# LAB 9: RESONANCE

Assume that we have a box of mass 1g on a frictionless table and attached to a spring. Initially, the spring is unstretched. We pull the box 3 cm to the right and let it go. We have seen that the motion of the spring is governed by the equation

$$my'' = -k_d y' - k_s y \tag{1}$$

where $m$ is the mass of the cube (which is 1), $k_d$ is the coefficient of friction and $k_s$ is the spring constant. This equation comes from Newton's law $ma = F$. The term on the right of this equality is the total force on the spring and the term on the left is $ma$.

## Exercises

(1) Assume that $m = 1$, $k_d = 0$, and $k_s = .25$ dynes/cm. Thus, it takes .25 dynes of force to stretch the spring 1cm. Convert equation (1) to a system, enter the system into pplane. Use the "Keyboard input" option to plot the orbit corresponding to 3cm stretching and 0 initial velocity. Use your graph to estimate the period of the motion. It might help to enter the "grid on" command into MATLAB. Would you consider this slow or fast oscillation? (The time is in seconds.)

(2) Find the general solution to the above system. Then find a formula for the solution $y$ that satisfies $y(0) = 3, y'(0) = 0$. Use your answer to find the exact value for the period of the motion you estimated in the previous exercise.

(3) The period of the oscillations is determined by the "stiffness" of the spring. (A stiff spring is one which takes a large force to stretch it.) How do you guess stiffness should relate the period of the motion: should stiff springs oscillate faster or slower? Test your guess by graphing the solution curve for a stiff spring and a non-stiff spring.

Note: This will require that you change $k_s$ in equation (1). How should you change it to model a stiffer spring?

(4) Imagine that our box is resting near our stereo that is generating a tone which is causing the box to vibrate. We model this as applying an external force $F(t) = (seed/10)\sin(\omega t)$ where $\omega$ is determined by the pitch of the tone. Now equation (1) becomes

$$my'' = -k_d y' - k_s y + (seed/10)\sin(\omega t) \tag{2}$$

Assume that $m = 1$, $k_s = .25$, and $k_d = 0$. The equivalent system is

$$\begin{aligned} y' &= v \\ v' &= -.25y + (seed/10)\sin(\omega t) \end{aligned} \tag{3}$$

The system corresponding to equation (1) would be obtained by setting *seed* $= 0$. The above system is a very different "animal" from the system corresponding to equation (1) due to the explicit presence of $t$ in the second equation. Thus, this system is non-autonomous while the system corresponding to equation (1) is autonomous.

In this part of the exercise, we want to plot the solution as a function of $t$ for several different values of $\omega$. Unfortunately, pplane requires an autonomous system. Thus, we must use some program such as ode45. For this, you must first create a function file to represent the right side of this equation. If $\omega = .3$, you would first create a file called, perhaps, "yy.m" containing the lines

```
function x=yy(t,u);
x(1;1)=u(2);
x(2;1)=-.25*u(1)+(seed/10)*sin(.3*t);
```

(In this function, u(1) represents $y$ and $u(2)$ represents $v$.)

To solve the differential equation you enter

```
[t,u]=ode45('yy',[0,40],[0,0]);
```

The "0,40" tells MATLAB to solve the equation for $0 \leq t \leq 40$. The term "[0,0]" tells MATLAB that the initial velocity is 0 and there is no initial stretching. (The box is initially at rest! Any motion of the box is due entirely to the vibrations.)

We can plot the result with the command

```
plot(t,u(:,1));
```

(If we wanted to plot $v$, we would use u(:,2).)

Plot the graphs of $y$ for $\omega = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8$, and $0.9$. (Don't get your graphs printed.) In each case use your graph to determine (approximately) the maximum displacement of the box from its rest position. To find the maximum, you may need to graph $y$ over a larger time interval. Construct (either by hand or on the computer) a graph of the maximum displacement as a function of $\omega$.

Note: You might want to put more tick marks onto the $y$-axis than MATLAB's default option provides. The following command puts marks from $-100$ to 100 at .5 intervals. It should be executed after the graph is drawn. (You could put it on the same line as the plot command to ease re-executing it.) You might also want to execute "grid on" which will draw a grid on the graph.

```
set(gca,'Ytick',-100:.5:100);
```

**Remark.** The *frequency* of a wave is the number of oscillations per unit period of time. The *period* is the time it takes to complete each oscillation. It follows that the frequency is the reciprocal of the period. Hence, the frequency of $f(t) = \sin(\omega t)$ is $\omega/(2\pi)$ cycles per second. Thus, $\omega$ is proportional to the frequency. The graph of maximum displacement against frequency is referred to as the *frequency response* of the system. It indicates how sensitive the system is to changes in the driving frequency. Your graph is essentially a graph of the frequency response of the box.

(5) In Exercise 4, you should find that the behavior at $\omega = .5$ is quite extraordinary. What you are observing is a phenomena call resonance. If we vibrate the box with just the right frequency, the oscillations become ever bigger and bigger. Prove that this is indeed correct by solving equation (2) with the stated initial conditions and $\omega = .5$. Notice that the peaks in the graph for this case seemed to lie on a straight line. Use your solution to prove that this is true. What is the slope of this line?

The frequency at which resonance occurs is called the *resonant frequency* of the system. Hence, the resonant frequency of our box is $.5/(2\pi)$ cycles per second. In general, the resonant frequency for a system described by $y'' + k_s y = 0$ is $\sqrt{k_s}/2\pi$ which is also the frequency at which the system oscillates without any external force.

(6) The behavior you observed in Exercise 4 would not happen in real life. One limiting factor is friction. Assume that in equation 1, $k_d = .3$. Construct a graph of the frequency response of the new system as you did in Exercise 4. Try to estimate as accurately as you can the value of $\omega$ for which the maximum displacement is greatest. (It is not $\omega = .5$.)

Note: The maximum does not change rapidly as $\omega$ moves away from .5. To judge the place where the maximum occurs, you may find it convenient to graph the curves for several different values of $\omega$, in different colors, on the same graph so that you can tell which curve is highest. You might also want to have grid on.

*Turn in only the graph of the frequency response with the maximum displacement indicated.*

The value of $\omega/2\pi$ at which the maximum occurs would be the resonant frequency for this system.

(7) In your write up, discuss what changing the spring constant does to the rate at which the spring oscillates. What is resonance? Discuss the effect of resistance on resonance. Does adding resistance change the resonant frequency? If so, does the frequency increase or decrease?

In this lab, we only investigated two values of $k_d$, 0 and .3. As a guess, what do you think a plausible frequency response curve would look like for a considerably smaller, but non-zero, value of $k_d$. Draw, on one sheet of paper, the graphs from Exercise 4, Exercise 6, and your guess. Indicate very carefully, where you expect the maximum of your guess to lie. Explain why you put it there.