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1. Bring the following matrix into row reduced echelon form: 10 pts

$$A = \begin{bmatrix} 3 & 6 & 1 & -4 & 0 & 11 \\ 2 & 4 & 1 & -3 & 0 & 8 \\ -1 & -2 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 6 & 1 & -4 & 0 & 11 \\ 2 & 4 & 1 & -3 & 0 & 8 \\ -1 & -2 & 0 & 1 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} -1 & -2 & 0 & 1 & 1 & 1 \\ 2 & 4 & 1 & -3 & 0 & 8 \\ 3 & 6 & 1 & -4 & 0 & 11 \end{bmatrix} \begin{array}{l} A_3 \to A_1 \\ \\ A_1 \to A_3 \end{array}$$

$$\rightarrow \begin{bmatrix} 1 & 2 & 0 & -1 & -1 & -1 \\ 2 & 4 & 1 & -3 & 0 & 8 \\ 3 & 6 & 1 & -4 & 0 & 11 \end{bmatrix} \begin{array}{l} A_1 \to -A_1 \\ \\ \\ \end{array} \rightarrow \begin{bmatrix} 1 & 2 & 0 & -1 & -1 & -1 \\ 0 & 0 & 1 & -1 & 2 & 10 \\ 0 & 0 & 1 & -1 & 3 & 14 \end{bmatrix} \begin{array}{l} \\ A_2 \to A_2 - 2A_1 \\ A_3 \to A_3 - 3A_1 \end{array}$$

$$\rightarrow \begin{bmatrix} 1 & 2 & 0 & -1 & -1 & -1 \\ 0 & 0 & 1 & -1 & 2 & 10 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \begin{array}{l} \\ \\ A_3 \to A_3 - A_2 \end{array} \rightarrow \begin{bmatrix} 1 & 2 & 0 & -1 & 0 & 3 \\ 0 & 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \begin{array}{l} A_1 \to A_1 + A_3 \\ A_2 \to A_2 - 2A_3 \\ \\ \end{array}$$

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2. Let A, X and B be as below. 9 pts

$$A = \begin{bmatrix} 2 & 1 & 1 & 2 \\ 1 & 0 & -1 & 2 \\ 4 & 3 & 5 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 3 \\ -3 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}.$$

(a) Find the general solution to the system $AX = B$.
Hint: **See the last page of the test.**

(b) What are the translation and spanning vectors?

a) $AX = B$

$$\therefore \begin{array}{ccccc} x & y & z & w & \\ \end{array}$$
$$\begin{bmatrix} 2 & 1 & 1 & 2 & 1 \\ 1 & 0 & -1 & 2 & 3 \\ 4 & 3 & 5 & 2 & -3 \end{bmatrix} \overset{RREF}{=} \begin{array}{c} \begin{array}{ccccc} x & y & z & w & \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 2 & 3 \\ 0 & 1 & 3 & -2 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$\therefore \begin{cases} x = 3 + z - 2w \\ y = -5 - 3z + 2w \end{cases} \quad \text{let } z = s,\ w = t$$

$$\therefore \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 3 + s - 2t \\ -5 - 3s + 2t \\ s \\ t \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 1 \\ -3 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

b) translation vector : $[3, -5, 0, 0]^t$

Spanning vector : $[1, -3, 1, 0]^t$, $[-2, 2, 0, 1]^t$

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3. Find a spanning set for the nullspace of the matrix A in Problem 2. 3 pts

The translation vectors span the nullspace

so $\begin{bmatrix} -1 \\ -3 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 2 \\ 0 \\ 1 \end{bmatrix}$ span the nullspace

4. Prove that the spanning set you found in Problem 3 is linearly independent. (The proof can be very short.) 3 pts

There is a zero in each of the matrices where there is a 1 in the other matrices (the third and fourth entries of the matrices).

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5. For which values of a, b, and c will the following system have a solution? 9 pts

$$\begin{aligned} x + 2\,y + 2\,z &= a \\ 3\,x - y + 2\,z &= b \\ x + 16\,y + 10\,z &= c \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & a \\ 3 & -1 & 2 & b \\ 1 & 16 & 10 & c \end{array}\right] \xrightarrow{=R} \left[\begin{array}{ccc|c} 1 & 2 & 2 & a \\ 0 & -7 & -4 & b-3a \\ 0 & 14 & 8 & c-a \end{array}\right] \begin{array}{l} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & a \\ 0 & -7 & -4 & b-3a \\ 0 & 0 & 0 & 2b-7a+c \end{array}\right] R_3 \to R_3 + 2R_2$$

FOR the system to have a solution, it must be consistent, so $2b - 7a + c = 0$.

