gamma(n)t} H_n(t) \tag{35}$$

converges in the $\mathcal{V}(u)$ topology for $t \leq t_o$. If $P(D)(e^{\gamma(n)t}H)(t) = 0$, then $\pi^j(H)$ is a $\mathcal{V}(u)$-valued solution to 18 for $t < t_o$.

Proof

Let

$$P^\alpha(D) = P(D + \alpha) = e^{-\alpha t} P(D) e^{\alpha t}$$

Equation 34 can be written

$$P^{\gamma(n)}(D) H_n = Q_n$$

where Q_n is may be assumed (by induction) to be a known polynomial, valued in $\mathcal{V}(u)$ for all $u > s$.

If $\gamma(n) \notin \mathcal{E}_o$, then $P^{\gamma(n)}(D)$ has trivial kernel in the space of polynomials. Hence, in this case, $P(D)^{\gamma(n)}$ maps the space of polynomials of a given degree invectively onto itself. Thus equation 34 has a unique solution H_n in the space of polynomials. It is clear that H_n is valued in $\mathcal{V}(u)$ for all $u > s$.

If $\gamma(n) = \tilde{\alpha}_i \in \mathcal{E}_o$, then

$$P^{\gamma(n)}(D) = D^{n_i} D_o$$

where D_o is bijective on the space of polynomials of a given degree. It follows that 34, together with condition (4), uniquely determines $H(n)$.

To prove convergence, for $r \in \mathbb{R}$, let $\mathcal{C}(u)_r$ be the set of $F \in C^\infty((-\infty, 0], \mathcal{V})$ for which the set 16 is a bounded subset of $\mathcal{V}(u)$. For such F, we define

$$\|F\|_{u,r} = \sup_{t \in (-\infty, 0]} e^{-rt} \|F(t)\|_u$$

Let $n_o \in \mathbb{N}_0^k$ be such that $\operatorname{re} \gamma(n) > \operatorname{re} \tilde{\alpha}_l$ for all $|n| \geq n_o$ where $n \in \mathbb{N}_0^k$ and $|n| = \sum n_j$. (Note that due to the ordering of the roots, this implies that $\operatorname{re} \gamma(n) > \operatorname{re} \tilde{\alpha}_i$ for all i.) Let

$$K_n(t) = e^{\gamma(n+n_o)t} H_{n_o+k}(t).$$

Then $K_0 \in \mathcal{C}_r$ where $r = \text{ re } \gamma(n_o)$ and equation 34 implies that

$$K_n(t) = \sum_{i=1}^{k} \Lambda^{(d)} N_i e^{\beta_i t} K_{n-e_i}(t). \tag{36}$$

For $q \in \mathbb{N}_0$, let

$$\rho(q,u,s) = \sup_{|n|=q} \|K_n\|_{u,s}.$$

Lemma 2. *For all $n \in \mathbb{N}_0$ and all $0 < u_o \leq u < v \leq v_o$,*

$$\rho(n,u,r+nb) \leq K \left(\frac{e^d C(u_o,v_o)}{b^d |u-v|^d} \right)^n \rho(0,v,r)$$

where

$$C(u_o,v_o) = \sum_i C_{N_i}(u_o,v_o).$$

and K is independent of u, v, and n.

Proof

For simplicity of notation, we let $C(u_o,v_o) = C$. It is easily seen that for $\text{re } \alpha = a$ and $r > a$,

$$\|\Lambda_\alpha F\|_{u,r} \leq (r-a)^{-1} \|F\|_{u,r}.$$

Let $m = \text{ re } \tilde{\alpha}_l$. Then for $r > m$,

$$\|\Lambda^{(d)} F\|_{u,r} \leq (r-m)^{-d} \|F\|_{u,r}.$$

Let $0 < u_o \leq u < v \leq v_o$ be given. It follows from the preceding equality that

$$\|(\Lambda^{(d)} N_i e^{\beta_i t}) F\|_{u,r+b} \leq \frac{C_{N_i}(u_o,v_o)}{(r-m+b)^d |u-v|^d} \|F\|_{v,r} \tag{37}$$

We apply this inequality to 36 with v replaced by $u+\epsilon$ where $\epsilon = (v-u)/n$, and r replaced by $r+(n-1)b$, finding

$$\rho(n,u,r+nb) \leq \frac{Cn^d}{b^d (n+(r-m)/b)^d |u-v|^d} \rho(n-1,u+\epsilon,r+(n-1)b)$$

We repeat $n-1$ more times, with (u,v) replaced by $(u+k\epsilon, u+(k+1)\epsilon)$, $k=1,2,\ldots,n-1$, finding

$$\rho(n,u,r+nb) \leq \left(\frac{C}{b^d|u-v|^d}\right)^n \frac{n^{nd}\Gamma((r-m)/b)^d}{\Gamma(n+(r-m)/b+1)^d}\rho(0,v,r)$$

Our lemma follows from Stirling's formula. □

The convergence of the series 35 follows since for all $t \leq 0$,

$$\|e^{\gamma(n)t}H_n(t)\|_u \leq e^{(r+|n|b)t}\rho(|n|,u,r+|n|b)$$

That $\pi^j(H)$ is a solution to 18 follows since a series such as 17 satisfies 18 if and only if 33 holds. □

Remark: The $G=0$ assumption is made only for convenience. In fact, suppose that F satisfies 18 with G a non-zero exponential polynomial. Then there is a polynomial Q such that

$$Q(D)P(D)F = Q(D)NF.$$

The reasoning from the remark following the proof of Theorem 3 allows us to reduce to the case of Theorem 3. Again, we leave the details to the reader as we don't require this generality.

The following theorem, together with the uniqueness of the asymptotic expansions, implies the convergence of the asymptotic expansions in the topology of $\mathcal{V}(s)$ for all $s>0$. In the case of direct limits this implies convergence in $\mathcal{V}$ since the injection of $\mathcal{V}(s)$ into $\mathcal{V}$ is continuous.

Theorem 4. *Let $F \in \mathcal{C}_r$ satisfy 18 with $G=0$. Then for $1 \leq i \leq l$ there exist unique $\mathcal{V}$-valued polynomials H_j satisfying $P(D)(e^{\tilde{\alpha}_i t}H_j)(t) = 0$ such that for for all $s>0$ there is a t_o (depending on s) such that in the $\mathcal{V}(s)$ topology*

$$F(t) = \sum_{j=1}^{l} \pi^j(H_j)(t)$$

for all $t \geq t_o$. Furthermore, $\deg H_j < n_j$, where n_j is the multiplicity of $\tilde{\alpha}_j$ as a root of P.

Proof

Let re α be minimal with respect to $F_\alpha \neq 0$. Then, as noted previously, $\alpha = \tilde{\alpha}_j$ is a root of P and $P(D)(e^{\alpha t}F_\alpha)(t) = 0$. Let $H_1 = F_\alpha$. Since $H_1 \in \mathcal{V}$, $H_1 \in \mathcal{V}(s)$ for all $s > 0$. Fix $s > 0$ and set

$$F_1(t) = F(t) - \pi^j(H_1)(t)$$

Then, for sufficiently large t, F_1 is a $\mathcal{V}(s)$-valued solution to 18 such that $(F_1)_{\tilde{\alpha}_j} = 0$. We repeat this argument l times, producing H_j such that

$$F_l = F - \sum_{j=1}^{l} \pi^j(H_j)$$

is a $\mathcal{V}(s)$-valued solution to 18 with all of its boundary functions zero. The following lemma shows that then $F_l = 0$, proving our theorem.

