1. Given that \( \det A = 7 \) where

\[
A = \begin{bmatrix}
    a & b & c \\
    d & e & f \\
    g & h & i
\end{bmatrix}
\]

find \( \det B \) where

\[
B = \begin{bmatrix}
    g-3d & h-3e & i-3f \\
    2a & 2b & 2c \\
    7d+10a & 7e+10b & 7f+10c
\end{bmatrix}
\]

Be sure to show all steps in the computation. 12 pts.

\[
-\det(A) = \begin{vmatrix}
    g & h & i \\
    d & e & f \\
    a & b & c
\end{vmatrix}
\]

\[
\det(A) = \begin{vmatrix}
    g & h & i \\
    a & b & c \\
    d & e & f
\end{vmatrix}
\]

\[
2\det(A) = \begin{vmatrix}
    g & h & i \\
    2a & 2b & 2c \\
    d & e & f
\end{vmatrix}
\]

\[
2(7)\det(A) = \begin{vmatrix}
    g & h & i \\
    7d & 7e & 7f \\
    10a & 10b & 10c
\end{vmatrix}
\]

\[
0
\]

\[
14\det(A)
\]

\[
= \det(B) = 14\sqrt{\det(A)}
\]

\[
= 14(7) = 98
\]
2. We wish to solve the equation $AX = B$ where $A$, $X$, and $B$ are given below. Express the value of $y$ as a ratio of the determinants of two specific matrices. Do not compute these determinants. 6 pts.

$$A = \begin{bmatrix} 2 & -1 & 1 & 2 \\ 1 & 2 & 1 & 3 \\ 4 & 5 & 3 & 7 \\ 1 & -3 & 0 & 5 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

$$y = \begin{bmatrix} z & 1 & 2 \\ 1 & 2 & 1 & 3 \\ 4 & 3 & 3 & 7 \\ 1 & 4 & 0 & 5 \end{bmatrix}$$
3. Let $A$, $B$, and $C$ be as below. Using only algebraic properties of the determinant, find numbers $b$ and $c$ such that $\det A = b \det B + c \det C$. Do not compute any determinants to solve this problem. Be sure to show all steps in your work. 8 pts.

$$A = \begin{bmatrix} 1 & -1 & 3 & 1 \\ 1 & 2 & 3 & 4 \\ 3 & 2 & 3 & 2 \\ 6 & 4 & 2 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & -1 & 3 & 1 \\ 2 & 1 & 2 & 1 \\ 3 & 2 & 1 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & -1 & 3 & 1 \\ 1 & 1 & 1 & 1 \\ 3 & 2 & 1 & 1 \end{bmatrix}$$

$$\det(B) + \det(C) = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 1 & -1 & 3 & 1 \\ 3 & 2 & 3 & 2 \\ 3 & 2 & 1 & 1 \end{vmatrix} \quad \text{By row additivity prop. in 3rd row.}$$

$$2 \left( \det(B) + \det(C) \right) = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 1 & -1 & 3 & 1 \\ 3 & 2 & 3 & 2 \\ 6 & 4 & 2 & 2 \end{vmatrix} \quad \text{Scalar prop in 4th row.}$$

$$-2 \left( \det(B) + \det(C) \right) = \det(A) \quad \text{By row exchange property, 2nd row to 1st}.$$ 

$$-2 \det(B) - 2 \det(C) = \det(A) \quad \text{Hence } b = c = -2.$$
4. Use row reduction to find the inverse of the following matrix $A$. Other techniques will not give credit. Be sure to show enough steps so that I know that you know what you are doing.  

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$

$$\begin{align*}
A & = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \\
& \xrightarrow{\frac{1}{2}R_1} \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \\
& \xrightarrow{R_3 - \frac{1}{2}R_1} \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{2} & 1 \end{bmatrix} \\
& \xrightarrow{R_3 + \frac{1}{2}R_2} \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ 0 & -\frac{1}{2} & 1 \end{bmatrix} \\
& \xrightarrow{R_1 - \frac{1}{2}R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \\
& \xrightarrow{R_2 - R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \\
A^{-1} & = \begin{bmatrix} 1 & -1 & -2 \\ -1 & 2 & 2 \\ 1 & -1 & -2 \end{bmatrix}
\end{align*}$$

Check: $$\begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & -1 & -2 \end{bmatrix} \begin{bmatrix} 1 & -1 & -2 \\ -1 & 2 & 2 \\ 1 & -1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \checkmark$$
5. Let $A$ be as below. Given that $A$ is invertible, express the $(3,1)$ entry of $A^{-1}$ as $(-1)^n$ times the ratio of the determinants of two matrices. Give $n$ and the two matrices. **Do not compute these determinants.**

\[
A = \begin{bmatrix}
a & b & c \\
d & e & f \\
g & h & i
\end{bmatrix}
\]

\[
\begin{align*}
(A)(A^{-1}) &= I \\
\Rightarrow [\begin{array}{ccc}
a & b & c \\
d & e & f \\
g & h & i
\end{array}] [\begin{array}{ccc}
a & b & c \\
d & e & f \\
g & h & i
\end{array}]^{-1} &= [\begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}]
\end{align*}
\]

\[
A^{-1} = \frac{[\begin{array}{ccc}
a & b & c \\
d & e & f \\
g & h & i
\end{array}]}{[\begin{array}{ccc}
a & b & c \\
d & e & f \\
g & h & i
\end{array}]^{-1}} = \frac{[\begin{array}{ccc}
1 & b & a \\
0 & e & d \\
0 & h & g
\end{array}]}{[\begin{array}{ccc}
a & b & c \\
d & e & f \\
g & h & i
\end{array}]}
\]

