1. Bring the following matrix into **row reduced echelon form**: 10 pts

\[
\begin{bmatrix}
-2 & -4 & -2 & -1 & -2 & -1 \\
1 & 2 & 1 & 1 & 1 & 1 \\
3 & 6 & 3 & 3 & 3 & 3
\end{bmatrix}
\]

\[
A_1 \begin{bmatrix}
-2 & -4 & -2 & -1 & -2 & -1 \\
1 & 2 & 1 & 1 & 1 & 1 \\
3 & 6 & 3 & 3 & 3 & 3
\end{bmatrix} \rightarrow
\begin{bmatrix}
1 & 2 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
A_2 - A_1 \begin{bmatrix}
1 & 2 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} \rightarrow
\begin{bmatrix}
1 & 2 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]
2. Let $A$, $X$ and $B$ be as below.  

\[
A = \begin{bmatrix} 
-3 & -6 & -6 & -9 \\
2 & 4 & 5 & 7 \\
1 & 2 & 2 & 3 
\end{bmatrix} \quad B = \begin{bmatrix} 
-21 \\
17 \\
7 
\end{bmatrix} \quad X = \begin{bmatrix} 
x \\
y \\
z \\
w 
\end{bmatrix}
\]

Find the general solution to the system $AX = B$ expressed in terms of translation and spanning vectors.

\[
\begin{bmatrix} 
-3 & -6 & -6 & -9 \\
2 & 4 & 5 & 7 \\
1 & 2 & 2 & 3 
\end{bmatrix} \begin{bmatrix} 
x \\
y \\
z \\
w 
\end{bmatrix} = \begin{bmatrix} 
-21 \\
17 \\
7 
\end{bmatrix}
\]

\[
x \\
y \\
z \\
w 
\]

\[
= \begin{bmatrix} 
1 & 2 & 2 & 3 & 7 \\
2 & 4 & 5 & 7 & 17 \\
1 & 2 & 2 & 3 & 7 
\end{bmatrix} \begin{bmatrix} 
x \\
y \\
z \\
w 
\end{bmatrix} = \begin{bmatrix} 
10 & 0 & 0 & 0 & 0 
\end{bmatrix}
\]

\[
x + 2y + 2z + 3w = 7 \\
z + w = 3 \\
z = 3 - w \\
x = 7 - 2y - 2z - 3w = 7 - 2y - (6 - 2w) - 3w \\
= 7 - 2y - 6 + 2w - 3w \\
= 7 - 2y - w \\
\]

\[
= \begin{bmatrix} 
x \\
y \\
z \\
w 
\end{bmatrix}^t = \begin{bmatrix} 
1, 0, 3, 0 
\end{bmatrix}^t e \\
+ y \cdot \begin{bmatrix} 
-2, 1, 0, 0 
\end{bmatrix}^t \cdot \begin{bmatrix} 
-1, 0, 1 
\end{bmatrix}
\]

so the translation vector is \([1, 0, 3, 0]^t\)

spanning vectors are \([-2, 1, 0, 0]^t, \begin{bmatrix} 
1, 0, -1, 1 
\end{bmatrix}^t\)
3. For which values of $a$, $b$, and $c$ will the following system have a solution? 

\[
\begin{align*}
    x + 2y + z - 3w &= a \\
    2x + 6y + 4z - 5w &= b \\
    x - 2y - 3z - 5w &= c
\end{align*}
\]

\[
\begin{bmatrix}
    1 & 2 & 1 & -3 & a \\
    2 & 6 & 4 & -5 & b \\
    1 & -2 & 3 & -5 & c \\
\end{bmatrix}
\begin{bmatrix}
    1 & 2 & 1 & -3 & a \\
    6 & 2 & 1 & b & 2a \\
    0 & 4 & 4 & -2 & c-a
\end{bmatrix}
\]

\[
\begin{bmatrix}
    1 & 2 & 1 & -3 & a \\
    6 & 2 & 1 & b & 2a \\
    0 & 0 & 0 & (c-a)+2(b-2a)
\end{bmatrix}
\]

\[(c-a)+2(b-2a)=0\]

\[c-a+2b-4a=0\]

\[-5a+2b+c=0\]

\[c=5a-2b\]

so $a$, $b$, $c$ will follow

\[c=5a-2b\]
4. Prove that there are no $3 \times 3$ matrices $A$ with all of its entries $> 0$ such that $AX = B$ is solvable if and only if $B$ belongs to the span of the vectors $U, V$ below.