Lemma 3. *Suppose that $F \in \mathcal{C}_{r_o}$ satisfies 18 with $G = 0$. If $F_\alpha = 0$ for all $\alpha \in \mathcal{E}_o$, then there is a t_o such that $F(t) = 0$ for all $t \leq t_o$.*

Proof It follows by induction from 33 that $F_\alpha = 0$ for all α. Then Theorem 3 implies that $F \in \mathcal{C}_r$ for all $r \in \mathbb{R}$. Since $\Lambda^{(d)}$ is a left inverse for $P(D)$ on $\mathcal{C}_r$ for sufficiently large r, we have

$$F = \Lambda^{(d)} N F.$$

Thus

$$F = (\Lambda^{(d)} N)^n F$$

for all $n \in \mathbb{N}$. Reasoning as in the proof of Lemma 2 using 37, we see that for all $0 < u_o \leq u < v \leq v_o$,

$$\|F\|_{u,r+nb} = \|(\Lambda^{(d)} N)^n F\|_{u,r+nb} \leq K \left(\frac{e^d C(u_o, v_o)}{b^d |u - v|^d} \right)^n \|F(t)\|_{v,r}$$

Our lemma follows by letting n tend to infinity in

$$\|F(t)\|_u \leq e^{(r+nb)t} \|F\|_{u,r+nb}$$

□

The following result follows from Theorem 3 and the uniqueness of the asymptotic expansions.

Theorem 5. *Assume that $F \in \mathcal{C}_r$ satisfies 18 where the N_i in 19 have $\deg N_i \leq \deg P$. Then for all $s > 0$ there is a $t_o \leq 0$ such that the expansion from Definition 4 converges pointwise in the $\mathcal{V}(s)$ topology for all $t \leq t_o$.*

Next we consider multi-variable expansions. By a $\mathcal{V}$-valued exponential series on $\mathbb{R}^n$ with exponents from $\mathcal{E} \in \mathbb{C}^n$ we mean a formal sum of the form

$$\tilde{F}(t) = \sum_{\gamma \in \mathcal{E}} e^{t \cdot \gamma} F_\gamma(t) \tag{38}$$

where the F_γ are $\mathcal{V}$-valued polynomials on $\mathbb{R}^n$. If $\mathcal{E}$ is finite, then $\tilde{F}(t)$ is an exponential polynomial.

Let $\beta \in \mathbb{R}^n$. We say that $F \in \mathcal{C}_\beta$ if

$$\{e^{-\beta \cdot t} F(t) \,|\, t \in (-\infty, 0]^n\} \tag{39}$$

is bounded in $\mathcal{V}$. We say that $F \in \mathcal{C}_\beta^\infty$ if for all multi-indecies j of length n,

$$D^j F = D_1^{j_1} D_2^{j_1} \dots D_n^{j_n} F \in \mathcal{C}_\beta. \tag{40}$$

For β and γ in $\mathbb{R}^n$, we say

$$\beta \succ \gamma$$

if, for all i, $\beta_i > \gamma_i$. If $B \subset \mathbb{R}^n$ is finite, we define $\inf B$, to be the vector c where $c_i = \min\{b_i \mid (b_1, \dots, b_n) \in B\}$.

Suppose that $F \in \mathcal{C}_\beta$. We say that a series as in 38 is an asymptotic expansion for F if for all $\gamma \in \mathbb{R}^n$, there is a finite subset $\mathcal{E}(\gamma) \subset \mathcal{E}$ such that

$$F - F^\gamma \in \mathcal{C}_\gamma$$

where F^γ is the sum in 38 with $\mathcal{E}$ replaced by $\mathcal{E}(\gamma)$.

Proposition 4. *Suppose that $F \in \mathcal{C}_\beta$ has an asymptotic expansion as in 38. Then both the subset $\mathcal{E}$ and the polynomials F_γ are uniquely determined.*

Proof

Our Proposition clearly follows from the following lemma.

Lemma 4. *Suppose that $\tilde{F}$ is an exponential polynomial as in 38 where $\tilde{F} \in \mathcal{C}_\gamma$. Then $F_\alpha = 0$ for all $\gamma \succ$ re α.*

Proof Let U be the set of $X \in (-\infty, 0)^n$ such that $X \cdot \alpha \neq X \cdot \beta$ for all $\alpha, \beta \in \mathcal{E}$. Since $\mathcal{E}$ is finite, U is an open, not necessarily convex, cone. For $X \in U$,

$$G(t) = \tilde{F}(-tX)$$

is an exponential polynomial on $(-\infty, 0]$ with exponents from $-X \cdot \mathcal{E}$ which belongs to $\mathcal{C}_{-X\cdot\gamma}$. Since $X \in U$,

$$G_{-X\cdot\alpha}(t) = F_\alpha(-tX)$$

for all $\alpha \in \mathcal{E}$. For $\gamma \succ$ re α and $X \in (-\infty, 0)^n$, $-X \cdot \gamma > -X \cdot$ re α. Hence, $G_{-X\cdot\alpha}(t) \equiv 0$. Our lemma follows since a polynomial which is zero on an open subset is zero. $\square$

Let the general variable $t \in \mathbb{R}^n$ be $t = (t_1, \ldots, t_n)$ and let $D_j = \frac{d}{dt_j}$. We assume that $F : (-\infty, 0]^n \to \mathcal{V}$ satisfies a system of differential equations of the form

$$P_j(D_j)F(t) = N_j F(t) + G_j(t) \tag{41}$$

for $1 \leq j \leq n$ where:

1. The P_j are polynomials of degree $d_j > 0$.
2. For $1 \leq j \leq n$,
$$N_j = \sum_{i=1}^{k} e^{t\cdot\beta_{i,j}} N_{ij} \tag{42}$$
where the N_{ij} are continuous linear operators on $\mathcal{V}$ and $\beta_{i,j} = (\beta^1_{i,j}, \ldots, \beta^n_{i,j}) \in \mathbb{C}^n$ satisfies
$$\text{re } \beta^k_{i,j} > b > 0$$
where b is independent of i, j, k.
3. The G_j are exponential polynomials in t with exponents from $\mathcal{E}_1 \in \mathbb{C}^n$.

