LAB 11 (THE PENDULUM)

Prior to lab. No specific preparation is required.

During the lab. The motion of a frictionless pendulum of length L is governed by the differential equation

$$y'' = -\frac{g}{L}\sin y.$$

where y is the angle (in radians) that the pendulum makes with the vertical, $g = 32.2 ft/sec^2$, and L is the length of the pendulum. In this lab, we study the motion of the pendulum.

Exercises

(1) Assume that $L = g/seed$ so that the equation simplifies to

(1) $$y'' = -(seed)\sin y$$

Convert this equation into a first order system by setting $v = y'$ and find the equilibrium points. (There are an infinite number of them.) Construct the corresponding phase plane portrait. (Put y on the horizontal axis.) Orbits with large v coordinate are unbounded. Make sure your scale is large enough to show this as well as several of the equilibria.

(2) What kind of equilibrium does $(0, 0)$ seem to be? Prove that you are right by finding the Jacobian matrix M_0 at $(0, 0)$ and its eigenvalues. Plot a phase plane portrait for the system $X' = M_0 X$, using the same range on v and y as in exercise 1, to demonstrate that the linear system approximates the nonlinear one near $(0, 0)$.

(3) There is also an equilibrium at $(\pi, 0)$. What kind of equilibrium does it seem to be? Prove that you are right by finding the Jacobian matrix M_1 at $(\pi, 0)$ and its eigenvalues. Plot a phase plane portrait for the system $X' = M_1 X$ using the same range on v and y as in exercise 1, to demonstrate that the linear system approximates the nonlinear one near $(\pi, 0)$. (Of course, the portrait for the linear system will be centered at $(0, 0)$ instead of $(\pi, 0)$.)

In the pplane options menu, select "Plot stable and unstable orbits". A cross will appear on the screen. Using the mouse, place the cross close to $(0, 0)$ and click. What is produced is the orbits corresponding to the straight line solutions. Demonstrate this by computing (using MATLAB) the eigenvectors for M_1 and plotting them (by hand) on your graph. Indicate which of these straight line solutions is stable and which is unstable. (Here, stable means that the orbit moves toward the equilibrium point and unstable means that it moves away from the equilibrium point.) How, from the eigenvalues, can you tell which is which?

(4) Reenter the system corresponding to equation (1) into pplane. Use the "Plot stable and unstable orbits" option to plot the stable and unstable orbits near $(\pi, 0)$. Get this plot printed. Draw the tangent lines to these orbits at $(\pi, 0)$. These tangent lines are the straight line solutions from the previous exercise, translated over to the point $(\pi, 0)$. Check this comparing their slope (computed from your graph) with the slope of the straight line solution as computed using the eigenvector. Show your computations.

The point here is that at each saddle point for a nonlinear system, there are four special orbits, called *separatrices*, whose tangent lines are parallel to the straight line solutions for the linear approximation. (Each one is a *separatrix.*)

(5) The reason for the term "separatrix" is that the separatrices separate the phase plane into regions where the solutions behave very differently. Demonstrate this by first plotting the stable and unstable orbits for each of the saddle points shown in the plot from exercise (1). Then plot the orbits corresponding to several initial conditions lying inside the regions bounded by the separatrices and several initial conditions lying outside these regions. Describe the motion of the pendulum for each type of initial condition.

Plot y vs. t for an initial condition lying outside of the regions bounded by the separatrices. The motion of the pendulum should be periodic. However, the graphs certainly does not seem to be periodic. Explain how the graph you are seeing really can describe periodic motion. What exactly is the pendulum doing?

(6) A child is sitting on a swing which is $g/seed$ feet long. Use your plot to estimate the lowest velocity that we could push her with to make her go over the top of the swing? (You may ignore resistance as well as the wrapping of the chain around the support post.)

(7) Let y be a solution to equation (1) and let $v = y'$. Show that

$$\frac{d(v^2/2 - (seed)\cos y)}{dt} = 0$$

(*Hint:* $d(v^2)/dt = 2v\,dv/dt$, $d(\cos y)/dt = -(\sin y)dy/dt$). It follows that there is a constant C (depending on the orbit) such that

$$v^2/2 - (seed)\cos y = C \qquad (2)$$

The significance of this equation is that it provides a formula for the orbits. Specifically, each orbit is described by

$$v = \pm\sqrt{2C + 2(seed)\cos y} \qquad (3)$$

As a demonstration of this, assume that at $y = 0$, the velocity is $v_o = 1$. Find C. Plot using fplot the functions in formula (3). For sake of comparison also generate (using pplane) the orbit corresponding to $y(0) = 0$ and $v(0) = 1$. The two plots should be identical.

Remark: The expression on the left side of the equality in formula (2) represents the total energy of the pendulum. Formula (2) states that energy is conserved.

(8) Find a formula for the separatrices near $(\pi, 0)$. (*Hint:* Use the fact that the separatrix approaches the point $(\pi, 0)$ to find C.) Use this to compute the exact value of v_o for which the point $(0, v_o)$ lies on the separatrix. This value of v_o is the exact answer to the value you estimated in Exercise 6. Explain.

(9) One learns in physics classes that pendulums are used for making clocks because their period depends only on their length. In particular, the period does not depend on how we start the pendulum moving. The "proof " of this fact usually is based on the linear approximation to the differential equation. Specifically, for y small, $\sin y \approx y$. Thus, for y small, the motion of the pendulum is described by the *linear* equation

$$y'' = -(seed)y \tag{4}$$

Find the general solution to this equation and use it to compute the period of the motion. As a check on your computation, plot several solutions to this equation corresponding to different initial conditions. (Use the y vs.t option of pplane.) Measure as accurately as possible the period of each solution and compare it with the predicted value.

(10) Is the period of the real pendulum equal to that of the linear approximation? To investigate this, approximate (using a graph) the period of the solution to equation (1) corresponding to the initial conditions $y(0) = 0$, $v(0) = v_o$ for several values of v_o. Choose a wide range of values of v_o ranging from values near 0 to values where the point $(0, v_o)$ is close to the separatrix. Do not turn in your plots. Rather, turn in a plot of the period verses initial velocity. Is it really true that the period does not depend on the initial velocity? If not, at what initial velocity does the actual value begin to differ significantly from that of the linear approximation?

(11) In your discussion discuss

(a) The different types of behavior one can expect for the pendulum depending on the initial velocity.

(b) The relation between the initial velocity and the period.

(c) The domain of validity of the linear approximation.