$$U = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad V = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}.$$

If $AX = B$ is solvable, $B$ has to belong to the columns space of $A$. Under the condition that $B$ belongs to the span of $U, V$, columns from $A$ have to be element of the span of $U$ and $V$, which are linear combination of $U, V$.

Since $aU + bV$

$$= a\begin{bmatrix} 1 \\ -1 \end{bmatrix} + b\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} a+2b \\ a+b \\ -a-2b \end{bmatrix}$$

To prove that no $3 \times 3$ matrices $A$ with all of its entries $> 0$ such that $AX = B$ is solvable if and only if $B$ belongs to the span of $U, V$.

The systems

$$\begin{cases} a+2b > 0 \\ a+b > 0 \\ -a-2b > 0 \end{cases}$$

need to be inconsistent.

Since $a+2b > 0.5$, then $-a-2b$ must be negative.

So the system is inconsistent, which proves the statement above.
5. Let
\[ X_1 = [1, 0, 1, 1]^t, \ X_2 = [3, 1, 0, 1]^t \]
\[ Y_1 = [4, 1, 1, 2]^t, \ Y_2 = [5, 1, 2, 3]^t, \ Y_3 = [2, 0, 2, 2]^t \]

Prove prove that \( X_1, X_2, \) and \( Y_3 \) span the same subspace of \( \mathbb{R}^3 \) as \( Y_1 \) and \( Y_2 \).  

Since
\[ Y_1 = X_1 + X_2 \ \
\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 1 & 1 \end{bmatrix} \]
\[ Y_2 = 2X_1 + X_2 \ \
2 \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 1 & 2 \end{bmatrix} \]
\[ Y_3 = 2X_1 \ \
\begin{bmatrix} 2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 2 \end{bmatrix} \]

the span of \( Y_1, Y_2, Y_3 \) \( = ax_1 + bx_2 + c y_3 \)
\[ = a(x_1 + x_2) + b(2x_1 + x_2) + c(2x_1) \]
\[ = (a+2b)y_1 + (a+b)y_2, \text{ which is a span of } x_1, x_2, \]
so all the elements belong to the span of \( Y_1, Y_2, Y_3 \)
also belong to the span of \( X_1, X_2 \)

Similarly,
\[ Y_1 = Y_2 - Y_1 \]
\[ Y_2 = Y_2 - Y_3 \]

the span of \( X_1, X_2 \) \( = \alpha x_1 + \epsilon x_2 \)
\[ = \alpha (y_2 - y_1) + \epsilon (y_2 - y_3) \]
\[ = (-\epsilon)y_1 + (\alpha + \epsilon)y_2 + (-\epsilon)y_3, \text{ which is a span of } \]
\[ y_1, y_2 \]
so all the elements belonging to the span of \( X_1, X_2 \) also belong to the span of \( Y_1, Y_2, Y_3 \)

Overall, the span of \( X_1, X_2 \) equal to the span of \( Y_1, Y_2, Y_3 \)
6. Let \( \mathcal{W} \) be the set of all functions in \( C^\infty(\mathbb{R}) \) such that \( f(1) = -3 \). Prove that none of the subspace properties (1)-(3) in Theorem 6 on p. 79 of the text) hold for \( \mathcal{W} \).

Property 1: \[ f(1) = -3 \]
let \( g(1) = 3 \)

\[ f(1) + g(1) = h(1) = -6 \neq -3 \]
so it doesn't follow the subspace property 1

Property 2: \[ f(1) = -3 \]
\[ k \cdot f(1) = W(1) = -3 k \neq -3 \text{ (except for } k = 1) \]
so it doesn't follow the subspace property 2

Property 3: \[ \text{let } k = 0 \]
\[ 0 \cdot f(1) = 0 \neq -3 \]
so it doesn't follow the subspace property 3

4 pts
7. Demonstrate your understanding of the test for independence by using it to test the following matrices for independence. (Other methods will not be accepted.) You MUST indicate clearly how any equations or matrices you use come from the test for independence and how you reach your conclusions.

