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Justify all answers. A correct answer without supporting justification is worth NO credit!

1. Find the characteristic polynomial $p(\lambda)$ for A where *5 pts.*

$$A = \begin{bmatrix} 2 & 3 & 0 \\ 0 & 7 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

$$\lambda I = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 2-\lambda & 3 & 0 \\ 0 & 7-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{bmatrix}$$

$$p(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 3 & 0 \\ 0 & 7-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{vmatrix}$$

$$= (2-\lambda)\left[(7-\lambda)(2-\lambda)\right] - 3(0) + 0$$

$$= (2-\lambda)^2(7-\lambda)$$

$$\therefore\ p(\lambda) = (2-\lambda)^2(7-\lambda)$$

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2. Given that the characteristic polynomial for the matrix A below is $p(\lambda) = -(\lambda-3)^2(\lambda-4)$, find a basis for the $\lambda = 3$ eigenspace. *6 pts.*

$$\begin{bmatrix} -1 & -2 & 2 \\ 0 & 3 & 0 \\ -10 & -5 & 8 \end{bmatrix}$$

$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

$p(\lambda) = 0$

$\therefore \lambda = 3$ or 4

$AX = \lambda X$.

$\lambda = 3$

$\therefore (A - 3I)X = 0$

$\therefore \begin{bmatrix} -4 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ -10 & -5 & 5 & 0 \end{bmatrix} \leftarrow$ Augmented matrix

$\begin{bmatrix} 2 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 1 & -1 & 0 \end{bmatrix}$ $R_1' = R_1/-2$, $R_3' = R_3/-5$

$\therefore \begin{array}{cccc} x & y & z & \\ \end{array}$
$\begin{bmatrix} 2 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $R_3' = R_3 - R_1$

$2x + y - z = 0$

[s and t are arbitrary constants]

let $y = s$; $z = t$

$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{t-s}{2} \\ s \\ t \end{bmatrix} = s\begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1/2 \\ 0 \\ 1 \end{bmatrix}$

$\therefore$ Eigen ~~vectors~~ = $\begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 0 \\ 1 \end{bmatrix}$

Basis : $\left\{ \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 0 \\ 1 \end{bmatrix} \right\}$

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3. Let

$$A = \begin{bmatrix} -2 & -1 & -1 \\ 2 & -5 & -2 \\ 1 & -1 & -4 \end{bmatrix}$$

(a) Verify that the vectors X, Y, and Z are eigenvectors for A where *6 pts.*

$$X = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, Y = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, Z = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$AX = \begin{bmatrix} -2 & -1 & -1 \\ 2 & -5 & -2 \\ 1 & -1 & -4 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 3 \end{bmatrix} = -3\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \lambda_1 X \qquad \lambda_1 = -3.$$

$$AY = \begin{bmatrix} -2 & -1 & -1 \\ 2 & -5 & -2 \\ 1 & -1 & -4 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ -10 \\ -5 \end{bmatrix} = -5\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \lambda_2 Y \qquad \lambda_2 = -5.$$

$$AZ = \begin{bmatrix} -2 & -1 & -1 \\ 2 & -5 & -2 \\ 1 & -1 & -4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ -3 \\ 0 \end{bmatrix} = -3\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \lambda_3 Z. \qquad \lambda_2 = -3$$

$\therefore$ X, Y, Z are eigenvectors with eigenvalues $-3, -5, -3$ respectively.

(b) Find an eigenvector W for A that has one positive and two negative entries. (*Note*: 0 is neither positive nor negative!) *2 pts.*

$W = -X - 2Z = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} -2 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix}$

w, X and Z have eigenvalue $= -3$

4. Suppose that A is a square matrix with characteristic polynomial $p(\lambda) = \lambda^2(\lambda+2)^3(\lambda^2-4)^4$.

(a) Is A invertible? Why? *3 pts.*

$AX = \lambda X$; we know $p(\lambda) = \det(A - \lambda I)$; from the data given, λ has a value 0. $\therefore \det A = p(0) = 0$; $\therefore \det A = 0$

$\therefore$ A is not invertible because A can only be invertible when $\det A \neq 0$. $\therefore \det A = 0$ and A is not invertible.