We factor P_j as in 21 where the α_i corresponding to P_j are denoted α^j_i. We assume that for each j, the α^j_i are ordered so that re $\alpha^j_i \leq$ re α^j_{i+1} for all i. Let

$$a^i_j = \begin{matrix} \text{re } \alpha^j_i & 1 \leq i \leq d_j \\ \infty & i > d_j \end{matrix} \tag{43}$$

We also let

$$a^i = (a^i_1, a^i_2, \ldots, a^i_n).$$

Let $\mathcal{E}^j_o$ be the set of roots of P_j and let

$$\mathcal{E}_o = \mathcal{E}^1_o \times \ldots \mathcal{E}^n_o \subset \mathbb{C}^n$$
$$I = \{\alpha + \sum_j \beta_{ij} k_{ij} \mid \alpha \in \mathcal{E}_o \cup \mathcal{E}_1, k_{ij} \in \mathbb{N}_0\}.$$

Let $\Lambda^{o,j}_\alpha$ and Λ^j_α be, respectively, the analogues of the operators Λ^o_α and Λ_α from 22 and 28 defined by integration in the jth variable. The analogs of the $\Lambda^{(i)}$ are the operators defined by

$$\Lambda^{(i)}_j = \prod_{k=1}^{i} (\Lambda^j_{\alpha^j_k}) \prod_{k=i+1}^{d_j} (\Lambda^{j,o}_{\alpha^j_k}). \tag{44}$$

where $1 \le i \le d_j$. For $i > d_j$, we set $\Lambda^{(i)}_j = \Lambda^{(d_j)}_j$.

$\Lambda^{j,o}_{\alpha^j_k}$ is defined on $\mathcal{C}^\infty_\beta$ for all β while $\Lambda^j_{\alpha^j_k}$ is defined on $\mathcal{C}^\infty_\beta$ for $\beta_j >$ re α^j_k. Hence $\Lambda^{(i)}_j$ is defined on $\mathcal{C}^\infty_\beta$ as long as

$$\beta_j > \text{ re } \alpha^j_k \quad 1 \le k \le i. \tag{45}$$

We also let

$$\Delta^i_j = I - \Lambda^{(i)}_j P(D_j) \tag{46}$$

and

$$N^i = \Lambda^{(i)}_1 N_1 - \Lambda^{(i)}_2 \Delta^i_1 N_2 - \cdots - \Lambda^{(i)}_n \Delta^i_1 \Delta^i_2 \ldots \Delta^i_{n-1} N_n.$$

Note that for $i \ge d_j$, $\Delta^i_j = 0$.

Since on its domain, Λ^j_α is a *two-sided* inverse for $(D_j - \alpha I)$, for $i < d_j$,

$$\Delta^i_j = I - \tilde{\Lambda}^{(i)}_j \tilde{P}^i_j(D_j)$$

where

$$\tilde{\Lambda}^{(i)}_j = \prod_{k=i+1}^{d_j} (\Lambda^{j,o}_{\alpha^j_k})$$

and

$$\tilde{P}^i_j(x) = \prod_{k=i+1}^{d_j} (x - \alpha^j_k)$$

Thus $\tilde{P}^i_j(D_j)\Delta^i_j = 0$. Hence, if $H \in \mathcal{C}^\infty_\beta$ where β satisfies 45, there are constants c_{kl} (independent of H) such that

$$\Delta^i_j H(t) = \sum_{k=i+1}^{d_j} \sum_{m=0}^{d_j-i-1} c_{km} D^m_j H(t^j) e^{\alpha^j_k t_j}. \tag{47}$$

where $t^j = (t_1, \ldots, t_{j-1}, 0, t_{j+1}, \ldots, t_n)$. It also follows that Δ^i_j is defined on $\mathcal{C}^\infty_\beta$ for all β and for $i < d_j$, $\Delta^i_j : \mathcal{C}^\infty_\beta \to \mathcal{C}^\infty_\gamma$ where

$$\gamma = (\beta_1, \ldots, \beta_{j-1}, a^{i+1}_j, \beta_{j-1}, \ldots, \beta_n) \tag{48}$$

For the next lemma, let $\mathbf{1} = (1, 1, \ldots, 1) \in \mathbb{R}^n$. We remind the reader that for $i > d_j$, $a^i_j = \infty$. (See 43.)

Lemma 5. *Let $\beta \in \mathbb{R}^n$ and $i \in \mathbb{N}$ be such that $\beta_j > \ \mathrm{re}\, \alpha^j_k$ for $1 \le k \le i$. Then $N^i : C^\infty_\beta \to C^\infty_\gamma$ where $\gamma = \inf\{\beta + b\mathbf{1}, a^{i+1}\}$.*

Proof This all follows easily from the observation that $N_i : \mathcal{C}^\infty_\beta \to \mathcal{C}^\infty_{\beta+b\mathbf{1}}$ along with 45, 31, and the comments following 47. □

Theorem 6. *Let $F \in \mathcal{C}^\infty_\beta$ satisfy 41. Then F has an asymptotic expansion with exponents from I.*

Lemma 6. *Let $E_j = P_j(D_j)$ and let β satisfy the hypothesis of Lemma 5 with respect to i. Then on $\mathcal{C}^\infty_\beta$,*

$$\begin{aligned} E_j N^i = N_i &+ \Lambda^{(i)}_1(E_j N_1 - E_1 N_j) + \Delta^i_1 \Lambda^{(i)}_2 (E_j N_2 - E_2 N_j) + \ldots \\ &+ \Delta^i_1 \ldots \Delta^i_{j-1} \Lambda^{(i)}_j (E_j N_{j-1} - E_{j-1} N_j) \end{aligned}$$

Proof For $j = 1$, our lemma claims that $E_1 N^i = N_1$ which is clear from $E_1 \Delta^i_1 = 0$ and $E_1 \Lambda^{(i)}_1 = I$. Thus we assume by induction that our result is known for all $k < j$.

Using $E_j \Delta^i_j = 0$ and $E_j \Lambda^{(i)}_j = I$, we find

$$\begin{aligned} E_j N^i = \Lambda^{(i)}_1 E_j N_1 &+ \Lambda^{(i)}_2 \Delta^i_1 E_j N_2 + \cdots + \Lambda^{(i)}_{j-1} \Delta^i_1 \Delta^i_2 \ldots \Delta^i_{j-2} E_j N_{j-1} \\ &+ \Delta^i_1 \Delta^i_2 \ldots \Delta^i_{j-1} N_j \end{aligned}$$

If we replace Δ^i_{j-1} with $I - \Lambda^{(i)}_{j-1} E_{j-1}$ and combine the last two terms, we obtain

$$E_j N^i = \Lambda^{(i)}_1 E_j N_1 + \Lambda^{(i)}_2 \Delta^i_1 E_j N_2 + \cdots + \Lambda^{(i)}_{j-2} \Delta^i_1 \Delta^i_2 \ldots \Delta^i_{j-3} E_j N_{j-2} + \\ \Delta^i_1 \Delta^i_2 \ldots \Delta^i_{j-2} N_j + \Delta^i_1 \Delta^i_2 \ldots \Delta^i_{j-2} \Lambda^{(i)}_{j-1} (E_j N_{j-1} - E_{j-1} N_j)$$

The last term on the right is as required for the lemma. The sum of the other terms equals

$$E_j(\Lambda^{(i)}_1 N_1 + \Lambda^{(i)}_2 \Delta^i_1 N_2 + \cdots + \Lambda^{(i)}_{j-2} \Delta^i_1 \Delta^i_2 \ldots \Delta^i_{j-3} N_{j-2} + \\ \Lambda^{(i)}_j \Delta^i_1 \Delta^i_2 \ldots \Delta^i_{j-2} N_j)$$

This has the same form as that described in our lemma except that the terms corresponding to the $j-1$st variable are omitted. We use the inductive hypothesis to simplify this expression, proving the lemma. □