(a) \[
\begin{bmatrix}
1 & 2 \\
1 & 0 \\
1 & 0
\end{bmatrix}, \begin{bmatrix}
4 & 1 \\
-1 & 1 \\
1 & -1
\end{bmatrix}, \begin{bmatrix}
-5 & -3 \\
1 & 1
\end{bmatrix}
\]

Dependency Equation:

\[
a\begin{bmatrix} 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 4 \\ 1 \end{bmatrix} + c\begin{bmatrix} -5 \\ -3 \end{bmatrix} = 0
\]

\[
\begin{bmatrix}
a+4b-5c & 2a+b-3c \\
a-b+c & b-c
\end{bmatrix} = \begin{bmatrix} 0 & 0 \\
0 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 4 & -5 & 0 \\
2 & 1 & -3 & 0 \\
1 & -1 & 1 & 0 \\
0 & 1 & -1 & 0
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

\[a = 0, \quad b = 0, \quad c = 0\]

Since the only solution to the equation \(a[1 \ 3]+b[4 \ 1]+c[-5 \ -3]=0\) is that \(a=b=c=0\),

then these three matrices are linearly independent.
(b) 

Dependency equation:

\[
\begin{align*}
\begin{pmatrix}
2 \\
1 \\
3 \\
-1
\end{pmatrix} a + 
\begin{pmatrix}
1 \\
4 \\
-3 \\
-3
\end{pmatrix} b + 
\begin{pmatrix}
1 \\
2 \\
1 \\
2
\end{pmatrix} c + 
\begin{pmatrix}
3 \\
1 \\
10 \\
-9
\end{pmatrix} d &= 0
\end{align*}
\]

\[
\begin{bmatrix}
2 & 1 & 1 & 3 & 0 \\
1 & 1 & 2 & 1 & 0 \\
3 & 4 & 1 & 10 & 0 \\
-1 & -3 & 2 & -9 & 0
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 2 & 0 \\
0 & 0 & 1 & -10 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
\begin{align*}
a + d &= 0 \\
b + 2d &= 0 \\
c - d &= 0
\end{align*}
\]

\[
\begin{align*}
a &= -d \\
b &= -2d \\
c &= d
\end{align*}
\]

Let \(d = 1\), then \(a = -1\) \(b = -2\) \(c = 1\)

\[
\begin{align*}
-1 \begin{pmatrix}
2 \\
1 \\
3 \\
-1
\end{pmatrix} -2 \begin{pmatrix}
1 \\
4 \\
-3 \\
-2
\end{pmatrix} + \begin{pmatrix}
1 \\
2 \\
1 \\
2
\end{pmatrix} + \begin{pmatrix}
3 \\
1 \\
10 \\
-9
\end{pmatrix} &= 0
\end{align*}
\]

Since there are infinite solutions with at least one positive number from \(a, b, c, d\), these four matrices are linearly dependent.
8. Find a basis for the column space of $A$ consisting of columns of $A$ where

$$A = \begin{bmatrix} 1 & -2 & 1 & 1 & -1 \\ 1 & -2 & 1 & 3 & 5 \\ 3 & -6 & 3 & 5 & 3 \\ 2 & -4 & 2 & 5 & 7 \end{bmatrix}$$

To find the linearly independent columns of $A$,

$$\begin{bmatrix} 1 & -2 & 1 & 1 & -1 \\ 1 & -2 & 1 & 3 & 5 \\ 3 & -6 & 3 & 5 & 3 \\ 2 & -4 & 2 & 5 & 7 \end{bmatrix} \xrightarrow{\text{REF}} \begin{bmatrix} 1 & 2 & 1 & 0 & -4 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Since the $A_1$ and $A_4$ are pivot columns, the first column and fourth column of the original matrix are linearly independent.