(b) What are the possible dimensions for the $\lambda = -2$ eigenspace of A? *Be careful! Look closely at $p(\lambda)$.* *4 pts.*

$\lambda^2 - 4 = 0$ $\quad (\lambda+2)^3 = 0$

$\lambda^2 = 4$ $\quad \therefore \lambda = -2$

$\lambda = \pm 2$

$(\lambda^2-4)^4 = [(\lambda+2)(\lambda-2)]^4 = (\lambda+2)^4(\lambda-2)^4$

$\therefore$ Dimensions for $\lambda = -2$ can be :

1, 2, 3, 4, 5, 6, 7

5. Let

$$A = \begin{bmatrix} -2 & 12 & -2 \\ -3 & 9 & 0 \\ -3 & 4 & 5 \end{bmatrix}$$

It is given that X_1, X_2, and X_3 are eigenvectors for A corresponding respectively to the eigenvalues 3, 4, and 5 where

$$X_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \quad X_2 = \begin{bmatrix} 5 \\ 3 \\ 3 \end{bmatrix} \quad X_3 = \begin{bmatrix} 4 \\ 3 \\ 4 \end{bmatrix}$$

Find an **explicit** diagonal matrix D and an **explicit** invertible matrix Q such that $A = QDQ^{-1}$. **Do not compute** Q^{-1}. *6 pts.*

$$Q = \begin{bmatrix} 2 & 5 & 4 \\ 1 & 3 & 3 \\ 1 & 3 & 4 \end{bmatrix}$$

X_1, X_2, X_3 are eigen vectors ; thus they are independent. Q has columns X_1, X_2, X_3. ∴ Q is a rank 3 matrix and is invertible because $\det Q \neq 0$

$$D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}.$$

6. Let A be as in Problem 5. Find **explicit** matrices B and C such that $A = (BCB^{-1})^3$. All that is asked for is B and C. **Do not compute** B^{-1} **or** BCB^{-1}. *4 pts*

$A = BCB^{-1}BCB^{-1}BCB^{-1} = BCCCB^{-1} \quad (BB^{-1} = I)$

$\therefore A = BC^3B^{-1}$

$$B = Q = \begin{bmatrix} 2 & 5 & 4 \\ 1 & 3 & 3 \\ 1 & 3 & 4 \end{bmatrix}$$

$C^3 = D$.

$$C = \sqrt[3]{D} = \begin{bmatrix} \sqrt[3]{3} & 0 & 0 \\ 0 & \sqrt[3]{4} & 0 \\ 0 & 0 & \sqrt[3]{5} \end{bmatrix}$$

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7. Find **all** values of a, b, and c for which the matrix A below is diagonalizable. *8 pts.*

$$A = \begin{bmatrix} 3 & a & b & 0 \\ 0 & 3 & 0 & c \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$$p(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 3-\lambda & a & b & 0 \\ 0 & 3-\lambda & 0 & c \\ 0 & 0 & 2-\lambda & 0 \\ 0 & 0 & 0 & 2-\lambda \end{vmatrix}$$

$\therefore p(\lambda) = (3-\lambda)^2(2-\lambda)^2$; $\therefore \lambda = 3$ or 2.

when $\lambda = 3$

$$A - 3I = \begin{bmatrix} 0 & a & b & 0 \\ 0 & 0 & 0 & c \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

rank of $A-3I$ is always 3. if $a \neq 0$ ✓

$\therefore$ we will only get one eigenvector from this. For A to be diagonalizable it needs two eigenvectors from $\lambda = 3$ and rank of $A-3I$ should have been 2. since multiplicity is 2.

$\therefore$ we let $a = 0$; then rank of $A - 3I$ is 2.

To check ; for $\lambda = 2$

$$A - 2I = \begin{bmatrix} 1 & a & b & 0 \\ 0 & 1 & 0 & c \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ if $a = 0$; $$A - 2I = \begin{bmatrix} 1 & 0 & b & 0 \\ 0 & 1 & 0 & c \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ rank of matrix $A-2I$ is 2 and we get two eigen vectors and that is what we need since multiplicity is 2

$\therefore$

$a = 0$

However b and c can have any value to make A diagonalizable. b and c can be any possible number. ✓

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8. Suppose that A is an $n \times n$ invertible matrix. Prove that if λ is an eigenvalue for A then λ^{-1} is an eigenvalue for A^{-1}. *7 pts.*

$AX = \lambda X$ (given)

$A^{-1}AX = A^{-1}\lambda X$ (A^{-1} exists)

$IX = \lambda A^{-1}X$ ($AA^{-1} = I$; λ is a scalar).

$X = \lambda A^{-1}X$

$A^{-1}X = \frac{1}{\lambda} X = \lambda^{-1}X$ $\lambda \neq 0$? -1

$\therefore A^{-1}X = \lambda^{-1}X$.

