1. Bring the following matrix into row reduced echelon form:  

$$A = \begin{bmatrix} 3 & 6 & 1 & -4 & 0 & 11 \\ 2 & 4 & 1 & -3 & 0 & 8 \\ -1 & -2 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 3 & 6 & 1 & -4 & 0 & 11 \\ 2 & 4 & 1 & -3 & 0 & 8 \\ -1 & -2 & 0 & 1 & 1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & -2 & 0 & 1 & 1 & 1 \\ 2 & 4 & 1 & -3 & 0 & 8 \\ 3 & 6 & 1 & -4 & 0 & 11 \end{bmatrix}$$  

$$A_3 \rightarrow R_1$$

$$\rightarrow \begin{bmatrix} 1 & 2 & 0 & -1 & -1 & -1 \\ 2 & 4 & 1 & -3 & 0 & 8 \\ 3 & 6 & 1 & -4 & 0 & 11 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 0 & -1 & -1 & -1 \\ 0 & 0 & 1 & -1 & 2 & 10 \\ 0 & 0 & 1 & -1 & 3 & 14 \end{bmatrix}$$  

$$A_1 \rightarrow R_1$$

$$\rightarrow \begin{bmatrix} 1 & 2 & 0 & -1 & -1 & -1 \\ 0 & 0 & 1 & -1 & 2 & 10 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 0 & -1 & -1 & -1 \\ 0 & 0 & 1 & -1 & 2 & 10 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$  

$$A_3 \rightarrow R_2 - 2R_3$$

$$A_2 \rightarrow R_2$$

$$A_1 \rightarrow R_1 + R_3$$
2. Let $A$, $X$ and $B$ be as below.

$$A = \begin{bmatrix} 2 & 1 & 1 & 2 \\ 1 & 0 & -1 & 2 \\ 4 & 3 & 5 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 3 \\ -3 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

(a) Find the general solution to the system $AX = B$.

_Hint:_ See the last page of the test.

(b) What are the translation and spanning vectors?

\[
A\begin{bmatrix} x \\ y \\ 2 \\ w \end{bmatrix} = B = \begin{bmatrix} 1 \\ 3 \\ -2 \\ -5 \end{bmatrix}
\]

\[
\begin{bmatrix} 2 & 1 & 1 & 2 \\ 1 & 0 & -1 & 2 \\ 4 & 3 & 5 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ 2 \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -2 \\ -5 \end{bmatrix}
\]

\[
x = 3 + z - 2w \\
y = -5 - 3z + 2w
\]

Let $z = s$, $w = t$

\[
\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 3 + s - 2t \\ -5 - 3s + 2t \\ s \\ t \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \\ s \\ t \end{bmatrix} + t\begin{bmatrix} -2 \\ 3 \\ 0 \\ 1 \end{bmatrix}
\]

b) Translation vector: $[3, -5, 0, 0]^t$

Spanning vector: $[1, -3, 1, 0]^t$, $[-2, 3, 0, 1]^t$
3. Find a spanning set for the nullspace of the matrix $A$ in Problem 2.  

The translation vectors span the nullspace

So

$\begin{bmatrix} 1 \\ -3 \\ -1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -2 \\ 0 \\ 1 \end{bmatrix}$ span the nullspace

4. Prove that the spanning set you found in Problem 3 is linearly independent. (The proof can be very short.)

There is a zero in each of the matrices where there is a 1 in the other matrices (the third and fourth entries of the matrices).
5. For which values of $a$, $b$, and $c$ will the following system have a solution? 9 pts

\[
\begin{align*}
x + 2y + 2z &= a \\
3x - y + 2z &= b \\
x + 16y + 10z &= c
\end{align*}
\]

\[
\begin{bmatrix}
1 & 2 & 2 & a \\
3 & -1 & 2 & b \\
1 & 10 & 10 & c
\end{bmatrix} \xrightarrow{R_2 \rightarrow R_2 - 3R_1} \begin{bmatrix}
1 & 2 & 2 & a \\
0 & -7 & -4 & b - 3a \\
0 & 14 & 8 & c - a
\end{bmatrix} \xrightarrow{R_3 \rightarrow R_3 - R_1} \begin{bmatrix}
1 & 2 & 2 & a \\
0 & -7 & -4 & b - 3a \\
0 & 0 & 2a - 7a + c
\end{bmatrix} \xrightarrow{R_3 \rightarrow R_3 + 2R_2}
\]