Corollary 1. *Let $F \in \mathcal{C}^\infty_\beta$ satisfy 41 where β satisfies the hypothesis of Lemma 5 with respect to i. Then $G = (I - N^i)F$ is an exponential polynomial.*

Proof Since $N_j F = E_j F - G_j$, it follows that $(E_j N_i - E_i N_j)F$ is an exponential polynomial. Hence, form Lemma 6, for each j, $H_j = P_j(D_j) N^i F$ is an exponential polynomial. Let

$$\tilde{G} = G - \Lambda^{(0)}_1 H_1 - \Lambda^{(0)}_2 \Delta^0_1 H_2 - \ldots \\ - \Lambda^{(0)}_n \Delta^0_1 \Delta^0_2 \ldots \Delta^0_{n-1} H_n.$$

it follows easily from the following lemma that $\tilde{G}$ is an exponential polynomial, proving our corollary. □

Lemma 7. *For all $1 \le i \le n$, $P(D_i)\tilde{G} = 0$.*

Proof Using the observations

$$P(D_i)\Lambda_i = I \\ \Lambda_i P(D_i) = I - \Delta_i \\ P(D_i) H_j = P(D_j) H_i \\ P(D_i)\Delta_i = 0$$

we see

$$P(D_i)G_0 = H_i - (I - \Delta_1^0)H_i - (I - \Delta_2^0)\Delta_1^0 H_i - \dots \\ - (I - \Delta_{i-1}^0)\Delta_1^0 \dots \Delta_{i-2}^0 H_i - \Delta_1^0 \Delta_2^0 \dots \Delta_{i-1}^0 H_i = 0$$

proving the lemma. □

We set

$$F_n = \sum_{k=0}^{n} (N^0)^k G \tag{49}$$

so that

$$F - F_n = (N^0)^{n+1} F.$$

Note that F_n is an exponential polynomial. It follows from Lemma 5 that for sufficiently large n, $F - F_n \in \mathcal{C}^\infty_{a^1}$. Then $F - F_n$ satisfies a similar system of differential equations as F, allowing us to assume that $F \in \mathcal{C}^\infty_{a^1}$. In this case $G_i = F - N_i F \in \mathcal{C}^\infty_{a^1}$.

Now we repeat the preceding argument using N^1 instead of N^0. This is allowed since now F, $P(D_i)F$, $N_i F$ and G_i all belong to the domain of $\Lambda_i^{(1)}$. We conclude that there is an exponential polynomial F_n such that $F - F_n \in \mathcal{C}^\infty_{a^2}$. We replace F by $F - F_n$, and continue the argument.

Once we reach the point where $F \in \mathcal{C}^\infty_{a^n}$, then the proof is finished just as in the one variable case. □

3 Invariant Operators

We begin with a few observations concerning the structure of homogeneous domains. We assume the notation from the introduction is still in force. Let $\mathcal{Q} \subset \mathcal{S}_c$ be the set of complex tangent vectors at e of type $(1,0)$ i.e. $\overline{\mathcal{Q}}$ is the Lie algebra of left invariant vector fields which annihilate holomorphic functions. Let $J : \mathcal{S} \to \mathcal{S}$ be the operator whose $\pm i$ eigenspaces are $\mathcal{Q}$ and $\overline{\mathcal{Q}}$ respectively. Thus, J is the complex structure on $\mathcal{S}$ corresponding with the identification of S with $\mathcal{D}$.

It is known that

$$J : \mathcal{S}_{ij} \to \mathcal{M}_{ij}$$
$$J : \mathcal{Z}_i \to \mathcal{Z}_i$$
$$J : \mathcal{A} \to \sum_i \mathcal{M}_{ii}$$

where the notation is explained below Proposition 1. Let $\mathcal{S}$, $\mathcal{M}$, and $\mathcal{Z}$ be, respectively, the spans of the $\mathcal{S}_{ij}$, $\mathcal{M}_{ij}$, and $\mathcal{Z}_{ij}$.

Since the Hermitian structure on X is invariant, it is determined by a Hermitian product H on $\mathcal{S}$. Let $g = \text{ re } H$. Then g defines a real scalar product on $\mathcal{S}$ which defines the Riemannian structure on X. The Kähler form on X is then defined by

$$\phi(X, Y) = g(X, JY).$$

The Kähler assumption is equivalent with the statements that ϕ is J-invariant, skew-symmetric, and closed–i.e.

$$\phi([X, Y], Z) = \phi([X, Z], Y) + \phi(X, [Y, Z]).$$

For $X \in \mathcal{S}$, we define

$$|X| = \sqrt{g(X, X)}$$

Many of the results in [4] were based on the assumption that there is a linear functional ν such that for all X and Y in $\mathcal{S}$,

$$\phi(X, Y) =< [X, Y], \nu > \tag{50}$$

The following lemma is certainly known, although we lack a reference.

Lemma 8. *The functional ν described above exists.*

Proof For each i we let $E_i = -JA_i \in \mathcal{M}_{ii}$. Then

$$[A_i, E_i] = E_i. \tag{51}$$

Let

$$E = \sum_1^d E_i$$

Then
$$JE = \sum_1^d A_i.$$
It follows that
$$\begin{aligned} \text{ad } JE\big|_{\mathcal{S}} &= 0 \\ \text{ad } JE\big|_{\mathcal{M}} &= I, \\ \text{ad } JE\big|_{\mathcal{Z}} &= I/2. \end{aligned} \tag{52}$$

We define ν to be zero on $\mathcal{Z}$ and $\mathcal{S}$ and,
$$< M, \nu >= g(M, E)$$
for $M \in \mathcal{M}$. We claim that formula 50 holds. To see this, consider first the case where $X \in \mathcal{S}$ and $Y \in \mathcal{M}$. Then $[X, Y] \in \mathcal{M}$ so
$$\begin{aligned} < [X, Y], \nu > &= g([X, Y], E) \\ &= -\phi([X, Y], JE) \\ &= -\phi([X, JE], Y) - \phi(X, [Y, JE]) = \phi(X, Y) \end{aligned}$$
as desired.

The equality for X and Y in $\mathcal{Z}$ is similar.

For X and Y in $\mathcal{M}$, we must show that $\phi(X, Y) = 0$. However,
$$\begin{aligned} \phi(X, Y) &= \phi([JE, X], Y) \\ &= \phi([JE, Y], X) + \phi(JE, [X, Y]) \\ &= \phi(Y, X) \end{aligned}$$
which must be zero due to the skew-symmetry of ϕ.

It follows from the J-invariance of ϕ that ϕ is also zero on $\mathcal{S} \times \mathcal{S}$ which is consistent with our definition of ν. □

Let π_Q be the projection to $\mathcal{Q}$ along $\overline{\mathcal{Q}}$. For each $Z \in \overline{\mathcal{Q}}$, we define an operator $M(Z) : \mathcal{Q} \to \mathcal{Q}$ by
$$M(Z)(X) = \pi_Q([Z, X]).$$
Then
$$\nabla_Z X = M(Z)(X).$$

(See the discussion following formula (1.7) in [4].)

