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1. Given that $\det A = 5$ where

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

find $\det B$ where

$$\begin{bmatrix} g - 3a & h - 3b & i - 3c \\ d & e & f \\ 6a - 6d & 6b - 6e & 6c - 6f \end{bmatrix}$$

*Be sure to show all steps in the computation.* *10 pts.*

$$\begin{vmatrix} g & h & i \\ d & e & f \\ 6a-6d & 6b-6e & 6c-6f \end{vmatrix} + 3\begin{vmatrix} a & b & c \\ d & e & f \\ 6a-6d & 6b-6e & 6c-6f \end{vmatrix}$$

$$6\begin{vmatrix} g, h, i \\ d, e, f \\ a, b, c \end{vmatrix} - 6\underbrace{\begin{vmatrix} g, h, i \\ d, e, f \\ d, e, f \end{vmatrix}}_{0} - 3\underbrace{\begin{vmatrix} a & b & c \\ d & e & f \\ 6a & 6b & 6c \end{vmatrix}}_{0} - 3\underbrace{\begin{vmatrix} a & b & c \\ d & e & f \\ -6d & -6e & -6f \end{vmatrix}}_{0}$$

$$-6\begin{vmatrix} a, b, c \\ d, e, f \\ g, h, i \end{vmatrix} = -6 \cdot 5 = -30$$

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2. Let $A$ be a $4 \times 4$ matrix which we write as

$$A = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix}$$

where $A_i$ are the rows of $A$. Prove the following statement *using only the row scalar, row additive, and row reversal properties* of the determinant. You may use other properties only if you first prove them using the stated properties.

$$\det A = -\frac{1}{3} \det \begin{bmatrix} A_2 \\ 3A_1 - 4A_4 \\ A_3 \\ A_4 \end{bmatrix}$$

*10 pts.*

$$= -\frac{1}{3} \begin{vmatrix} A_2 \\ 3A_1 \\ A_3 \\ A_4 \end{vmatrix} - \frac{1}{3} \begin{vmatrix} A_2 \\ -4A_4 \\ A_3 \\ A_4 \end{vmatrix}$$ row additive

$$= -1 \begin{vmatrix} A_2 \\ A_1 \\ A_3 \\ A_4 \end{vmatrix} + \frac{4}{3} \begin{vmatrix} A_2 \\ A_4 \\ A_3 \\ A_4 \end{vmatrix}$$ row scalar

by the row reversal property

$$\det A = \begin{vmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{vmatrix}$$ row reversal property

$$\begin{vmatrix} A_2 \\ A_4 \\ A_3 \\ A_4 \end{vmatrix} = - \begin{vmatrix} A_2 \\ A_4 \\ A_3 \\ A_4 \end{vmatrix}$$ so the determinant must be 0

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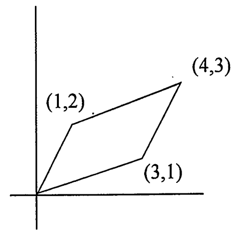


3. Find a $2 \times 2$ matrix $M$ such that multiplication by $M$ transforms the above parallelogram onto itself and transforms $(3, 1)$ onto $(1, 2)$. *10 pts.*

and $(1,2) \to (3,1)$

$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ $\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$a + 2b = 3$ $\quad 4a + 3b = 4$ $\quad 3a + b = 1$ $\quad 0 = 0$

$c + 2d = 1$ $\quad 4c + 3d = 3$ $\quad 3c + d = 2$

$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 3 & 4 \\ 3 & 1 & 1 \end{pmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -8 \\ 0 & -5 & -8 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -8 \\ 0 & 0 & 0 \end{bmatrix}$

$b = 1 - 3a$

$a = 3 - \frac{16}{5} = -\frac{1}{5}$

$1 = 4a - 9a$

$5b = -8$

$-5a = 1$

$b = \frac{8}{5}$

$\begin{bmatrix} 1 & 2 & 1 \\ 4 & 3 & 3 \\ 3 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -5 & -1 \\ 0 & -5 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -5 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

$-5d = -1$

$d = \frac{1}{5}$

$c = 1 - \frac{2}{5}$

$c = \frac{3}{5}$

$\begin{bmatrix} -\frac{1}{5} & \frac{8}{5} \\ \frac{3}{5} & \frac{1}{5} \end{bmatrix}$

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4. Use row reduction to find the inverse of the following matrix $A$. **Other techniques will not give credit.** Be sure to show enough steps so that I know that you know what you are doing. *10 pts.*

$$A = \begin{bmatrix} 0 & 0 & 3 \\ 2 & 1 & 0 \\ 1 & 2 & 6 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 0 & 0 & 3 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 1 & 2 & 6 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 2 & 6 & 0 & 0 & 1 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 1 & 0 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 2 & 6 & 0 & 0 & 1 \\ 0 & -3 & -12 & 0 & 1 & -2 \\ 0 & 0 & 3 & 1 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 2 & 6 & 0 & 0 & 1 \\ 0 & 1 & 4 & 0 & -\frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 1 & \frac{1}{3} & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -2 & 0 & 1 \\ 0 & 1 & 0 & -\frac{4}{3} & -\frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 1 & \frac{1}{3} & 0 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ 0 & 1 & 0 & -\frac{4}{3} & -\frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 1 & \frac{1}{3} & 0 & 0 \end{array}\right] \quad A^{-1} = \begin{bmatrix} \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{4}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & 0 & 0 \end{bmatrix}$$