**For the system to have a solution, it must be consistent, so $2a - 7a + c = 0$.**
6. Create a $3 \times 5$ matrix $A$ with none of its entries equal to 0 such that $AX = B$ is solvable if and only if $B$ belongs to the span of $U$ and $V$ where

$$U = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}, \quad V = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}. \quad \text{9 pts}$$

Explain why your answer works. Note that we are asking about $B$, not $X$.

\[ \text{span of } U \text{ and } V = aU + bV \]

\[ = a[1, -2, 4]^T + b[1, 0, -1]^T \]

\[ = [a+b, -2a, 4a-b]^T \]

\[ \therefore a=1, \ b=1, \ [2, -2, 3]^T \]
\[ a=2, \ b=1, \ [3, -4, 7]^T \]
\[ a=1, \ b=2, \ [3, -2, 2]^T \]
\[ a=2, \ b=2, \ [4, -4, 6]^T \]
\[ a=3, \ b=1, \ [4, -6, 11]^T \]

\[ A = \begin{bmatrix} 2 & 3 & 3 & 4 & 4 \\ -2 & -4 & -2 & -4 & -6 \\ 3 & 7 & 2 & 6 & 11 \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}, \quad AX = B \]

\[ AX = B \text{ is solvable if and only if } B \text{ belong to the column space of } A \]

\[ \text{column space of } A = x_1A_1 + x_2A_2 + x_3A_3 + x_4A_4 + x_5A_5 \]
\[ = \text{span of } U \text{ and } V = B \]

\[ AX = B \text{ is solvable if and only if } B \text{ belong to the span of } U \text{ and } V \]

\[ \therefore \text{We have matrix } A \text{ created in this way.} \]
7. Prove that if $A$ satisfies the requirements of Problem 6, then the following equation will not be solvable: 

$$AX = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

For $AX = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ to be solvable, $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ must belong to the column space, or the span of the columns of $A$. The column space of $A$ is spanned by $\begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

So for some scalars $a$ and $b$, $a\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + b\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ for the equation to be solvable.

$$R = \begin{pmatrix} 1 & 1 & 1 \\ -2 & 0 & 1 \\ 4 & -1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & -5 & -3 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 2 & 3 \\ 0 & -5 & -3 \end{pmatrix} \rightarrow R_2 \rightarrow R_3 - 4R_2$$

$\begin{pmatrix} 0 & 2 & 3 \\ 0 & -5 & -3 \end{pmatrix} \rightarrow R_3 \rightarrow R_3 - 5R_2$

$0 \neq 9$, so the system is inconsistent. $[1,1,1]^T$ does not belong to the span of $U + V$, so it does not belong to the column space of $A$, which is why the system $AX = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is not solvable.
8. Create a system of three equations in four unknowns (reader’s choice) such that the solution space is a plane in $\mathbb{R}^4$ that does not pass through 0. Do not make any coefficients equal 0. Explain why your example works.

\[
\begin{bmatrix}
1 & 1 & 1 & 1 \\
2 & 3 & 4 & 3 \\
3 & 8 & 4 & 5
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z \\
w
\end{bmatrix} =
\begin{bmatrix}
a_1 \\
a_2 \\
a_3
\end{bmatrix}
\]

\[
x + y + z + w = 1
\]
\[
2x + 7y + 3z + 4w = 3
\]
\[
3x + 8y + 4z + 5w = 4
\]

For a system of 3 equations in 4 unknowns to have a solution space as a plane in $\mathbb{R}^4$, it must have 2 free variables, which means it must be a rank 2 system. This means that 1 of the equations must be linearly dependent on the other two equations, which are linearly independent of each other. Also, since it is a non-homogeneous system, it will not pass through 0. The third equation is linearly dependent on the first two equations.

9. Change one single number in the system created in problem 8 to obtain an inconsistent system. Explain why the system is inconsistent.

\[
\begin{align*}
A_1 & : x + y + z + w = 1 \\
A_2 & : 2x + 7y + 3z + 4w = 3 \\
A_3 & : 3x + 8y + 4z + 5w = 5
\end{align*}
\]

$A_1 + A_2 = 3x + 8y + 4z + 5w = 4$

but $A_3 = 3x + 8y + 4z + 5w = 5$

The system is inconsistent because $3x + 8y + 4z + 5w$ cannot equal both 4 and 5.
10. For a certain $3 \times 3$ matrix $A$ it is observed that

$$
A \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = A \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}.
$$

Find a non-zero element $Z$ of the nullspace of $A$.