Since the connection is real, it follows that

$$\nabla_X Z = M(X)(Z),$$

where

$$M(X)Z = \overline{M(\overline{X})\overline{Z}}.$$

From Theorem (1.9) of [4], on $\mathcal{Q}$, for Z and W in $\mathcal{Q}$,

$$\begin{aligned} R(Z,\overline{W}) &= -M^*(Z)M(\overline{W}) + M(\overline{W})M^*(Z) \\ &\quad - M^*(M(\overline{W})Z) - M(M(Z)\overline{W}) \end{aligned} \tag{53}$$

where $M^*(Z)$ is the adjoint of $M(\overline{Z})$ on $\mathcal{Q}$ with respect to the Hermitian form.

For $X \in \mathcal{S}$, let $X^{\mathcal{Q}} = X - iJX \in \mathcal{Q}$ and $X^{\overline{\mathcal{Q}}} = \overline{X^{\mathcal{Q}}} \in \overline{\mathcal{Q}}$. Then $\mathcal{Q} = \mathcal{S}^{\mathcal{Q}}$. Let $A = a_1 A_j + \cdots + a_d A_d$.

Lemma 9. *For $A \in \mathcal{A}$,*

$$M(A^{\overline{\mathcal{Q}}})X^{\mathcal{Q}} = \frac{1}{2}(a_j + a_k)X^{\mathcal{Q}} \quad X \in \mathcal{S}_{jk} + \mathcal{M}_{jk}$$
$$M(A^{\overline{\mathcal{Q}}})X^{\mathcal{Q}} = \frac{1}{2}a_j X^{\mathcal{Q}} \quad X \in \mathcal{Z}_j$$

Proof Let $JA = M \in \sum_i \mathcal{M}_{ii}$. Let $X \in \mathcal{S}_{jk}$ and let $Y = JX \in \mathcal{M}_{jk}$. Then, since the span of the $\mathcal{M}_{jk}$ is abelian, we have (mod $\overline{\mathcal{Q}}$)

$$\begin{aligned} &= [A + iM, X + iY] - 2i[A + iM, Y] \\ &\equiv -2i[A, Y] = -i(a_j + a_k)Y \\ &\equiv \frac{1}{2}(a_j + a_k)(X - iY) \end{aligned}$$

proving the first equality for $X \in \mathcal{S}_{ij}$. The equality for $X \in \mathcal{M}_{ij}$ follows from the complex linearity of $M(\overline{Z})$. The second equality is a similar argument. □

Proposition 5. *For $A \in \mathcal{A}$*

$$R(A^{Q}, A^{\overline{Q}})X^{\mathcal{Q}} = -(a_i^2 + a_j^2)X^{\mathcal{Q}} \quad X \in \mathcal{S}_{ij} + \mathcal{M}_{ij}$$
$$R(A^{Q}, A^{\overline{Q}})X^{\mathcal{Q}} = -a_i^2 X^{\mathcal{Q}} \quad X \in \mathcal{Z}_i$$

Proof From Lemma 9, $M^*(A^{\mathcal{Q}}) = M(A^{\overline{\mathcal{Q}}})$. Hence, our proposition follows from formula 53 and Lemma 9. □

For $X \in (\mathcal{S})$, we let $\delta(X) \in (\mathcal{A})$ be the $(\mathcal{A})$ component in the decomposition 6. Then

$$\delta(X) = \sum_{|k|\leq l} C_k A_1^{k_1} A_2^{k_2} \dots A_d^{k_d}$$

where $k = (k_1, k_2, \dots, k_d)$ is a multi-index of length d and $|k| = k_1 + \dots + k_d$. The minimum value of l for which such an inequality holds is referred to as the A-degree of X and is denoted $\deg_A(X)$.

Let

$$E_i = \frac{A_i}{|A_i|}$$

We identify $\mathcal{A}$ with $\mathbb{R}^d$ via the orthogonal mapping

$$(x_1, \dots, x_d) \rightarrow x_1 E_1 + \dots + x_d E_d.$$

For $l \geq \deg_A(X)$ we define the symbol $\sigma_l(X)$ to be the polynomial on $\mathcal{A}$

$$\sigma_l(X)(A) = \sum_{|k|=l} C_k x_1^{k_1} x_2^{k_2} \dots x_d^{k_d}.$$

(If $l > \deg_A(X)$, $\sigma_l(X) = 0$, while $\sigma_l(X)$ is undefined if $l < \deg_A(X)$.)

Let $\omega \in T_{geo}(X)$ have degree $2k$. We identify D^ω from formula 4 with an element of $(\mathcal{S})$. Our first goal is to compute $\sigma(D^\omega)$.

Proposition 6. *Let $\omega \in T_{geo}(X)$ have degree $2k$. Then*

$$\sigma_{2k}(D^\omega)(A) = 2^{-k}\omega(A^{\overline{\mathcal{Q}}}, A^{\mathcal{Q}}, \dots, A^{\overline{\mathcal{Q}}}, A^{\mathcal{Q}})$$

Proof The spaces $\mathcal{S}_{ij}$, $\mathcal{M}_{ij}$, and $\mathcal{Z}_j$ are all mutually orthogonal. (See [4].) Furthermore,

$$\begin{aligned} H(E_i^{\overline{Q}}, E_i^{\overline{Q}}) &= g(E_i, E_i) + g(JE_i, JE_i) \\ &= 2g(E_i, E_i) = 2 \end{aligned}$$

We may choose an orthonormal basis Z_i for $\mathcal{P}$ such that

1. $Z_i = 2^{-1/2} E_i^{\overline{Q}}$ for $1 \leq i \leq d$.

2. $Z_i \in (\mathcal{N}_S)_c$ for $i > d$.

Let $W_1, W_2, \ldots, W_{2k} \in \mathcal{P}$. The differential operator

$$f \to \nabla^{2k} f(\overline{W}_1, W_2, \ldots, \overline{W}_{2k-1}, W_{2k})$$

is degree $2k$ with leading term

$$L = \overline{W}_1 W_2 \ldots \overline{W}_{2k-1} W_{2k}.$$

If $W_i \in (\mathcal{N}_S)_c$ for any i, then $L \in (\mathcal{S}_c)(\mathcal{N}_S)_c$ and $\sigma_{2k}(L) = 0$. Hence, only those terms in 4 where all of the Z_{i_j} equal $E_{i_j}^{\overline{Q}}$ can contribute to $\sigma_{2k}(D^\omega)$.

Thus, assume that the operator L above is such that $W_j = 2^{-1/2} E_{i_j}^{\overline{Q}}$ for all j. Since $JE_i \in \mathcal{N}_S$, the leading part of $\delta(L)$ is $2^{-k} Ei_1 Ei_2 \ldots E_{i_{2k}}$. Finally, from formula 4

$$\sigma_{2k}(D^\omega)(A) = 2^{-k} \sum_{i,j} \omega(E_{i_1}^{\overline{Q}}, E_{j_1}^{Q}, \ldots, E_{i_k}^{\overline{Q}}, \ldots, E_{j_k}^{Q}) x_{i_1} x_{j_1} \ldots x_{i_k} x_{j_k}$$

which is equivalent with the stated formula. □

For the sake of the next proposition, we remind the reader that $\nu_{ij} = \dim \mathcal{M}_{ij} = \dim \mathcal{S}_{ij}$ and $\nu_i = \dim \mathcal{Z}_i$.