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6. Create a 3×5 matrix A with none of its entries equal to 0 such that $AX = B$ is solvable if and only if B belongs to the span of U and V where 9 pts

$$U = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}, \quad V = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

Explain why your answer works. *Note that we are asking about B, not X.*

$\because$ span of U and V = $aU + bV$

$= a[1,-2,4]^t + b[1,0,-1]^t$

$= [a+b, -2a, 4a-b]^t$

$\therefore a=1,\ b=1,\ [2,-2,3]^t$

$a=2,\ b=1,\ [3,-4,7]^t$

$a=1,\ b=2,\ [3,-2,2]^t$

$a=2,\ b=2,\ [4,-4,6]^t$

$a=3,\ b=1,\ [4,-6,11]^t$

$$\therefore A = \begin{array}{ccccc} A_1 & A_2 & A_3 & A_4 & A_5 \end{array} \begin{bmatrix} 2 & 3 & 3 & 4 & 4 \\ -2 & -4 & -2 & -4 & -6 \\ 3 & 7 & 2 & 6 & 11 \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}, \quad AX = B$$

$\because$ AX=B is solvable if and only if B belong to the column space of A ✓

$\because$ column space of A = $x_1A_1 + x_2A_2 + x_3A_3 + x_4A_4 + x_5A_5$

= span of U and V = B

$\because$ AX=B is solvable if and only if B belongs to the span of U and V

$\therefore$ We have matrix A created in this way.

9 8

7. Prove that if A satisfies the requirements of Problem 6, then the following equation will not be solvable: 3 pts

$$AX = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

For $AX = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ to be solvable, $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ must belong to the column space, or the span of the columns of A. The column space of A is spanned by $\begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} \equiv u$ and $v = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

so for some scalars a & b, $au + bv = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ for the equation to be solvable

$$R = \begin{array}{c} \begin{array}{ccc} a & b & \end{array} \\ \begin{bmatrix} 1 & 1 & 1 \\ -2 & 0 & 1 \\ 4 & -1 & 1 \end{bmatrix} \end{array} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & -5 & -3 \end{bmatrix} \begin{matrix} \\ R_2 \to R_2 + 2R_1 \\ R_3 \to R_3 - 4R_1 \end{matrix}$$

$$\to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 9 \end{bmatrix} R_3 \to 2R_3 + 5R_2$$

$0 \neq 9$, so the system is inconsistent. $[1,1,1]^t$ does not belong to the span of u & v, so it does not belong to the column space of A, which is why the system $AX = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is not solvable.

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8. Create a system of three equations in four unknowns (reader's choice) such that the solution space is a plane in $\mathbb{R}^4$ that does not pass through $\mathbf{0}$. *Do not make any coefficients equal 0.* Explain why your example works. 7 pts

$$\begin{array}{c}x \quad y \quad z \quad w \\ \left[\begin{array}{ccccc} 1 & 1 & 1 & 1 & 1 \\ 2 & 7 & 3 & 4 & 3 \\ 3 & 8 & 4 & 5 & 4 \end{array}\right]\end{array} \begin{array}{l} = A_1 \\ A_2 \\ A_3 \end{array} \quad A_3 = A_1 + A_2$$

$x + y + z + w = 1$
$2x + 7y + 3z + 4w = 3$
$3x + 8y + 4z + 5w = 4$

For a system of 3 equations in 4 unknowns to have a solution space as a plane in $\mathbb{R}^4$, it must have 2 free variables, which then means it must be a rank 2 system. This means that 1 of the equations must be linearly dependent on the other two equations, which are linearly independent of each other. Also, since it is a non homogeneous system, it will not pass through $\mathbf{0}$.
The third equation is linearly dependent on the first two equations.

