

Lab 6

In this lab, you will perform a systematic study of the solutions to the equation

$$(1) \quad y'' = -by' - cy.$$

Notice that if $b \geq 0$ and $c \geq 0$, then this is the equation for the mass 1 box on a spring where c is the spring constant and b is the damping. We will also, however, study this equation in the “non-physical” cases where either b or c are negative.

1. Express the above equation as a linear system $X' = AX$ and find $p(\lambda) = \det(A - \lambda I)$. (Ans: $p(\lambda) = \lambda^2 + b\lambda + c$.)
2. The roots of the characteristic polynomial are

$$\lambda_{\pm} = (-b \pm \sqrt{b^2 - 4c})/2$$

They will be imaginary if $b^2/4 < c$. Using "fplot", plot the function $y = x^2/4$ for $-4 \leq x \leq 4$, $-4 \leq y \leq 4$. Get your plot printed and (using a pen perhaps) label the axes b and c . (Alternatively, you can get MATLAB to do the labeling. See the help for plot.)

- a Indicate on your picture the set of points (b, c) for which the eigenvalues for the corresponding system are not real. Call this region NR (for “not real”).
 - b Values of b and c outside NR yield real eigenvalues. Indicate on your graph the regions where (a) both eigenvalues are positive (Call this region “PP”) (b) both are negative (Call this region “NN”) and (c) where one is positive and one is negative. (Call this region “PN”) (Hint: Your answer will depend on which quadrant of the bc -plane (b, c) lies in. In the first quadrant, $c > 0$ and $b > 0$. Then, $b^2 - 4c < b^2$. What does this say about the sign of $-b \pm \sqrt{b^2 - 4c}$? Do a similar analysis for the other quadrants.)
3. Pick a point (b, c) in region PP (student’s choice) and enter the system corresponding to equation (1) into pplane. Plot several orbits and describe their behavior. Specifically, consider the following questions:
 - a Is the origin a source, a sink, a center, or a saddle. (An equilibrium point is a sink if all solutions which begin sufficiently close to it converge to it. It is a source if all solutions sufficiently close to it move away from it. It is a center if all solutions which begin sufficiently close to it “loop around” it. i.e. they return to their initial position after a finite amount of time. Finally, the equilibrium point is a saddle if some solutions converge to it and some move away from it.)
 - b If the origin is a source (respectively a sink) do the solutions spiral as they move away from (respectively, toward) it? Indicate the point you used on your $b - c$ plane and draw (in a small box near this point) a small sketch of the phase plane portrait for the corresponding system.
 4. Find the general solution to the system you studied in Exercise 2. (Same b and c .) Note: You can get MATLAB to do all of the work. Try this: Enter the following into MATLAB

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A=[1,3;3,1];
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$$[B,D]=\text{eig}(A)$$

The each column of B is an eigenvector for A and the non-zero entries of D are the corresponding eigenvalues.

5. Use your answer to the previous exercise to prove that your answers to Exercise 3-a would be valid for any solution to the system studied in Exercise 3.
6. Now choose a point (b, c) in region NN and repeat Exercises 3 and 4.
7. Now choose a point (b, c) in NP and repeat Exercises 3 and 4. It should turn out that the origin is a saddle. To prove this, you must consider the straight line solutions.
8. The behavior in region NR is somewhat varied. Find examples of points (b, c) for which the corresponding system has (a) a spiral sink (b) a spiral source and (c) a center at the origin. In each case, prove that your answer is correct by finding the general solution for the system in question. (Yes, MATLAB can compute complex eigenvalues.) Indicate on your picture from Exercise (a) which points in region NR would produce which kind of behavior.

For your summary, discuss what kinds of behavior can be expected for the orbits of the system corresponding to equation (1) depending upon which region (b, c) comes from. Relate this behavior to the nature of the eigenvalues of the matrix which describes the system.

Next consider a general linear system $X' = AX$ where A is a 2×2 matrix. Assume that A has two distinct eigenvalues. On the basis of the above work, describe what you feel the nature of the orbits would be in terms of the nature of the eigenvalues of A . For example, if both eigenvalues are negative, is the origin a source, sink or what. Why? If the eigenvalues are complex, under what conditions would the origin be a spiral sink and under what conditions would it be a spiral source? Under what conditions would it be a saddle? A center?