LAB 8

This lab analyzes the Lorentz system which is the system

$$\begin{aligned} x' &= \sigma(y - x) \\ y' &= \rho x - y - xz \\ z' &= -\beta z + xy \end{aligned} \tag{1}$$

Outside of lab (either before or after).

In the Lorentz system let $\sigma = 10$, $\rho = 28$, and $\beta = \frac{8}{3}$. Verify that in this case the equilibrium points are $(0, 0, 0)$, $(6\sqrt{2}, 6\sqrt{2}, 27)$, and $(-6\sqrt{2}, -6\sqrt{2}, 27)$.

During lab.

(1) Let $\sigma = 10$, $\rho = 28$, and $\beta = \frac{8}{3}$. Plot the Lorentz system with the initial value $[x_o, y_o, z_o] = [0, 1, 0]$. For this you will need to create a function file (call it lor.m) which contains the lines

```
function up=lor(t,u)
up(1,1)=-10*u(1)+10*u(2);
up(2,1)=28*u(1)-u(2)-u(1)*u(3);
up(3,1)= u(1)*u(2)-(8/3)*u(3);
```

(The symbol "up" is chosen to remind you of u'.) Once this file has been saved, enter

```
[t,u]=ode45('lor', [0,20],[0;1;0]); plot3(u(:,1),u(:,2),u(:,3));
```

The command "ode45" solves the system defined by the function lor. The "0,20" tells us that the time interval is $0 \leq t \leq 20$ and $[0, 1, 0]$ is the initial condition.

The beautiful butterfly shape you are seeing is just one orbit. The "wings" of the butterfly seem to be approaching planes. The purpose of this lab is to find formulas for these planes.

(2) Note that each "wing" seems to have a hole in it. We guess that there is an equilibrium point at the hole's center which, form the work done outside of lab, must be $(6\sqrt{2}, 6\sqrt{2}, 27)$. It appears to be approaching a spiral source in the plane. To prove it we will use the linear approximation to the system.

Specifically, let $u = x - 6\sqrt{2}$, $v = y - 6\sqrt{2}$ and $w = z - 27$. Assuming that x, y and z satisfy the system (1) with the stated values of σ, ρ and

β, find a system satisfied by u, v and w. Then show that this system is approximated by the linear system $X' = AX$ where

$$A = \begin{bmatrix} -10 & 10 & 0 \\ 1 & -1 & -6\sqrt{2} \\ 6\sqrt{2} & 6\sqrt{2} & -\frac{8}{3} \end{bmatrix}$$

(3) For the matrix A from the previous exercise, use the MATLAB command "[P,D]=eig(A)" to find the eigenvectors and eigenvalues for A. (The columns of P will be the eigenvectors and the diagonal entries of D are the eigenvalues of A.) The first and second columns of P are complex. In MATLAB, the i^{th} column of P is denoted "P(:,i)". Let "R=real(P(:,2))" and "I=imag(P(:,2))".

(4) Plot (in $\mathbb{R}^3$) the orbit of $X' = AX$ over the interval $0 \le t \le 20$ with R as initial data. (Use "ode45" as above, after creating the appropriate function file.) Then plot (on the same graph) the orbit of $X' = AX$ over the interval $0 \le t \le 20$ with I as initial data. (For this you can simply use the up arrow key to return to the plot command and change "R" to "I" and reenter the line. Don't forget to use "hold on" which should be executed *after* the first graph is drawn.) Notice that both orbits seem to lie in a single plane. Prove that this is indeed the case. (*Hint:* Show that the solution X to $X' = AX$ with initial data $X(0) = R$ can be expressed as a function times R plus another function times I. Repeat for the solution with initial data $X(0) = I$.)

The plane in question is the span of the vectors R and I. Find an equation for it in the form $au + bv + cw = 0$.

(5) On the same figure as in the previous exercise, get MATLAB to plot the orbit whose initial data is ten times the real eigenvector. Let W denote this eigenvector. If, for example, this is the first column of P, then $W = 10 * P(:,1)$ and we would replace R by W in the "ode45" command. The orbit seems to be a line. Prove that this is indeed the case. Does the solution tend toward or away from the origin as $t \to \infty$?

(6) Clear the previous figure with the command "cla" and plot the solution with initial data (R+(seed/10)*W). Get your plot printed and draw arrows indicating the direction of travel along the orbit. Do you think that the spirals actually lie in a plane? The behavior you are seeing is called a "spiral saddle." The plane is an "attractor" for the linear system because all orbits are attracted to it. It turns out that the linear approximation to the Lorentz system at the other equilibrium point also has a spiral saddle which is attracted to a plane.

(7) Clear the previous graph with "cla" , issue the command "hold off" and then plot the phase portrait of the Lorentz system over the interval $1 \le t \le 30$ with initial data $[0, 1, 0]$. Note the way that the orbit randomly hops from spiral to spiral. The "butterfly" shape you see is an attractor for the orbits because all of the orbits are drawn to it. It is referred to as a "strange attractor" because it is unlike attractors found in linear systems.

(8) In linear algebra, if T, X and Y are vectors in $\mathbb{R}^3$, then the set of all points of the form $T+rX+sY$, where r and s vary over all real numbers, is a plane through the point T. This description is called the *parametric* description of the plane. If $T = (6\sqrt{2}, 6\sqrt{2}, 27)$, $X = R$ and $Y = I$ where R and I are as in the previous exercise, then one of the butterfly wings is attracted to the corresponding plane. Explain why this is true on the basis of the work done in the previous exercise.

(9) Find a description of the plane from the preceding exercise in the form $ax + by + cz = d$.

(10) Find both a parametric description and a description in the form $ax + by + cz = d$ for the plane which is approached by the other butterfly wing.

(11) You can change the viewing angle of your graph with the command "view(a,b)" where "a" is the viewing angle in the xy-plane and "b" is the angle of elevation (both measured in degrees). Look at the Lorentz attractor from a variety of viewpoints and print the two that you think are the most revealing.

(12) Construct the plots (with respect to t) of x, y and z with the initial conditions (a) $(x_o, y_o, z_0) = (0, 1.1, 0)$ and (b)$(x_o, y_o, z_0) = (0, 1.01, 0)$. Comment on how long it takes for two solutions of the Lorentz system with very close initial conditions to separate and any consistency (or lack there of) in the size of oscillations of the coordinates over large time intervals Note: To plot, say, the x-coordinate of the solution generated above against t, you can simply enter the command "plot(t,u(:,1))". To see the detail of interest you may need to recompute the solution with a different t range.