9. Change one single number in the system created in Problem 8 to obtain an inconsistent system. Explain why the system is inconsistent. 3 pts

$A_1 \quad x + y + z + w = 1$
$A_2 \quad 2x + 7y + 3z + 4w = 3$
$A_3 \quad 3x + 8y + 4z + 5w = 5$

$A_1 + A_2 = 3x + 8y + 4z + 5w = 4$
but $A_3 = 3x + 8y + 4z + 5w = 5$

The system is inconsistent because $3x + 8y + 4z + 5w$ cannot equal both 4 and 5.

10. For a certain 3×3 matrix A it observed that

$$A \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = A \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}.$$

want $AZ=0$ so then Find a non-zero element Z of the nullspace of A. 3 pts

Z belongs to null space

$$A \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$$

$$A \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$$

$$A \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} - A \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} - \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$$

$$A \begin{bmatrix} 2-3 \\ 3-4 \\ 1-2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$ so $\begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$ belongs to the nullspace

11. Suppose that the element Z found in Problem 11 spans the nullspace of A. Explain why the general solution to the system $AX = [1, -2, -1]^t$ is

$$X = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

4 pts

The general solution is a particular solution plus the span of the nullspace. So for $AX=B$ when $X = T + Z$ with T being a particular solution and Z being the nullspace

$AX = B$

$A(T+Z) = B$

$AT + AZ = B$

since $AZ = 0$ because Z is the nullspace of A

$AT = B$

$\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$ was a particular solution and $\begin{bmatrix} 1 \end{bmatrix}$ is part of the nullspace, so the general solution is $X = [2, 3, 1]^t + t[1, 1, 1]^t$

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A span is a subspace.

12. Let $X_1 = [1,3,7]^t$, $X_2 = [3,1,-2]^t$, $Y_1 = [7,5,3]^t$ and $Y_2 = [4,4,5]^t$. Prove prove that X_1 and X_2 span the same subspace of $\mathbb{R}^3$ as Y_1 and Y_2. For this you need to prove (a) that every element of the span of X_1 and X_2 also belongs to the span of Y_1 and Y_2 and, conversely, (b) that every element of the span of Y_1 and Y_2 also belongs to the span of X_1 and X_2. 9 pts

$X_1 = \begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix}$ $X_2 = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}$ $Y_1 = \begin{bmatrix} 7 \\ 5 \\ 3 \end{bmatrix}$ $Y_2 = \begin{bmatrix} 4 \\ 4 \\ 5 \end{bmatrix}$

$X_1 + 2X_2 = Y_1$

$X_1 + X_2 = Y_2$

let $u = aY_1 + bY_2$ which is the span of Y_1 and Y_2, (any scalar a and b)

then $u = a(X_1 + 2X_2) + b(X_1 + X_2)$

$u = (a+b)X_1 + (2a+b)X_2$

$u = cX_1 + dX_2$ (where $c = a+b$, $d = 2a+b$)

which is the span of X_1 and X_2

So every element in the span of Y_1 and Y_2 belongs to the span of X_1 and X_2.

Therefore X_1 & X_2 span the same subspace as Y_1 & Y_2.

$2Y_2 - Y_1 = X_1$

$Y_1 - Y_2 = X_2$

let $v = aX_1 + bX_2$ which is the span of X_1 & X_2 (any scalar a & b)

then $v = a(2Y_2 - Y_1) + b(Y_1 - Y_2)$

$v = (-a+b)Y_1 + (2a-b)Y_2$

$= cY_1 + dY_2$ (where $c = -a+b$, $d = 2a-b$)

which is the span of Y_1 & Y_2. So every element in the span of X_1 & X_2 belongs to the span of Y_1 and Y_2

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13. Let $\mathcal{W}$ be the set of matrices of the form below where a, b and c range over all real numbers.