$\therefore A^{-1}$ has an eigenvalue λ^{-1}

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9. We use the ordered basis $\mathcal{B} = \{[1,1]^t, [1,3]^t\}$ to define coordinates for $\mathbb{R}^2$. Find the $\mathcal{B}$ coordinate vector for $X = [3,4]^t$. *5 pts.*

$$P_B = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}$$

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{array}\right] = \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 2 & -1 & 1 \end{array}\right] \quad R_2' = R_2 - R_1$$

$$= \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & -1/2 & 1/2 \end{array}\right] \quad R_2' = R_2/2 \quad = \left[\begin{array}{cc|cc} 1 & 0 & 3/2 & -1/2 \\ 0 & 1 & -1/2 & 1/2 \end{array}\right]$$

$R_1' = R_1 - R_2$

$$\therefore \quad P_B^{-1} = \frac{1}{2}\begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\therefore \quad C_B = \frac{1}{2}\begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}$$

$C_B X = X'$ where X' is the B coordinate vector,

$$\therefore \quad X' = \frac{1}{2}\begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 5/2 \\ 1/2 \end{bmatrix}.$$

$$\therefore \quad X' = \begin{bmatrix} 5/2 \\ 1/2 \end{bmatrix}.$$

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10. Let $\mathcal{P}_2$ be the space of all polynomial functions of the form ax^2+bx+c where $a, b, c \in \mathbb{R}$. Let $L : \mathcal{P}_2 \mapsto \mathcal{P}_2$ be the linear transformation defined by

$$L(y) = 2y' + y$$

We use the standard ordered basis $\mathcal{B} = \{1, x, x^2\}$ for both the domain and target space of L. Find the matrix M that represents L in these bases. *6 pts.*

$B = \{1, x, x^2\}$

$L(y) = 2y' + y$

$\therefore L(1) = 2\times 0 + 1 = 1$

$L(x) = 2\times 1 + x = 2 + x$

$L(x^2) = 2 \cdot 2x + x^2 = 4x + x^2$

$$\therefore M = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

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11. Let all information be as in problem 10 except that we now use the standard ordered basis $\mathcal{B}_1 = \{1, x, x^2\}$ for the domain and the ordered basis $\mathcal{B}_2 = \{1, (x+3)^2, (x+3)\}$ for the target space of L. Find the matrix M that represents L in these bases. *8 pts.*

$B = \{1, x, x^2\}$

$L(y) = 2y' + y$

$L(1) = 2 \times 0 + 1 = 1$

$L(x) = 2 \times 1 + x = 2 + x = -1 + (x+3)$

$L(x^2) = 2 \cdot 2x + x^2 = 4x + x^2$

$= (x+3)^2 - 2(x+3) - 3$

$$\therefore M = \begin{bmatrix} 1 & -1 & -3 \\ 0 & 0 & 1 \\ 0 & 1 & -2 \end{bmatrix} \quad -0$$

Grading Error: The answer is correct.

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12. Show that the set $\mathcal{B}$ formed by the following vectors is an orthogonal basis for $\mathbb{R}^3$.

$$\mathcal{B} = \{[1,1,-1]^t, [1,-2,-1]^t, [1,0,1]^t\}$$

5 pts.

$x_1 = [1, 1, -1]^t$; $x_2 = [1, -2, -1]^t$; $x_3 [1, 0, 1]^t$

$x_1 \cdot x_2 = 1 - 2 + 1 = 0$

$x_1 \cdot x_3 = 1 + 0 - 1 = 0$

$x_2 \cdot x_3 = 1 + 0 - 1 = 0$

Since the dot products are zero, the vectors are orthogonal. And since the vectors are independent, B is the basis for $\mathbb{R}^3$.

$\therefore$ B is an orthogonal basis.

13. Order the basis $\mathcal{B}$ from problem 12 as listed in that problem.

(Theorem 6.4 on p. 312)

(a) Use orthogonality to find the $\mathcal{B}$ coordinate vector for $X = [x, y, z]^t$.
Other methods will not give credit. *7 pts.*

$$x_1' = \frac{X \cdot P_1}{P_1 \cdot P_1}$$

$$x_1' = \frac{[x,y,z]^t \cdot [1,1,-1]^t}{[1,1,-1]^t \cdot [1,1,-1]^t} = \frac{x+y-z}{1+1+1} = \frac{x+y-z}{3}$$

$$x_2' = \frac{X \cdot P_2}{P_2 \cdot P_2}$$

$$x_2' = \frac{[x,y,z]^t \cdot [1,-2,-1]^t}{[1,-2,-1]^t \cdot [1,-2,-1]^t} = \frac{x-2y-z}{1+4+1} = \frac{x-2y-z}{6}$$

$$x_3' = \frac{X \cdot P_3}{P_3 \cdot P_3}$$

$$x_3' = \frac{[x,y,z]^t \cdot [1,0,1]^t}{[1,0,1]^t \cdot [1,0,1]^t} = \frac{x+z}{1+1} = \frac{x+z}{2}$$

$$\therefore X' = \begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} \frac{x+y-z}{3} \\ \frac{x-2y-z}{6} \\ \frac{x+z}{2} \end{bmatrix}$$