You could also use the formula $A^{-1}_{31} = (-1)^n 4 \cdot \text{det } A_{13} / \text{det } A$ where $A_{13}$ is $A$ with its first row and third column removed.
6. Use the linearity properties to prove that the transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ defined below is linear. *Other techniques will not be accepted.*

$$T([x, y]^t) = [x + y, 2x + y, x - 3y]^t.$$  \hspace{1cm} \text{If linear,} \quad T([x + y]) = T([x]) + T([y])$$

and $$T(c[ x ]) = cT([ x ]).$$

\(\text{\textbullet}\) \quad T \left( \left[ \begin{array}{c} x_1^1 \\ x_2^1 \\ \end{array} \right] + \left[ \begin{array}{c} y_1^1 \\ y_2^1 \\ \end{array} \right] \right) = T \left( \left[ \begin{array}{c} x_1^1 \\ y_1^1 \\ \end{array} \right] \right) + T \left( \left[ \begin{array}{c} y_1^1 \\ y_2^1 \\ \end{array} \right] \right)$$

$$\left[ \begin{array}{c} x_1+y_1^1 + x_2 + y_2^1 \\ 2(x_1+y_1) + (x_2+y_2) \\ x_1+y_1^1 - 3(x_2+y_2) \end{array} \right] = \left[ \begin{array}{c} x_1+y_1^1 + x_2 + y_2^1 \\ 2x_1 + x_2 \\ x_1 - 3x_2 \end{array} \right] + \left[ \begin{array}{c} y_1+y_2^1 \\ 2y_1 + y_2 \\ y_1 - 3y_2 \end{array} \right]$$

$$\left[ \begin{array}{c} x_1+y_1^1 + x_2 + y_2^1 \\ 2x_1 + x_2 \\ x_1 - 3x_2 \end{array} \right] = \left[ \begin{array}{c} x_1+y_1^1 + x_2 + y_2^1 \\ 2x_1 + x_2 \\ x_1 - 3x_2 \end{array} \right]$$

\(\text{\textbullet}\) \quad T \left( c \left[ \begin{array}{c} x_1^2 \\ x_2^2 \end{array} \right] \right) = cT \left( \left[ \begin{array}{c} x_1^1 \\ x_2^1 \end{array} \right] \right)$$

$$T \left( \left[ \begin{array}{c} cx_1^2 \\ cx_2^2 \end{array} \right] \right) = \left[ \begin{array}{c} cx_1^2 + cx_2^2 \\ cx_1^2 + cx_2^2 \\ cx_1^2 - 3cx_2^2 \end{array} \right] = c \left[ \begin{array}{c} x_1^1 + x_2^1 \\ 2x_1 + x_2 \\ x_1 - 3x_2 \end{array} \right]$$

$$\left[ \begin{array}{c} cx_1^2 + cx_2^2 \\ cx_1^2 + cx_2^2 \\ cx_1^2 - 3cx_2^2 \end{array} \right] = \left[ \begin{array}{c} cx_1^2 + cx_2^2 \\ cx_1^2 + cx_2^2 \\ cx_1^2 - 3cx_2^2 \end{array} \right]$$

Hence, $T$ is a linear transformation.
7. Prove that there is no linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ satisfying
$T([2, -1]^t) = [1, 2, 3]^t$, $T([1, 1]^t) = [2, 3, 4]^t$ and, $T([4, 1]^t) = [3, 4, 5]^t$.  

\[ \text{If linear,} \]
\[
T\left([4,1]^t\right) = T\left([2,-1]^t + 2[1,1]^t\right) = T([2,-1]^t) + T(2[1,1]^t) \\
= T([2,-1]^t) + 2T([1,1]^t), \text{ by linearity properties.} \\

\begin{bmatrix} 3, 4, 5 \end{bmatrix}^t \neq \begin{bmatrix} 1, 2, 3 \end{bmatrix}^t + 2 \begin{bmatrix} 2, 3, 4 \end{bmatrix}^t \\
\begin{bmatrix} 3, 4, 5 \end{bmatrix}^t \neq \begin{bmatrix} 5, 8, 11 \end{bmatrix}^t
\]

Given the transformation data corresponding to $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$, $T$ cannot be a linear transformation because it does not satisfy the linearity properties.
8. Suppose that $A$ is an $n \times n$ matrix such that $A^4 - 2A^2 + 5I = 0$. Prove that $A^2$ is invertible and $(A^2)^{-1} = -\frac{1}{5}(A^2 - 2I)$. 10 pts.