Proposition 7. *Let $D^k = D^{\omega^k}$ where ω^k is as in 3 and let $A = a_1 A_1 + \cdots + a_d A_d$. Then*

$$\sigma_{2k}(D^k)(A) = 2^{-k}\Big(\sum_{1 \le i \le j \le d} \nu_{ij}(a_i^2 + a_j^2)^k + \sum_{1 \le i \le d} \nu_i a_i^{2k}\Big).$$

Proof This follows immediately from Propositions 5 and 6. □

Next we consider a general co-finite ideal $\mathcal{I}$.

If $\mathcal{I}$ is an ideal in $\mathcal{I}_{geo}$, then we will (without comment) set

$$\mathcal{J} = (\mathcal{S})\mathcal{I}$$
$$\mathcal{K} = \mathcal{J} + (\mathcal{S})\mathcal{N}_S$$

Note that $\mathcal{K}$ is an algebra since $(\mathcal{S})\mathcal{N}_S$ is an ideal in $(\mathcal{S})$. In the case of $\mathcal{I}_{geo}$, we will denote $\mathcal{K}$ by $\mathcal{K}_{geo}$ and $\mathcal{J}$ by $\mathcal{J}_{geo}$.

Now let Pol($\mathcal{A}$) be the space of polynomial functions on $\mathcal{A}$. For any subset $\mathcal{V} \subset (\mathcal{S})$, let

$$\tilde{\mathcal{V}} = \text{span}_{\mathbb{C}}\{\sigma_k(X) \mid X \in \mathcal{V},\ \sigma_k(X) \text{ defined}\}.$$

Then $\tilde{\mathcal{K}}$ is an ideal in Pol($\mathcal{A}$) since, if X and Y are elements of $(\mathcal{S})$ of A-degrees k and l respectively, then

$$\sigma_{k+l}(XY) = \sigma_k(X)\sigma_l(Y).$$

Since σ factors through δ, it is clear that $\tilde{\mathcal{I}}$ is co-finite in $\tilde{\mathcal{I}}_{geo}$.

Proposition 8. *$\tilde{\mathcal{J}}$ is co-finite in Pol($\mathcal{A}$).*

Proof We consider first the case where $\mathcal{I} = \mathcal{I}_{geo}$ so that $\tilde{\mathcal{I}}$ contains all of the elements $\sigma_{2k}(D^k)$ from Proposition 7. Let

$$m = \sum_{i \le j} \nu_{ij} = \dim \mathcal{M}$$

$$f = \sum_{1}^{d} \nu_i = \dim \mathcal{Z}$$

Let $A_{ij} = a_i^2 + a_j^2$. We embed $\mathcal{A}$ into $\mathbb{R}^{f+m}$ using the mapping ϕ where

$$\begin{aligned} \phi(A) = (&a_1^2, \ldots, a_1^2, a_2^2, \ldots, a_2^2, \ldots, a_d^2 \ldots, a_d^2, \\ &A_{11}, \ldots, A_{11}, A_{12}, \ldots, A_{12}, \ldots, A_{dd}, \ldots, A_{dd}) \end{aligned} \tag{54}$$

where we only use the A_{ij} for $i \le j$, a_i^2 is repeated ν_i times, and and A_{ij} is repeated ν_{ij} times. Then for $t = \phi(A)$

$$\sigma_{2k}(D^k)(A) = t_1^k + t_2^k + \cdots + t_{f+m}^k$$

Let $Q_k(t)$ be the polynomial on the right side of the above equality. The Q_k, $0 \le k \le f+m$, generate the algebra of all symmetric polynomials on $\mathbb{R}^{f+m}$. (See [5], pp. 2-4.) Hence $\tilde{\mathcal{I}}_{geo}$ contains all polynomials $p \circ \phi$ where p is an arbitrary, non-constant, symmetric polynomial.

The elementary symmetric polynomials $S_j(t)$ in $f+m$ variables are defined by the equality

$$\prod_{i=1}^{f+m} (x + t_i) = \sum_{k=0}^{f+m} S_{f+m-k}(t) x^k \tag{55}$$

Letting $x = -t_i$ in 55 shows that

$$(-1)^{f+m} t_i^{f+m} = -\sum_{k=0}^{f+m-1} (-1)^k S_{f+m-k}(t) t_i^k \tag{56}$$

Composing with ϕ and choosing i so that $t_i = A_{jj} = 2a_{jj}^2$ shows that $a_j^{2(f+m)} \in \tilde{\mathcal{J}}_{geo}$. Hence, the monomials $a_1^{n_1} a_2^{n_2} \dots a_d{}^{n_d}$, $n_i < 2(f+m)$, span $\mathrm{Pol}(\mathcal{A})/\tilde{\mathcal{J}}_{geo}$, proving our proposition in this case.

To prove the general case, suppose that $\mathcal{I}$ is a co-finite ideal of $\mathcal{I}_{geo}$. From 56, for all $l \geq 2(f+m)$, there are polynomials P_{ij}^l such that

$$a_j^l = \sum_{k=0}^{f+m-1} P_{ij}^l (S_1 \circ \phi, \dots, S_{m+f} \circ \phi) a_j^{2k} \tag{57}$$

Since $\tilde{\mathcal{I}}_{geo}/\tilde{\mathcal{I}}$ is finite dimensional, the $P_{ij}^l(S_1 \circ \phi, \dots, S_{m+f} \circ \phi)$ span a finite dimensional subspace of $\tilde{\mathcal{I}}_{geo}/\tilde{\mathcal{I}}$. Hence, $\{a_j^l \mid 1 \leq j \leq d, l \geq 0\}$ spans a finite dimensional subset mod $\tilde{\mathcal{J}}$. In particular, for each j there is an l such that

$$a_j^l \equiv \sum_1^{l-1} C_k a_j^k \mod \tilde{\mathcal{J}}$$

for some scalars C_k. Our proposition follows as before. □

Corollary 2. *For any homogeneous, symmetric polynomial p in $f+m$ variables, there is a polynomial P and a $k \in \mathbb{N}$ such that*

$$p \circ \phi = \sigma_k(P(D^1, \dots, D^{f+m})).$$

Proof Let p be homogeneous of degree q. Choose P so that

$$p = P(Q_1, \dots, Q_{f+m}).$$

Since the Q_i are homogeneous of degree i,

$$P(t) = \sum C_i t^i$$

where $i = (i_1, \dots, i_{f+m})$ ranges over a set of multi-indices such that

$$i_1 + 2i_2 + \cdots + (f+m) i_{f+m} = q.$$

Then,

$$\begin{aligned}\sigma_{2q}(P(D^1,\ldots,D^{f+m})) &= \sigma_{2q}\left(\sum C_i(D^1)^{i_1}\ldots(D^{f+m})^{i_{f+m}}\right)\\ &= P(\sigma_2(D^1),\ldots,\sigma_{2(f+m)}(D^{f+m}))\\ &= P(Q_1\circ\phi,\ldots,Q_{f+m}\circ\phi)\\ &= p\circ\phi\end{aligned}$$

as desired. □

We grade $(\mathcal{S})$ by degree. For any $\mathcal{V}\subset(\mathcal{S})$ we let $\mathcal{V}_l$ be the set of $X\in\mathcal{V}$ with $\deg(X)\leq l$. We say that an element $X\in(\mathcal{S})$ is *non-expansive* if $\deg(X)=\deg(\delta(X))$. Note that the product of two non-expansive elements is non-expansive. We say that a subspace $\mathcal{V}\subset(\mathcal{S})$ is *non-expansive* if it is spanned by a (possibly infinite) set of non-expansive elements. It is clear from Proposition 7 that the D^k are non-expansive, implying that $\mathcal{I}_{geo}$ is non-expansive.

