Final

Last Name:____

First Name:____

No notes, calculators, or other electronic devices such as cell phones are allowed during the exam. Violation of this policy will result in an automatic 0 on the test. There should be nothing on your desk other than the test and something to write with. Please put all work and answers on the test sheets. If extra space is needed, use the backs of the pages.

Justify all answers. A correct answer without supporting justification is worth NO credit!

1. Find the characteristic polynomial $p(\lambda)$ for the following matrix A. *You need not simplify your answer, so long as it does not involve determinants. I know that you can multiply polynomials!* *5 pts.*

$$A = \begin{bmatrix} 2 & 0 & 3 \\ 5 & -4 & 0 \\ 0 & 1 & 2 \end{bmatrix}.$$

2. Given that the characteristic polynomial for the matrix A below is $p(\lambda) = (\lambda^2 - 10\,\lambda + 20)\,(\lambda - 3)^2$, find a basis for the $\lambda = 3$ eigenspace.

6 pts.

$$A = \begin{bmatrix} 4 & 0 & 2 & 1 \\ 1 & 4 & 1 & 1 \\ 1 & 1 & 4 & 1 \\ 1 & 1 & 1 & 4 \end{bmatrix}$$

3. Let

$$A = \begin{bmatrix} -8 & 4 & -2 \\ -6 & 2 & -3 \\ -8 & 8 & -8 \end{bmatrix}$$

Verify that the vectors X, Y, and Z are eigenvectors for A where *6 pts.*

$$X = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, Y = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, Z = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

4. Let A be as in Problem 3. Find an eigenvector $W = [x, y, z]^t$ for A such that $y = 0$. *3 pts.*

5. Let

$$A = \begin{bmatrix} 1 & 6 & -2 \\ -3 & 4 & 0 \\ -8 & 8 & 1 \end{bmatrix}, \; B = \begin{bmatrix} 12 \\ 20 \\ 54 \end{bmatrix}$$

It is given that X_1, X_2, and X_3 are eigenvectors for A corresponding respectively to the stated eigenvalues λ.

$$X = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \; (\lambda = 1), \; Y = \begin{bmatrix} 2 \\ 3 \\ 8 \end{bmatrix} \; (\lambda = 2), \; Z = \begin{bmatrix} 1 \\ 3 \\ 8 \end{bmatrix} \; (\lambda = 3)$$

Find numbers a, b, and c such that

$$A^{100}B = aX + bY + cZ.$$

Hint: Compute $X + 2Y + Z$. *7 pts.*

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6. Let A be as below. Given that the characteristic polynomial of A is $p(\lambda) = (\lambda - 3)^2 + 3$, find **explicit** matrices Q and D where D is diagonal such that $A = QDQ^{-1}$. All that is asked for is Q and D. **Do not compute Q^{-1} or QDQ^{-1}.** *Note:* Your answer might involve complex numbers. *7 pts*

$$A = \begin{bmatrix} 3 & -3 \\ 1 & 3 \end{bmatrix}$$

7. Suppose that A is an $n \times n$ diagonalizable matrix such that the only eigenvalues of A are 1, -1, and 0. Prove that $A^3 = A$. *8 pts.*

8. Prove that there are no values of a and b for which the matrix A below is diagonalizable. *8 pts.*

$$A = \begin{bmatrix} 2 & 1 & 1 & a \\ 0 & 4 & 0 & b \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

9. We use the ordered basis $\mathcal{B} = \{[1,1,0]^t, [0,1,1]^t, [0,0,1]^t\}$ to define coordinates for $\mathbb{R}^3$. Find the $\mathcal{B}$ coordinate vector for $X = [1,2,3]^t$. *7 pts.*

10. Recall that $\mathcal{P}_n$ be the space of all polynomial functions of the form $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ where $a_k \in \mathbb{R}$, $k = 0\ldots, n$. Let $L : \mathcal{P}_1 \mapsto \mathcal{P}_2$ be the linear transformation defined by

$$L(y) = 2y' + xy$$

We use the ordered basis $\mathcal{B}_1 = \{1, x+1\}$ for the domain and the ordered basis $\mathcal{B}_2 = \{x^2, x, 1\}$ for the target space of L. Find the matrix M that represents L in these bases. *7 pts.*

11. Assume that $L : \mathcal{P}_2 \mapsto \mathcal{P}_2$ is the linear transformation that is described by the matrix M below relative to the ordered bases $\mathcal{B}_1 = \{1, x, x^2\}$ for the domain and $\mathcal{B}_2 = \{1, (x+1), (x+1)^2\}$ for the target space

$$M = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

Find scalars a, b, and c such that

$$L(a + bx + cx^2) = 1 + 2x.$$

7 pts.

12. Find scalars a, b, and c that the set $\mathcal{B}$ formed by the following vectors is an orthogonal basis for $\mathbb{R}^3$.

$$\mathcal{B} = \{[1, 1, -3]^t, [1, 2, a]^t, [b, c, 1]^t\}$$

6 pts.

13. Let $\mathcal{W}$ be the subspace of $\mathbb{R}^4$ spanned by the set $\mathcal{B}$ below.

$$\mathcal{B} = \{[1,1,1,1]^t, [1,1,1,-3]^t\}$$

(a) Find the projection of $Z = [1,2,1,1]^t$ to $\mathcal{W}$. *4 pts.*

(b) Find vectors X and Y such that $Z = X + Y$ where $X \in \mathcal{W}$ and Y is orthogonal to $\mathcal{W}$. *3 pts.*

14. Let $\mathcal{W}$ be the subspace of $\mathbb{R}^4$ spanned by the set $\mathcal{B}$ below. We want to apply the Gram-Schmidt process to produce an orthogonal basis $\mathcal{P} = \{P_1, P_2, P_3\}$ for $\mathcal{W}$.

$$\mathcal{B} = \{[1,1,0,1]^t, [0,2,1,0]^t, [1,1,1,1]^t\}$$

(a) Compute the first two Gram-Schmidt basis elements P_1 and P_2. *3 pts.*

(b) Assume that your answer to part 14a was $P_1 = [1, -2, 0, 0]^t$ and $P_2 = [2, 1, 0, 1]^t$ (which is not correct). What would you obtain for P_3 if you continue to follow the Gram-Schmidt process using these incorrect answers? *5 pts.*

15. Prove the following result which is Theorem 2 on p. 294. Here it is assumed that all of the entries a_{ij} of A are real numbers and all of the λ_i are real numbers. *8 pts*

Theorem. *Let A be an $n \times n$ matrix and let Q_1, Q_2, …, Q_k be eigenvectors for A in $\mathbb{R}^n$ corresponding to the eigenvalues λ_i. Suppose that the λ_i are all different. Then $\{Q_1, Q_2, \ldots, Q_k\}$ is a linearly independent subset of $\mathbb{R}^n$.*