If $A^2$ is invertible, some $B$ exists such that $A^2B = I$, $B = (A^2)^{-1}$.

\[ A^4 - 2A^2 + 5I - 5I = 0 - 5I \]

\[ A^4 - 2A^2 + 0 = -5I \]

\[ -\frac{1}{5} (A^4 - 2A^2) = (-5I)(-\frac{1}{5}) \]

\[ -\frac{1}{5} A^4 + \frac{2}{5} A^2 = I \]

\[ A^2 (-\frac{1}{5} A^2 + \frac{2}{5} I) = I \]

Hence, from the above statement, $B = -\frac{1}{5} A^2 + \frac{2}{5} I = (A^2)^{-1}$.

$B = -\frac{1}{5} (A^2 - 2I) = (A^2)^{-1}$  \checkmark

QED
Let $A$ be a $2 \times 3$ matrix having rank 2. Prove that any non-zero $3 \times 3$ matrix $B$ such that $AB = 0$ must have rank 1. **Hint:** Write $B = [B_1, B_2, B_3]$ where $B_1$, $B_2$ and $B_3$ are columns.

By definition, any $B$ such that $AB = 0$ belongs to the nullspace of $A$. The dimension of the nullspace of $A$ is equal to the number of columns of $A$ minus its rank. Thus, the dimension of the nullspace of $A$ is $3 - 2 = 1$.

Consider $AB = [AB_1, AB_2, AB_3]$ by definition of matrix multiplication. $[AB_1, AB_2, AB_3] = 0$, such that $AB_i = [0]$ by dimensions of $A$ and $B$ being multiplied. Thus, each $B_i$ must belong in the nullspace of $A$ so that $AB_i = 0$ is satisfied. The dimension of the nullspace of $A$ is established to be 1 above. Thus, only one unique vector may exist within $B = [B_1, B_2, B_3]$.

Therefore, the rank of $B$ must equal 1.

Thus each of the $B_i = c_i U$ where $U$ is a basis of the nullspace.
10. Suppose that $A$ is an $n \times n$ matrix such that $A^2$ is invertible. Prove that $A$ is invertible.  

If $A^2$ is invertible, some matrix $B$ exists such that $A^2 B = I$, where $B = (A^2)^{-1}$.

Consider $A^2 B = I$.

$(A A)B = I$

$A (AB) = I$  \text{By Associative Property.}$

Therefore, a form of the given may be written such that $AC = I$, where $C = A^{-1}$.

Hence, by definition $A$ is invertible, and its inverse is equal to $AB$.

Note. You cannot use $A^{-1}$ in your proof since it is not given that $A^{-1}$ exists. This is what you must prove.
11. You were asked to learn the proof of Theorem 3 in the text. Part (b) of this theorem states that if $A$ is an $n \times n$ matrix for which there exists an $n \times n$ matrix $B$ such that $AB = I$ then $A$ is invertible and $B = A^{-1}$. Prove this result using the same argument that was used on the study sheet. 10 pts.

Given: $A$ is $n \times n$ matrix and some $B$ exists such that $AB = I$.

To prove $A$ is invertible, $A$ must certainly have an inverse and it must be shown that $B = A^{-1}$.

1. Some inverse of $A$ exists if $A$ is nonsingular — that is, the rank of $A$ must equal $n$.

$AB = I$, by the dimensions of $A$ and $B$, $I$ must be an $n \times n$ matrix. The rank of $n \times n$ $I$ is equal to $m$. The rank of $A$ cannot exceed its number of columns, $m = \text{rank}(AB) \leq \text{rank}(A) \leq m$.

Hence, $\text{rank}(A) = m$. ✓

2. Given: $AB = I$

$A^{-1}(AB) = A^{-1}(I)$ ✓

$(A^{-1}A)B = A^{-1}$

$I\ B = A^{-1}$

$B = A^{-1}$ ✓

Hence, $A$ is invertible and $B = A^{-1}$. QED
12. Suppose that $U$, $V$ and $W$ are vector spaces and that $T : V \rightarrow W$ and $S : V \rightarrow W$ are linear transformations. Prove that $S \circ T$ is a linear transformation. (Recall that for all $X \in W$, $S \circ T(X) = S(T(X)).$) 8 pts.

$X, Y \in U$, $T$ and $S$ are each linear transformations which satisfy linearity properties. If $S \circ T(X)$ is linear, must satisfy 1 and 2.

1. $S \circ T(X + Y) = S \circ T(X) + S \circ T(Y)$
   
   $S(T(X + Y)) = S(T(X)) + S(T(Y))$  
   $S(T(X)) + S(T(Y)) = S(T(X)) + S(T(Y))$  
   $\checkmark$

2. $S \circ T(cX) = cS \circ T(X)$
   
   $S(T(cX)) = cS(T(X))$  
   $cS(T(X)) = cS(T(X))$  
   $\checkmark$

Hence, $S \circ T$ is a linear transformation.