We say that a not necessarily co-finite ideal $\mathcal{I}\subset\mathcal{I}_{geo}$ is $\mathcal{N}$-co-finite if it satisfies the conclusion of Proposition 8. It turns out that the general theory we will develop requires only that $\mathcal{I}$ be $\mathcal{N}$-co-finite and non-expansive. The ability to work in this generality is important due to the following lemma, which allows us to replace a co-finite ideal by a non-expansive, $\mathcal{N}$-co-finite one.

Lemma 10. *Let $\mathcal{I}\subset\mathcal{I}_{geo}$ be a co-finite ideal. Then there is a $\mathcal{N}$-co-finite, non-expansive, ideal $\mathcal{I}_1\subset\mathcal{I}$.*

Proof From Lemma 2, there are non-expansive elements $E_k\in\mathcal{I}_{geo}$ such that

$$\sigma_{2k}(E_k)=S_k\circ\phi$$

for $1\leq k\leq f+m$. From the co-finite condition, for each k there is a non-zero monic polynomial $P_k\in\mathbb{R}[x]$ such that $P_k(E_k)\in\mathcal{I}$. Let $\mathcal{I}_1$ be the ideal in $\mathcal{I}_{geo}$ generated (as an ideal) by the elements $P_k(E_k)$, $1\leq k\leq f+m$. $\mathcal{I}_1$ is non-expansive since it is spanned by products of the $P_k(E_k)$ and D^j, both of which are non-expansive.

To see that $\mathcal{I}_1$ is $\mathcal{N}$-co-finite, let $d_k=\deg P_k$. Note that

$$(S_k\circ\phi)^{d_k}=\sigma_{2kd_k}(P_k(E_k))\in\tilde{\mathcal{I}}_1.$$

Hence, any polynomial in the $S_k \circ \phi$ is equivalent to one of degree less than $\sum d_j$ in the $S_k \circ \phi$, mod $\tilde{\mathcal{I}}_1$. Our lemma follows from the reasoning following 57. $\square$

Let

$$\mathcal{P}_l = (\mathcal{A})_l / (\mathcal{A})_l \cap (\mathcal{J}_l + (\mathcal{S})_{l-1}\mathcal{N}_S)$$

Proposition 9. *Let* $_=\{X_1, \ldots, X_k\} \subset (\mathcal{S})$ *be a set of non-expansive elements such that there are* l_i *such that* $\{\sigma_{l_i}(X_i)\}$ *projects to a basis for* $Pol(\mathcal{A})/\tilde{\mathcal{J}}$ *where* $\mathcal{I}$ *is* $\mathcal{N}$*-co-finite and non-expansive. (It is clear that such* X_i *exist.) Then* $\mathcal{B}_l$ *projects to a basis of* $\mathcal{P}_l$ *for all* l*. Hence* $\mathcal{B}$ *projects to a basis of* $\mathcal{P}$*. Furthermore*

$$(\mathcal{A})_l \cap (\mathcal{J} + (\mathcal{S})\mathcal{N}_S) = \mathcal{J}_l + (\mathcal{S})_{l-1}\mathcal{N}_S$$

Proof Let $X \in (\mathcal{S})_l$. Let $l_o = \deg(\delta(X)) \leq l$. Then there are scalars c_i, elements $B_j \in (\mathcal{S})$, and non-expansive elements $I_j \in \mathcal{I}$, with $\deg(\delta(B_j I_j)) = l_o$, such that

$$\sigma_{l_o}(X) - \sum_i c_i \sigma_{l_o}(X_i) = \sum_j \sigma_{l_o}(B_j I_j).$$

We may in fact choose $B_j \in (\mathcal{A})$ since δ is zero on $(\mathcal{S})\mathcal{N}_S$.

Let $J_j = B_j I_j$, a non-expansive element. From the non-expansive property, X_i and J_j belong to $(\mathcal{S})_l$.

Let

$$X_1 = X - \sum_i c_i X_i - \sum_j J_j.$$

Then $X_1 \in (\mathcal{S})_l$ and $\delta(X_1)$ has lower degree than $\delta(X)$.

We may repeat this argument with X_1 in place of X. It follows by induction that there are elements J_j in $\mathcal{J}_l$ and constants c_i as above such that $\delta(X_1) = 0$. Then $X_1 \in ((\mathcal{S})\mathcal{N}_S)_l$, proving the first part of our proposition.

To prove the last statement, let $A \in (\mathcal{J} + (\mathcal{S})\mathcal{N}_S)_l$. From the preceding argument

$$A = \sum c_i X_i + J + Y$$

where the c_i are scalars, $J \in \mathcal{J}_l$, and $Y \in (\mathcal{S})\mathcal{N}_S$. $A \in \mathcal{J} + (\mathcal{S})\mathcal{N}_S$ implies that the c_i are all 0. Hence $Y = A - J$ belongs to $((\mathcal{S})\mathcal{N}_S)_l = (\mathcal{S})_{l-1}\mathcal{N}_S$, proving out result. □

Remark: If X is a Riemannian symmetric space and $\mathcal{I}$ is $D_G(X)$, then the finite dimensionality of $\mathcal{P}$ is known. (See the discussion in Section 5 of [1].) The existence of a non-expansive spanning set for $\mathcal{I}$ follows from the observation that the Harish Chandra homomorphism preserves degree. The analogue of Lemma 10 follows from a similar argument. These comments are used to prove Theorem 2 in the Riemannian symmetric case.

4 Explicit Expansions

In this section we prove the existence and convergence of the asymptotic expansions. We refer the reader to §1 for our the notation.

Proposition 10. *Let F be an $\mathcal{I}$-harmonic function on S which satisfies 2 and let $F_A \in \mathcal{H}_{-\omega}(\pi_r)$ be defined by formula 11 where $A \in \mathcal{A}^+$. Let $s > 0$ be given. Then for each $A \in \mathcal{A}^+$ and each $\gamma \in < A, \mathcal{E} >$, there exists an unique $\mathcal{H}_{-\omega}(\pi_r)$-valued function $F^\gamma(A,t)$ (not depending on s) which is polynomial of bounded degree in t and a $t_o \geq 0$ (which may depend on s) such that for all $t \geq t_o$,*

$$F_{tA} = \sum_{\gamma \in <A,\mathcal{E}>} F_\gamma(A,t)e^{\gamma t} \tag{58}$$

where the convergence is in $(\mathcal{H}^s_\omega(\pi_r))^*$. Furthermore for all $t \in \mathbb{R}^+$, $F_\gamma(A,t) \in C^{-\infty}(\pi_r)$ and the above equality is valid in the asymptotic topology on $C^{-\infty}(\pi_r)$ -valued functions.

Proof From Lemma 10 we may *assume that $\mathcal{I}$ is non-expansive and $\mathcal{N}$-co-finite.* Then Proposition 9 shows that $\mathcal{P}$ is finite dimensional. Let $\mathcal{P}$ be decomposed as in 8.