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5. Let the transformation $T : \mathbb{R}^3 \mapsto \mathbb{R}^2$ be defined by

$$T([x, y, z]^t) = [x + y + z, x + 2y - z]^t.$$

(a) Use the linearity properties to prove that $T$ is linear. *Other techniques will not be accepted.* *10 pts.*

$X = [x_1, y_1, z_1]$ $\quad Y = [x_2, y_2, z_2]$

$T(X) + T(Y) = [x_1 + y_1 + z_1, x_1 + 2y_1 - z_1]^t + [x_2 + y_2 + z_2, x_2 + 2y_2 - z_2]^t$

$= [(x_1 + x_2) + (y_1 + y_2) + (z_1 + z_2), (x_1 + x_2) + 2(y_1 + y_2) - (z_1 + z_2)]$

$T(X + Y) = [(x_1 + x_2) + (y_1 + y_2) + (z_1 + z_2), (x_1 + x_2) + 2(y_1 + y_2) - (z_1 + z_2)]^t$

$T(cX) = [(cx_1) + (cy_1) + (cz_1), (cx_1) + 2(cy_1) - (cz_1)]^t$

$= c[x_1 + y_1 + z_1, x_1 + 2y_1 - z_1]^t$

$= cT(x)$

The transformation satisfies both properties of a linear transformation.

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(b) Find a matrix $A$ such that $T(X) = AX$ for all $X \in \mathbb{R}^3$. *5 pts.*

$$1 \quad \begin{bmatrix} x+y+z \\ x+2y-z \end{bmatrix} = x\begin{bmatrix} 1 \\ 1 \end{bmatrix} + y\begin{bmatrix} 1 \\ 2 \end{bmatrix} + z\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$A = \underline{\overset{2\times 3}{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \end{bmatrix}}} \; \underbrace{\overset{3\times 1}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}}_{= X}$$

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6. Suppose that $A$ and $B$ are $n \times n$ matrices such that $A^3 - 2BA^2 + 5I = \mathbf{0}$. Prove that $A$ is invertible. *10 pts.*

$A^3 - 2BA^2 \cancel{+5I - 5I} = 0 - 5I$

$-A^3 + 2BA^2 = 5I$

$\frac{1}{5}(-A^3 + 2BA^2) = I$

$-2$

$A\left[\frac{1}{5}(-A^2 + 2BA)\right] = I$ $\quad [\frac{1}{5}(-A^2 + 2BA)]A = I$

$A^{-1}$

Therefore, A is invertible and its inverse is above

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7. Let $A$ be a $3 \times 4$ matrix having rank 3. Prove that there is a $4 \times 3$ matrix $B$ such that $AB = I$ where $I$ is the $3 \times 3$ identity matrix. *Hint:* Write $B = [B_1, B_2, B_3]$ and $I = [I_1, I_2, I_3]$ where $B_1$, $B_2$, $B_3$ and $I_1$, $I_2$, and $I_3$ are columns. *10 pts.*

$AB=I \rightarrow [AB_1, AB_2, AB_3] = I$

by the rank nullity theorum

rank A + null A = n $\quad n=4$

3 + null A = 4

null A = 1

therefore there a exists an infinite # of X $n \times 1$ such that

$AX = C \quad C \in \mathbb{R}^n$

Since $I_1 - I_3$ are within $\mathbb{R}^n$ there exists an $n \times 1$ matrix $B_1 - B_3$ such that $AB_1 = I_1$ etc

8. In Problem 7, is there only one $4 \times 3$ matrix $B$ matrix that satisfies $AB = I$? Prove your answer in terms of theorems from the text. *5 pts.*

As mentioned in part 7. The rank nullity thm applied shows there to be an infinite # of solutions to $AX=C$. So if there are an infinite # of solutions for each $B_i$ there are an infinite # of $B$s as well.

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9. You were asked to learn the proof of Theorem 3.10 on p. 188 the text. One part of this theorem states that if $A$ is an $n \times n$ matrix for which there exists an $n \times n$ matrix $B$ such that $AB = I$ then $A$ is invertible and $B = A^{-1}$. Prove this result *using the same argument that was used in the text.* *10 pts.*

Proof: A is invertible

$\text{rank}(AB) \leq \text{rank } A$

$\text{rank } I \leq \text{rank } A$
$\overset{\|}{n} \Rightarrow \text{rank} A = n$ and A is invertible

Proof: $B = A^{-1}$

$AB = I$

$A^{-1}(AB) = A^{-1}I$

$(A^{-1}A)B = A^{-1}$

$IB = A^{-1}$

$\underline{B = A^{-1}}$

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10. Let $T : \mathcal{V} \to \mathcal{W}$ be a linear transformation where $\mathcal{V}$ and $\mathcal{W}$ are vector spaces. Let $\{X_1, X_2, X_3\}$ be a set of three elements in $\mathcal{V}$ such that the set $\{T(X_1), T(X_2), T(X_3)\}$ is linearly independent. Prove that the set $\{X_1, X_2, X_3\}$ is linearly independent. *10 pts.*

$\cancel{aT(X_1)+bT(X_2)+cT(X_3)=0}$ ~~only when a=b=c=0~~

$\cancel{T(cX_1)+T(bX_2)+T(cX_3)=0}$

$\cancel{T(cX_1+bX_2+cX_3)=0}$

To prove $cX_1+bX_2+cX_3=0$ only when $a=b=c=0$

$=T(cX_1+bX_2+cX_3)=T(0) \to \quad 0T(0)$

$=T(cX_1)+T(bX_2)+T(cX_3)=0$

$=cT(X_1)+bT(X_2)+cT(X_3)=0$ given, therefore the first statement holds as well