3 pts

We want $AZ = 0$ so then $Z$ belongs to nullspace

$$
A \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix},
$$

$$
A \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix},
$$

$$
A \begin{bmatrix} 2 \\ 3 \\ 1 \\ -1 \\ -1 \end{bmatrix} - A \begin{bmatrix} 3 \\ 4 \\ 2 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} - \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}.
$$

11. Suppose that the element $Z$ found in Problem 11 spans the nullspace of $A$. Explain why the general solution to the system $AX = [1, -2, -1]^T$ is

$$
X = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.
$$

4 pts

The general solution is a particular solution plus the span of the nullspace. So for $AX = B$ when $X = T + Z$ with $T$ being a particular solution and $Z$ being the nullspace.

\[ AX = B, \]
\[ A(T+Z) = B, \]
\[ AT + AZ = B, \]
\[ \text{Since } AZ = 0 \text{ because } Z \text{ is the nullspace of } A, \]
\[ AT = B. \]
A span is a subspace.

12. Let \( X_1 = [1, 3, 7]^t \), \( X_2 = [3, 1, -2]^t \), \( Y_1 = [7, 5, 3]^t \) and \( Y_2 = [4, 4, 5]^t \). Prove that \( X_1 \) and \( X_2 \) span the same subspace of \( \mathbb{R}^3 \) as \( Y_1 \) and \( Y_2 \). For this you need to prove (a) that every element of the span of \( X_1 \) and \( X_2 \) also belongs to the span of \( Y_1 \) and \( Y_2 \) and, conversely, (b) that every element of the span of \( Y_1 \) and \( Y_2 \) also belongs to the span of \( X_1 \) and \( X_2 \).

\[
\begin{align*}
X_1 &= \begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix} \\
X_2 &= \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} \\
Y_1 &= \begin{bmatrix} 7 \\ 5 \\ 3 \end{bmatrix} \\
Y_2 &= \begin{bmatrix} 4 \\ 4 \\ 5 \end{bmatrix}
\end{align*}
\]

\( X_1 + 2X_2 = Y_1 \)
\( X_1 + X_2 = Y_2 \)

Let \( U = aY_1 + bY_2 \) which is the span of \( Y_1 \) and \( Y_2 \), (any scalar \( a \) and \( b \))

Then \( U = a(X_1 + 2X_2) + b(X_1 + X_2) \)

\[
U = (a + b)X_1 + (2a + b)X_2
\]

\( U = cX_1 + dX_2 \) (where \( c = a + b, d = 2a + b \))

which is the span of \( X_1 \) and \( X_2 \)

So every element in the span of \( Y_1 \) and \( Y_2 \) belongs to the span of \( X_1 \) and \( X_2 \).

\[
\begin{align*}
2Y_2 - Y_1 &= X_1 \\
Y_1 - Y_2 &= X_2
\end{align*}
\]

\( U = aX_1 + bX_2 \) which is the span of \( X_1 \) and \( X_2 \) (any scalar \( a \) and \( b \))

Then \( U = a(2Y_2 - Y_1) + b(Y_1 - Y_2) \)

\[
\begin{align*}
U &= (-a + b)Y_1 + (2a - b)Y_2 \\
&= (cY_1 + dY_2) \quad \text{(where } c = -a + b, d = 2a - b \text{)}
\end{align*}
\]

which is the span of \( Y_1 \) and \( Y_2 \). So every element in the span of \( Y_1 \) and \( Y_2 \) belongs to the span of \( X_1 \) and \( X_2 \).
13. Let \( W \) be the set of matrices of the form below where \( a, b \) and \( c \) range over all real numbers.

\[
\begin{bmatrix}
  a + b + 3c, & -2a + b, & 3a + b + 5c
\end{bmatrix}
\]

Prove that \( W \) is a subspace of \( M(1,3) \). (Reason directly from either the definition of “subspace” or Theorem 6 on p. 79 of the text. Do not use the theorem that says that spans are subspaces!)

9 pts

Let \( u, v \) be elements of the subspace where

\[
\begin{align*}
  u &= \begin{bmatrix} a + b + 3c, & -2a + b, & 3a + b + 5c \end{bmatrix} \\
  v &= \begin{bmatrix} a' + b' + 3c', & -2a' + b', & 3a' + b' + 5c' \end{bmatrix}
\end{align*}
\]

where \( a, b, c, a', b', c' \) are all scalars.