For $A \in \mathcal{A}$ and $t \in \mathbb{R}$, let

$$P^A(t) = \prod_{\alpha \in \mathcal{E}_o} (t - < A, \alpha >)^{n_\alpha}.$$

P^A is a real polynomial since the roots of $\mathcal{P}$ occur in conjugate pairs with equal multiplicity. The following lemma is clear from 9 since $\mathcal{P}$ has a unit.

Lemma 11. *For $A \in \mathcal{A}$, let $\tilde{A} \in \mathcal{P}$ be the projection of A. Then $P^A(\tilde{A}) = 0$.*

It follows from the preceding lemma that

$$P^A(A) = X^A + J^A$$

where $X^A \in (\mathcal{S})\mathcal{N}_S$ and $J^A \in \mathcal{J}$. From Proposition 9, we may take $\deg X^A \leq \deg P^A$. We may also assume that X^A depends linearly on $P^A(A)$ and thus polynomially on A. Thus X^A is a sum of terms of the form

$$p(A)X_1X_2\dots X_l \tag{59}$$

where the p are polynomials in A and each X_k belongs to either $\mathcal{S}_{ij}$ or $\mathcal{M}_{ij}$ for some $i \leq j$ depending on k. Furthermore, since $X^A \in (\mathcal{S})\mathcal{N}_S$, there is at least one X_k in each term for which $i < j$.

Since F is annihilated by $\mathcal{J}$ it follows that F satisfies

$$P^A(A)F(x) = X^AF(x) \tag{60}$$

where elements of the enveloping algebra are identified with left invariant differential operators on S.

For $X \in (\mathcal{S})$, let $X(t) = \text{Ad}\,(\exp(tA))X$. We replace x by $x(\exp tA)$ in 60 discovering that

$$P^A(D)F_{tA} = X^A(t)F_{tA} \tag{61}$$

where $D = \dfrac{d}{dt}$. Expanding X^A in a sum of terms of the form of 59, we find that F_{tA} satisfies an equation of the form

$$P^A(D)F_{tA} = \sum_k e^{<A,\beta_k>t}X_i^AF_{tA} \tag{62}$$

where $\beta_k \in \text{span}_{\,\mathbb{N}}\Sigma$ and $X_i^A \in (\mathcal{S})$ depends polynomially on A. In particular, $A \in \mathcal{A}^+$ implies that $< A, \beta_k >> 0$.

We interpret 62 as an $\mathcal{H}_{-\omega}(\pi_r)$ valued differential equation. We claim that in this case, the hypotheses of Theorem 5 are satisfied. We note first that for all $\phi \in \mathcal{H}_{\omega}(\pi_r)$, $F \in \mathcal{H}_{-\omega}(\pi_r)$, and $X \in (\mathcal{S})$,

$$\begin{aligned} < \pi_r(g^{-1})\phi, XF > &=< \pi_r(g^{-1})\phi, \pi_r^*(X)F > \\ &=< \pi_r(X^*)\pi_r(g^{-1})\phi, F > \\ &= X(< \pi_r(g^{-1})\phi, F >). \end{aligned}$$

It follows from Lemma 1, p. 459 of [3] and the definition of the $\mathcal{H}_{-\omega}(\pi_r)$ topology, that if X has degree d as an element of $(\mathcal{S})$ then it has degree $\leq d$ as an operator on $\mathcal{H}_{-\omega}(\pi_r)$.

From the example on.p 282 of [17], there are positive constants C and r' such that

$$e^{\tau(x)} \leq C \| \text{ Ad } (x) \|^{r'}$$

where $\|\cdot\|$ denotes the operator norm with respect to any conveniently chosen norm on $\mathcal{L}$. In particular, if $x = \exp tA$

$$e^{\tau(\exp tA)} \leq Ce^{r''t}$$

for some constant r''. Hence, from inequalities 2 and 10,

$$\{e^{-r''t} F_{tA} \,|\, t \in (-\infty, 0]\}$$

is bounded in $L_r^1(S)^*$ and, thus, in $\mathcal{H}_{-\omega}(\pi_r)$.

Formula 58 now follows immediately from Theorem 5. The statement about $C^{-\infty}(\pi_r)$ follows from Theorem 3 together with the uniqueness of the coefficients. □

Next we prove Theorem 2:

Proof (Of Theorem 2)

Let F be $\mathcal{I}$-harmonic and let $B_1, B_2, \ldots, B_r$ be a basis for $\mathcal{A}$ contained in $\mathcal{A}^+$. For $t \in (-\infty, 0)^n$, let

$$G_F(t) = F_{B(t)}$$

where $B(t) = t_1 B_1 + t_2 B_2 \cdots + t_d B_d$. Then letting $D_i = \dfrac{d}{dt_i}$ and reasoning as in 62, we find

$$P^{B_j}(D_j) G_F = \sum_k e^{<B(t), \beta_k>} X_k^j G_F$$

where $\beta_k \in \Delta$ and $X^j \in (\mathcal{S})$.

We consider the preceding set of equations as a $C^{-\infty}(\pi_r)$ valued system. Theorem 6 shows that under asymptotic convergence

$$G_F(t) = \sum_{\delta \in \mathcal{E}_1} e^{t \cdot \delta} F_\delta(t)$$

where $\mathcal{E}_1 \subset \mathbb{R}^d$ and $F_\delta(t)$ depends polynomially on t.

Let $A = B(s)$ where the $s_i > 0$. Then, under asymptotic convergence,

$$F_{tA} = G_F(ts) = \sum_{\delta \in \mathcal{E}_1} e^{t(s \cdot \delta)} F_\delta(ts). \tag{63}$$

The uniqueness of coefficients in asymptotic expansions shows that for the F_γ as in 58,

$$F_\gamma(A, t) = \sum_{\gamma = s \cdot \delta} F_\delta(ts)$$

It follows that the series 63 converges in $\mathcal{H}_{-\omega}(\pi_r)$ for all $t > t_o$. From the proof of Lemma 2, t_o depends continuously on A. In particular, we may choose a value of t_o so that the series for $t \to F_{tA/|A|}$ converges for all for $t > t_o$ and all A which are positive linear combinations of the B_i. Hence, the series in question converges at $t = 1$ if $|A| \geq t_o$. For such A, we define

$$F_\delta(A) = F_\delta(s).$$

This finishes the proof of Theorem 2 for homogeneous domains, with the exception of the covariance property (c). This, however, is a simple consequence of the equality

$$R(\exp B(s)) F_{B(t)} = F_{B(s+t)}.$$

The proof of 2 in the case of Riemannian symmetric spaces is almost identical. See the remarks at the end of §2. □

Remark We still need to discuss the Poisson transformation. This however, is more or less immediate from Theorems 3 and 4. Explicitly, let $A \in \mathcal{A}^+$. Theorem 4 allows us to construct F_{tA} for $t \geq t_o$ using the boundary values and the operators π^j. Hence, we can construct

$$F = \pi_r(exp - t_o A) F_{t_o A}.$$

If F is harmonic, then the result will of course be independent of the choice of A. Conversely, one might hope that if the result is independent of A then F would be harmonic. This, however, is the subject of future research.

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