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(b) **Use the answer to problem 13a** to find the coordinate matrix $C_{\mathcal{B}}$ for the basis. *Other methods will not give credit.*

4 pts.

$$x' = \begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} \frac{x+y-z}{3} \\ \frac{x-2y-z}{6} \\ \frac{x+z}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1/3 & 1/3 & -1/3 \\ 1/6 & -1/3 & -1/6 \\ 1/2 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

$$x' = C_B x$$

$$\therefore C_B = \begin{bmatrix} 1/3 & 1/3 & -1/3 \\ 1/6 & -1/3 & -1/6 \\ 1/2 & 0 & 1/2 \end{bmatrix}.$$

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14. We want to apply the Gram-Schmidt process to the following ordered basis $\mathcal{B}$ of $\mathbb{R}^3$ to produce an orthogonal basis $\mathcal{P} = \{P_1, P_2, P_3\}$ of $\mathbb{R}^3$.

$$\mathcal{B} = \{[1,2,3]^t, [0,1,1]^t, [1,1,1]^t\}$$

(a) Compute the first two Gram-Schmidt basis elements P_1 and P_2. *3 pts.*

$P_1 = X_1 = [1,2,3]^t$

$Y_2 = \frac{X_2 \cdot P_1}{P_1 \cdot P_1} P_1 = \frac{0+2+3}{1+4+9}[1,2,3] = \frac{5}{14}[1,2,3]^t$

$P_2 = X_2 - Y_2 = \left[-\frac{5}{14}, 1-\frac{10}{14}, 1-\frac{15}{14}\right]^t$

$= \frac{1}{14}[-5, 4, -1]^t$

$\therefore P_1 = [1,2,3]^t$

$P_2 = \frac{1}{14}[-5,4,-1]^t$

$X_3 = [1, 1, 1]^t$

(b) Assume that your answer to part 14a was $P_1 = [1, -1, 1]^t$ and $P_2 = [1, 1, 0]^t$ (which is not correct). What would you obtain for P_3 if you continue to follow the Gram-Schmidt process using these incorrect answers? *5 pts.*

$$Y_3 = \frac{X_3 P_1}{P_1 \cdot P_1} P_1 + \frac{X_3 P_2}{P_2 \cdot P_2} P_2$$

$$Y_3 = \frac{1-1+1}{1+1+1} [1, -1, 1]^t + \frac{1+1+0}{1+1+0} [1, 1, 0]^t$$

$$= \frac{1}{3} [1, -1, 1]^t + [1, 1, 0]^t$$

$$= [4/3, \frac{2}{3}, \frac{1}{3}]^t$$

$$\therefore P_3 = X_3 - Y_3 = [1, 1, 1]^t - \frac{1}{3} [4, 2, 1]^t$$

$$= [-1/3, 1/3, 2/3]^t$$

$$P_3 = \frac{1}{3} [-1, 1, 2]^t$$

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Theorem 6.4 on p. 312

15. Prove the following theorem, which is part of ~~Theorem 4 on p. 318~~, which you were supposed to learn for this test. *6 pts.*

Theorem 1. *Let $\mathcal{B} = \{P_1, P_2, \ldots, P_n\}$ be an ordered orthogonal basis for $\mathbb{R}^n$ and let $X \in \mathbb{R}^n$. Then*

$$X = x_1' P_1 + x_2' P_2 + \cdots + x_n' P_n \tag{1}$$

where

$$x_i' = \frac{X \cdot P_i}{P_i \cdot P_i} \tag{2}$$

x_i' exist? -1

$X = x_1' P_1 + x_2' P_2 + \ldots + x_n' P_n$.

Multiply both sides by P_1

$\therefore \quad X \cdot P_1 = x_1' P_1 \cdot P_1 + x_2' P_2 \cdot P_1 + \ldots + x_n' P_n \cdot P_1$

Since $\mathcal{B}$ is an orthogonal basis,

$P_2 \cdot P_1 = 0$; $\ldots$ $P_n \cdot P_1 = 0$.

$\therefore \quad X \cdot P_1 = x_1' P_1 \cdot P_1 + 0 + 0 \ldots + 0$

$X \cdot P_1 = x_1' P_1 \cdot P_1$

$\therefore \; x_1' = \dfrac{X \cdot P_1}{P_1 \cdot P_1}$

General case:

$\therefore$ If $X = x_1' P_1 + \ldots + x_i' P_i + \ldots + x_n' P_n$

Multiplying both sides by P_i ✓

$\therefore \; X \cdot P_i = 0 + \ldots + x_i' P_i \cdot P_i + 0 \ldots + 0$.

$\therefore \; X \cdot P_i = x_i' P_i \cdot P_i$

$\therefore \; x_i' = \dfrac{X \cdot P_i}{P_i \cdot P_i}$