Then \( u + v = \begin{bmatrix} (a+a') + (b+b') + 3(c+c'), & -2(a+a') + (b+b'), & 3(a+a') + (b+b') + 5(c+c) \end{bmatrix} \]

which is in the proper form to belong to \( W \) where \( a'' = a + a' \)

\( b'' = b + b' \), \( c'' = c + c' \)

Proving subspace property 1.

Let \( k \) be any scalar, then

\[
\begin{align*}
  ku &= \begin{bmatrix} ka + kb + 3kc, & -2ka + kb, & 3ka + kb + 5kc \end{bmatrix} \\
  &= \begin{bmatrix} a^* + b^* + 3c^*, & -2a^* + b^*, & 3a^* + b^* + 5c^* \end{bmatrix}
\end{align*}
\]

which is in proper form to belong to \( W \) where \( a^* = ka, b^* = kb, \) and \( c^* = kc \)

Proving subspace property 2. and when \( k = 0 \), then \( 0 \in W \), proving subspace property 3.
14. (a) Find a spanning set for the set $W$ from Problem 13.

(b) Find a linearly independent spanning set for the set $W$ from Problem 13.

\[ \begin{bmatrix} a + b + 3c \\ -2a + b \\ 3a + b + 5c \end{bmatrix} \]

\[ = \begin{bmatrix} a \\ -2a \\ 3a \end{bmatrix} + \begin{bmatrix} b \\ b \\ b \end{bmatrix} + \begin{bmatrix} 3c \\ 0 \\ 5c \end{bmatrix} \]

\[ = a \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} + b \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c \begin{bmatrix} 3 \\ 0 \\ 5 \end{bmatrix} \]

A spanning set is \[ \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} 3 \\ 0 \\ 5 \end{bmatrix} \]

B. \[ A + 2B = C \]

So a linearly independent spanning set is \[ \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \]
15. Let \( W \) be the set of all \( 2 \times 2 \) matrices of the form shown below where \( a \) and \( b \) range over all real numbers. Prove that none of the subspace properties (1)-(3) in Theorem 6 on p. 79 of the text) hold for \( W \).

\[
X + Y \in W, \\
x = \begin{bmatrix} 1 & a+b \\ 0 & 3b \end{bmatrix}, \\
y = \begin{bmatrix} 1 & a'+b' \\ 0 & 3b' \end{bmatrix},
\]

where \( a, b, a', b' \) are scalars.

Then \( X + Y = \begin{bmatrix} 2 & (a+a')+(b+b') \\ 0 & 3(b+b') \end{bmatrix} \)

\[
= \begin{bmatrix} 2 & a''+b'' \\ 0 & 3b'' \end{bmatrix}, \quad \text{where } a'' = a+a', \quad b'' = b+b'
\]

which does not belong to \( W \) because \( 2 \neq 1 \) which goes against subspace property 1.

For the scalar 2,

\[
2X = \begin{bmatrix} 2 & 2a+2b \\ 0 & b b' \end{bmatrix}
\]

which does not belong to \( W \) because of the 2 in the 13 entry that must be a 1, so this goes against subspace property 2.

Also, subspace property 3 says that \( 0 \in W \), but \( \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \) has a 0 where there should be a 1 to belong to the subspace.
16. Let $X_1$, $X_2$, and $X_3$ be elements of some vector space. Prove that $W = \text{span} \{X_1, X_2, X_3\}$ is a subspace. **Reason directly from either the definition of "subspace" or Theorem 6 on p. 79 of the text. Do not use the theorem that says that spans are subspaces!**

For any scalars $a_1, a_2, a_3, b_1, b_2, b_3 \in \mathbb{R}$,

$U = a_1 X_1 + a_2 X_2 + a_3 X_3$ \[ V = b_1 X_1 + b_2 X_2 + b_3 X_3 \]

which are both elements of the span of $X_1, X_2, X_3$.

Then, $sU + tV = sa_1 X_1 + sa_2 X_2 + sa_3 X_3 + tb_1 X_1 + tb_2 X_2 + tb_3 X_3$

$= (sa_1 + tb_1) X_1 + (sa_2 + tb_2) X_2 + (sa_3 + tb_3) X_3$

$= f X_1 + g X_2 + h X_3$

(with $f = sa_1 + tb_1$, $g = sa_2 + tb_2$, $h = sa_3 + tb_3$) which is a subspace by the definition of subspace.