LAB 10: OFF LINE SPRINGS

Prior to lab. Read the last page.

During the lab. Let notation be as on the last pages. *Throughout this lab we will assume that* $M = L = K_s = 1$ so that the equation governing the motion is

$$y'' = -y + y/\sqrt{y^2 + a^2} \tag{1}$$

(1) Express the above equation as a first order system. Show that if $a \geq 1$, then this system has only one equilibrium point at $y = 0$. The equilibrium point at $y = 0$ is stable in the sense that if our mass is sitting at $y = 0$ and we give it a slight nudge, it will stay near $y = 0$. Explain. (Hint: At $y = 0$, is the spring stretched or compressed? Note that the length of the spring is 1.)

(2) Show that if $a < 1$, then this system has three equilibrium points. Draw a diagram similar to that on the back pages showing these points and explain their physical significance. The equilibrium point at $y = 0$ is unstable in the sense that if our mass is sitting at $y = 0$ and we give it a slight nudge, it will not stay near $y = 0$. Explain.

(3) Assume that $0 < a < 1$. Find a linear approximation to the system from the preceding exercise near each equilibrium point. What kind of equilibrium point does the linear approximation predict?

(4) Now, let $a = .5$. Use pplane to plot the orbits corresponding to $y(0) = 1$ and $y'(0) = v_o$ for several values of v_o. You should discover that the behavior is quite different for small and large v_o. Describe physically what type of motion is occurring in each case. Try to approximate (to within one decimal) the value of v_o at which the behavior changes.

(5) Modify the above differential equation to include a small amount of damping (i.e., a small amount of friction). Once again, use pplane to plot the orbits corresponding to $y(0) = 1$ and $y'(0) = v_o$ for several values of v. In each case, describe the motion of the mass. Notice that all of the orbits tend to one or the other of the equilibrium points. All orbits for which v_o is sufficiently small should converge to the point on the right. (Give a physical reason for why this should be true.) However, if we have a sufficiently large initial velocity, determining where we wind up is very difficult. As a demonstration of this, find values $v_1 < v_2 < v_3 < v_4$ such that the solution with $y'(0) = v_i$, $y(0) = 1$, seems to tend to the right equilibrium point for v_1 and v_3 and to the left point for v_2 and v_4.

    Remark: What you are seeing is an example of Chaos theory. Arbitrarily small changes in the input data can produce a dramatic effect in the final result. We say "seems" above because small inaccuracies in our differential equation solver can actually change the point to which our solution tends; hence these pictures are quite unreliable in their specifics.

(6) Next, we shall study resonance in this system. Assume that there is no damping, but we are applying an external force of magnitude $.4 \sin \omega t$.

On the basis of Lab 9, we expect that (since there is no damping) there should be one specific value of ω for which resonance occures. Specifically, for this value of ω, the size of the oscillations should become larger and larger. Try to approximate the value of ω at which resonance occurs by plotting the orbits for the solutions corresponding to the initial values $y(0) = 1$, $y'(0) = .45$ and $\omega = .1, .2, \ldots, .9$. Turn in the plot for which resonance seems to occur.

Remark: The system required for this exercise is non-autonomous. This means that you will need to use "ode45" (or "ode23") instead of "pplane". (See Lab 9 for information on how to do this.) I suggest using the time interval from 0 to 300. Lab 9 did not discuss how to plot orbits. This, however, is simple. If the output of ode45 has been called "[t,u]", then the command " plot(u(:,2),u(:,1));" will do the job.

This works because u is a large matrix such that the entries in the first column are values of y at various times and the entries in the second column are the corresponding values of v. Thus, plotting the second column against the first produces a plot of y against v, which is the orbit.

(7) The behavior of the system from the preceding exercise is quite interesting for small values of ω as well. Describe the motion of the mass for $\omega = .1$. Do you think that the orbit will eventually choose one or the other equilibrium points to circle around? Defend your answer by graphing a larger time interval. Plot the velocity of the mass as a function of time. (See Lab 9). Indicate on your graph the places where the acceleration is the greatest. (Recall that acceleration is the derivative of the velocity.) The acceleration indicates the amount of force acting on the mass. Do you think that a bug ridding on the mass would have a smooth ride?

(8) For your lab report, discuss (in physical terms) the possible behavior of the system studied for various initial velocities, depending on the presence or absence of friction. Discuss also its behavior under a sinusoidal driving force. Will this system exhibit resonance? If so, for what value of ω?

Discuss also how predictable you think the behavior of the system will be. Specifically, if I give you the initial position and velocity and I provide you with a computer, will you be able to tell me with a reasonable degree of accuracy, what y will be at, say, $t = 300$? Does you ability to make this prediction depend on whether or not there is an external driving force?

A box of mass m which is free to slide along a frictionless rod, is attached to a spring of natural length L by a pivot. The other end of the spring is attached with a pivot at a fixed point P, a units above the rod, as shown in Figure 1. Let y denote the signed distance from the box to the point on the rod directly below P. Then the force on the box due to the stretching of the spring is

$$F_s = K_s(\sqrt{y^2 + a^2} - L)$$

where K_s is the spring constant. Only the component of the force along the rod contributes to the motion of the box. Hence, setting force equal to mass times acceleration yields

$$\begin{aligned} My'' &= -K_s(\cos\phi)(\sqrt{y^2 + a^2} - L) \\ &= -K_s\frac{y}{\sqrt{y^2 + a^2}}(\sqrt{y^2 + a^2} - L) \\ &= K_s(-y + \frac{Ly}{\sqrt{y^2 + a^2}}) \end{aligned}$$