So all other columns could be expressed as a linear combination of $A_1$ and $A_4$.

Then $A_1$ and $A_4$ are the elements belong to the span of all the columns of $A$.

So $A_1, A_4$ are a basis of $A$.

The basis:

$$\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$$
9. Let $A$ be the matrix in Problem 8. Find a basis for the column space of $A$ with the property that each basis element has a one in a position where all of the other elements have a 0.

$$
A^t = \begin{bmatrix}
1 & 1 & 3 & 2 \\ -2 & -2 & 6 & 4 \\ 1 & 1 & 3 & 2 \\ 1 & 3 & 5 & 5 \\
1 & 3 & 5 & 7
\end{bmatrix}
$$

Since the row space of $A^t$ equal to the column space of $A$.

To follow the property that each basis element has a one in a position where all the other elements have a 0.

The reduced row echelon form of $A^t$ have the basis row space which meet the property.

$A^t \rightarrow \begin{bmatrix} 1 & 1 & 2 & \frac{1}{2} \\ 0 & 1 & 1 & \frac{3}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$R_1$ and $R_2$ are the basis for row space of $A^t$, So $A_1, A_2$ are the basis for column space of $A$.

Therefore, \( \begin{bmatrix} 1 \\ 0 \\ 2 \\ \frac{1}{2} \end{bmatrix} \) and \( \begin{bmatrix} 0 \\ 1 \\ 1 \\ \frac{3}{2} \end{bmatrix} \) are the basis for the column space with the property.
10. $A$ is a $4 \times 5$ matrix which has rank 3. You are given that $AX_1 = AX_2 = 0$ where

$$X_1 = [1, 1, 1, 1, 1]^t$$
$$X_2 = [2, 2, 1, 1, 1]^t$$

(a) Prove that $\{X_1, X_2\}$ spans the nullspace of $A$. 

By the rank-nullity theorem, $\text{rank}(A) + \text{null}(A) = n$. Since $n = 5$ and $\text{rank}(A) = 3$, $A$ must have a 2-dimensional nullspace. Since $X_1$ and $X_2$ are linearly independent upon inspection and both correspond to an element of the nullspace, $\{X_1, X_2\}$ must span the nullspace.

(b) Can we be certain that there is an $X \in \mathbb{R}^5$ such that $AX = [1, 1, 1, 1, 1]^t$? Explain.

Matrix $A$ is always solvable if $\text{rank}(A) = n$, as a consequence of rank-nullity. Since $\text{rank}(A) = 3$ and $n = 5$, $3 \neq 4$ and so we cannot be sure that there is such an $X$ that $A$ is solvable for.

(c) Suppose that $X \in \mathbb{R}^5$ satisfies $AX = [1, 1, 1, 1, 1]^t$. Is there only one such $X$? Prove your answer.

Matrix $A$ has a unique solution if $\text{rank}(A) = n$, as another consequence of above. Since $\text{rank}(A) = 3$ and $n = 5$, $3 \neq 5$ and so we cannot be sure that there is only one such $X$. In fact, since $X$ has a nullspace, it may be able to vary with solutions.
11. Suppose \( \mathcal{V} \) is a vector space and \( \{X_1, X_2, X_3\} \) a linearly independent subset of \( \mathcal{V} \). Let \( Y_1 = X_1 + X_3 \), \( Y_2 = 2X_2 - X_3 \) and \( Y_3 = X_1 - X_3 \). Prove that \( \{Y_1, Y_2, Y_3\} \) is linearly independent.