$$[\ a+b+3c,\ \ -2a+b,\ \ 3a+b+5c\ ]$$

Prove that $\mathcal{W}$ is a subspace of $M(1,3)$. *(Reason directly from either the definition of "subspace" or Theorem 6 on p. 79 of the text. Do not use the theorem that says that spans are subspaces!)* 9 pts

let u + v be elements of the subspace where

$u = [a+b+3c,\ -2a+b,\ 3a+b+5c]$ $v = [a'+b'+3c',\ -2a'+b',\ 3a'+b'+5c']$

where a, b, c, a', b', c' are all scalars

then $u+v = [(a+a')+(b+b')+3(c+c'),\ -2(a+a')+(b+b'),\ 3(a+a')+(b+b')+5(c+c')]$

$= [a''+b''+3c'',\ -2a''+b'',\ 3a''+b''+5c'']$

which is in the proper form to belong to W where $a''=a+a'$ $b''=b+b'$, $c''=c+c'$

proving subspace property 1

let k be any scalar, then

$ku = [ka+kb+3kc,\ -2ka+kb,\ 3ka+kb+5kc]$

$= [a^*+b^*+3c^*,\ -2a^*+b^*,\ 3a^*+b^*+5c^*]$

which is in proper form to belong to W where $a^*=ka$, $b^*=kb$, and $c^*=kc$

proving subspace property 2, and when $k=0$, then $0 \in W$, proving subspace property 3.

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14. (a) Find a spanning set for the set $\mathcal{W}$ from Problem 13.
(b) Find a linearly independent spanning set for the set $\mathcal{W}$ from Problem 13. 5 pts

A. $[a+b+3c, -2a+b, 3a+b+5c]$

$= [a, -2a, 3a] + [b, b, b] + [3c, 0, 5c]$

$= a\underbrace{[1, -2, 3]}_{A} + b\underbrace{[1, 1, 1]}_{B} + c\underbrace{[3, 0, 5]}_{C}$

A spanning set is $[1, -2, 3]$, $[1, 1, 1]$ and $[3, 0, 5]$

B. $A + 2B = C$

So a linearly independent spanning set is $[1, -2, 3]$ and $[1, 1, 1]$

15. Let $\mathcal{W}$ be the set of all 2×2 matrices of the form shown below where a and b range over all real numbers. Prove that none of the subspace properties((1)-(3) in Theorem 6 on p. 79 of the text) hold for $\mathcal{W}$. 6 pts

$$\begin{bmatrix} 1 & a+b \\ 0 & 3b \end{bmatrix}$$

$X + Y \in W$

$$X = \begin{bmatrix} 1 & a+b \\ 0 & 3b \end{bmatrix}$$

$$Y = \begin{bmatrix} 1 & a'+b' \\ 0 & 3b' \end{bmatrix}$$

where $a, b, a' + b'$ are scalars

then $$X+Y = \begin{bmatrix} 2 & (a+a')+(b+b') \\ 0 & 3(b+b') \end{bmatrix}$$

$$= \begin{bmatrix} 2 & a''+b'' \\ 0 & 3b'' \end{bmatrix} \quad \text{where } a'' = a+a', \quad b'' = b+b'$$

which does not belong to W, because $2 \neq 1$ which goes against subspace property 1.

for the scalar 2,

$$2X = \begin{bmatrix} 2 & 2a+2b \\ 0 & 6b \end{bmatrix}$$

which does not belong to W because of the 2 in the entry that must be a 1. So this goes against subspace property 2.

Also, subspace property 3 says that $0 \in W$, but $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ has a 0 where there should be a 1 to belong to the [illegible] property 3 [illegible] because

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16. Let X_1, X_2 and X_3 be elements of some vector space. Prove that $\mathcal{W} = \text{span}\ \{X_1, X_2, X_3\}$ is a subspace. *Reason directly from either the definition of "subspace" or Theorem 6 on p. 79 of the text. Do not use the theorem that says that spans are subspaces!* 8 pts

for any scalars $a_1, a_2, a_3, b_1, b_2, b_3, s, t$

$$u = a_1X_1 + a_2X_2 + a_3X_3$$

$$V = b_1X_1 + b_2X_2 + b_3X_3$$

which are both elements of the span of $X_1, X_2, \& X_3$

then $$sU + tV = sa_1X_1 + sa_2X_2 + sa_3X_3 + tb_1X_1 + tb_2X_2 + tb_3X_3$$

$$= (sa_1 + tb_1)X_1 + (sa_2 + tb_2)X_2 + (sa_3 + tb_3)X_3$$

$$= fX_1 + gX_2 + hX_3$$

(with $f = sa_1 + tb_1$, $g = sa_2 + tb_2$, $h = sa_3 + tb_3$) which is a subspace by the definition of subspace.