Exam 3

Last Name:____

First Name:____

No notes, calculators, or other electronic devices such as cell phones are allowed during the exam. Violation of this policy will result in an automatic 0 on the test. There should be nothing on your desk other than the test and something to write with. Please put all work and answers on the test sheets. If extra space is needed, please the backs of the pages.

Justify all answers. A correct answer without supporting justification is worth NO credit!

1. Find the characteristic polynomial $p(\lambda)$ for A where *5 pts.*

$$A = \begin{bmatrix} 2 & 3 & 0 \\ 0 & 7 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

2. Given that the characteristic polynomial for the matrix A below is $p(\lambda) = -(\lambda - 3)^2(\lambda - 4)$, find a basis for the $\lambda = 3$ eigenspace. *6 pts.*

$$\begin{bmatrix} -1 & -2 & 2 \\ 0 & 3 & 0 \\ -10 & -5 & 8 \end{bmatrix}$$

3. Let

$$A = \begin{bmatrix} -2 & -1 & -1 \\ 2 & -5 & -2 \\ 1 & -1 & -4 \end{bmatrix}$$

(a) Verify that the vectors X, Y, and Z are eigenvectors for A where *6 pts.*

$$X = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, Y = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, Z = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

(b) Find an eigenvector W for A that has one positive and two negative entries. (*Note*: 0 is neither positive nor negative!) *2 pts.*

4. Suppose that A is a square matrix with characteristic polynomial $p(\lambda) = \lambda^2(\lambda+2)^3(\lambda^2-4)^4$.

(a) Is A invertible? Why? *3 pts.*

(b) What are the possible dimensions for the $\lambda = -2$ eigenspace of A? *Be careful! Look closely at* $p(\lambda)$. *4 pts.*

5. Let

$$A = \begin{bmatrix} -2 & 12 & -2 \\ -3 & 9 & 0 \\ -3 & 4 & 5 \end{bmatrix}$$

It is given that X_1, X_2, and X_3 are eigenvectors for A corresponding respectively to the eigenvalues 3, 4, and 5 where

$$X_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \quad X_2 = \begin{bmatrix} 5 \\ 3 \\ 3 \end{bmatrix} \quad X_3 = \begin{bmatrix} 4 \\ 3 \\ 4 \end{bmatrix}$$

Find an **explicit** diagonal matrix D and an **explicit** invertible matrix Q such that $A = QDQ^{-1}$. **Do not compute Q^{-1}.** *6 pts.*

$Q =$

$D =$

6. Let A be as in Problem 5. Find **explicit** matrices B and C such that $A = (BCB^{-1})^3$. All that is asked for is B and C. **Do not compute** B^{-1} **or** BCB^{-1}. *4 pts*

$B =$

$C =$

7. Find **all** values of a, b, and c for which the matrix A below is diagonalizable. *8 pts.*

$$A = \begin{bmatrix} 3 & a & b & 0 \\ 0 & 3 & 0 & c \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

8. Suppose that A is an $n \times n$ invertible matrix. Prove that if λ is an eigenvalue for A then λ^{-1} is an eigenvalue for A^{-1}. *7 pts.*

9. We use the ordered basis $\mathcal{B} = \{[1,1]^t, [1,3]^t\}$ to define coordinates for $\mathbb{R}^2$. Find the $\mathcal{B}$ coordinate vector for $X = [3,4]^t$. *5 pts.*

10. Let $\mathcal{P}_2$ be the space of all polynomial functions of the form ax^2+bx+c where $a, b, c \in \mathbb{R}$. Let $L : \mathcal{P}_2 \mapsto \mathcal{P}_2$ be the linear transformation defined by

$$L(y) = 2y' + y$$

We use the standard ordered basis $\mathcal{B} = \{1, x, x^2\}$ for both the domain and target space of L. Find the matrix M that represents L in these bases. *6 pts.*

11. Let all information be as in problem 10 except that we now use the standard ordered basis $\mathcal{B}_1 = \{1, x, x^2\}$ for the domain and the ordered basis $\mathcal{B}_2 = \{1, (x+3)^2, (x+3)\}$ for the target space of L. Find the matrix M that represents L in these bases. *8 pts.*

12. Show that the set $\mathcal{B}$ formed by the following vectors is an orthogonal basis for $\mathbb{R}^3$.

$$\mathcal{B} = \{[1, 1, -1]^t, [1, -2, -1]^t, [1, 0, 1]^t\}$$

5 pts.

13. Order the basis $\mathcal{B}$ from problem 12 as listed in that problem.

   (a) Use orthogonality to find the $\mathcal{B}$ coordinate vector for $X = [x, y, z]^t$.
   Other methods will not give credit. *7 pts.*

(b) **Use the answer to problem 13a** to find the coordinate matrix $C_{\mathcal{B}}$ for the basis. *Other methods will not give credit.*

4 pts.

14. We want to apply the Gram-Schmidt process to the following ordered basis $\mathcal{B}$ of $\mathbb{R}^3$ to produce an orthogonal basis $\mathcal{P} = \{P_1, P_2, P_3\}$ of $\mathbb{R}^3$.

$$\mathcal{B} = \{[1, 2, 3]^t, [0, 1, 1]^t, [1, 1, 1]^t\}$$

(a) Compute the first two Gram-Schmidt basis elements P_1 and P_2. *3 pts.*

(b) Assume that your answer to part 14a was $P_1 = [1, -1, 1]^t$ and $P_2 = [1, 1, 0]^t$ (which is not correct). What would you obtain for P_3 if you continue to follow the Gram-Schmidt process using these incorrect answers? *5 pts.*

15. Prove the following theorem, which is part of Theorem 4 on p. 318, which you were supposed to learn for this test. *6 pts.*

Theorem 1. *Let $\mathcal{B} = \{P_1, P_2, \ldots, P_n\}$ be an ordered orthogonal basis for $\mathbb{R}^n$ and let $X \in \mathbb{R}^n$. Then*

$$X = x_1' P_1 + x_2' P_2 + \cdots + x_n' P_n \tag{1}$$

where

$$x_i' = \frac{X \cdot P_i}{P_i \cdot P_i} \tag{2}$$