Dependency equations: \( aY_1 + bY_2 + cY_3 = 0 \)

\[
\alpha (X_1+X_3) + b (2X_2-X_3) + c (X_1-X_3) = 0
\]

\((\alpha+c)X_1 + 2bX_2 + (\alpha-b-c)X_3 = 0\)

since \( \{X_1, X_2, X_3\} \) is linearly independent subset of \( \mathcal{V} \),

\[
\begin{aligned}
\alpha+c &= 0 \\
2b &= 0 \\
\alpha-b-c &= 0
\end{aligned}
\]

\[
\begin{bmatrix}
1 & 0 & 1 \\
0 & 2 & 0 \\
1 & -1 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & -1 & -2
\end{bmatrix}
\]

So, \( \alpha = 0 \) \( b = 0 \) \( c = 0 \)

Since \( \alpha = b = c = 0 \) is the only solution for

\( aY_1 + bY_2 + cY_3 = 0 \)

\( \{Y_1, Y_2, Y_3\} \) is linearly independent.
12. Let \( \{X_1, X_2, X_3, \ldots, X_n\} \) be a linearly independent subset of a vector space \( V \). Assume that \( W \) is an element of \( V \) such that 
\( \{W, X_1, X_2, X_3, \ldots, X_n\} \) is linearly dependent. Prove that then \( W \in \text{span} \{X_1, X_2, X_3, \ldots, X_n\} \). Your proof should follow the proof of Theorem 3 in Section 2.2 which you were supposed to learn for this test. 10 pts

Given: \( \exists X_1, X_2, X_3, \ldots, X_n \) is linearly independent.

Thus: \( aX_1 + bX_2 + cX_3 + \ldots + nX_n = 0 \) by definition of independence.

If \( \exists W, X_1, X_2, X_3, \ldots, X_n \) is linearly dependent, some constants exist such that
\[ c_1W + c_2X_1 + c_3X_2 + \ldots + c_nX_n = 0. \]

Suppose \( c_1 = 0 \). Then \( c_2X_1 + c_3X_2 + \ldots + c_nX_n = 0 \) results, but all \( c_i = 0 \) because set of \( X_i \) is independent.

Thus \( c_1 \neq 0 \). \( W \) may be written as
\[ W = -\frac{c_2X_1 + c_3X_2 + \ldots + c_nX_n}{c_1}, \]

thus proving it is an element of \( \text{span} \{X_1, X_2, X_3, \ldots, X_n\} \) because it is a linear combination of \( X_i \).
Appendix: Some matrices $A$ and their reduced forms $R$.

\[
A = \begin{bmatrix}
2 & 1 & 1 & 3 \\
1 & 1 & 2 & 1 \\
3 & 4 & 1 & 10 \\
-1 & -3 & 2 & -9
\end{bmatrix},
R = \begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 2 \\
0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
A = \begin{bmatrix}
1 & -2 & 1 & 1 & -1 \\
1 & -2 & 1 & 3 & 5 \\
3 & -6 & 3 & 5 & 3 \\
2 & -4 & 2 & 5 & 7
\end{bmatrix},
R = \begin{bmatrix}
1 & -2 & 1 & 0 & -4 \\
0 & 0 & 0 & 1 & 3 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
A = \begin{bmatrix}
1 & 1 & 3 & 2 \\
-2 & -2 & -6 & -4 \\
1 & 1 & 3 & 2 \\
1 & 3 & 5 & 5 \\
-1 & 5 & 3 & 7
\end{bmatrix},
R = \begin{bmatrix}
1 & 0 & 2 & 1/2 \\
0 & 1 & 1 & 3/2 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
A = \begin{bmatrix}
1 & 4 & -5 \\
2 & 1 & -3 \\
1 & -1 & 1 \\
0 & 1 & -1
\end{bmatrix},
R = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}
\]

\[
A = \begin{bmatrix}
-3 & -6 & -6 & -9 & -21 \\
2 & 4 & 5 & 7 & 17 \\
1 & 2 & 2 & 3 & 7
\end{bmatrix},
R = \begin{bmatrix}
1 & 2 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & 3 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
A = \begin{bmatrix}
-3 & -6 & -6 & -9 \\
2 & 4 & 5 & 7 \\
1 & 2 & 2 & 3
\end{bmatrix},
R = \begin{bmatrix}
1 & 2 & 0 & 1 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
A = \begin{bmatrix}
1 & 1 & 2 & 0 \\
4 & -1 & 1 & 1 \\
-5 & 1 & -3 & -1
\end{bmatrix},
R = \begin{bmatrix}
1 & 0 & 0 & 2/7 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & -1/7
\end{bmatrix}
\]

15