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**Justify all answers.** A correct answer without supporting justification is worth NO credit! *Please note that there are some matrices and their echelon forms at the end of the test. You should not need to do any row reduction on this test!*

1. Demonstrate your understanding of the test for independence by using it to test the following matrices for independence.(Other methods will not be accepted.) *You MUST indicate clearly how any equations or matricies you use come from the test for independence.*

(a)

$$\underset{X_1}{\begin{bmatrix} -2 & 3 \\ 1 & 3 \end{bmatrix}} \underset{X_2}{\begin{bmatrix} 0 & 1 \\ 1 & 7 \end{bmatrix}} \underset{X_3}{\begin{bmatrix} -4 & 3 \\ -1 & -15 \end{bmatrix}} \underset{X_4}{\begin{bmatrix} -14 & 18 \\ 4 & 0 \end{bmatrix}}$$

*5pts*

Assume $aX_1 + bX_2 + cX_3 + dX_4 = 0$

$$a\begin{bmatrix} -2 & 3 \\ 1 & 3 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 7 \end{bmatrix} + c\begin{bmatrix} -4 & 3 \\ -1 & -15 \end{bmatrix} + d\begin{bmatrix} -14 & 18 \\ 4 & 0 \end{bmatrix} = 0$$

$$\begin{bmatrix} -2a-4c-14d & 3a+b+3c+18d \\ a+b-c+4d & 3a+7b-15c \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

So $\begin{cases} -2a-4c-14d=0 \\ 3a+b+3c+18d=0 \\ a+b-c+4d=0 \\ 3a+7b-15c=0 \end{cases} \rightarrow \begin{bmatrix} -2 & 0 & -4 & -14 & 0 \\ 1 & 1 & -1 & 4 & 0 \\ 3 & 1 & 3 & 18 & 0 \\ 3 & 7 & -15 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 2 & 7 & 0 \\ 0 & 1 & -3 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

So $\begin{cases} a = -2c-7d \\ b = 3c+3d \\ c = c \\ d = d \end{cases}$

① if $c=1, d=0 \Rightarrow a=-2, b=3$

$\therefore -2X_1+3X_2+X_3=0 \Rightarrow X_3=2X_1-3X_2$

② If $c=0, d=1 \Rightarrow a=-7, b=3$

$\therefore -7X_1+3X_2+X_4=0 \Rightarrow X_4=7X_1-3X_2$

$X_1, X_2$ are pivot matrixes
$X_3, X_4$ are free matrices

$\therefore X_1, X_2$ are linearly independent., $X_3=2X_1-3X_2$, $X_4=7X_1-3X_2$, $X_1, X_2, X_3, X_4$ are linearly dependent.

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2. Suppose that $X_1, X_2, X_3, X_4, X_5$, and $X_6$ are elements of some vector space. The equation

$$x_1X_1 + x_2X_2 + x_3X_3 + x_4X_4 + x_5X_5 + x_6X_6 = \mathbf{0}$$

yields a system of linear equations in the $x_i$ with augmented matrix $A$. The row reduced form of $A$ is

$$R = \begin{bmatrix} 1 & 2 & 0 & 4 & 0 & -3 & 0 \\ 0 & 0 & 1 & 2 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1$ $x_2$ $x_3$ $x_4$ $x_5$ $x_6$

(a) Find a basis for $\mathcal{W} = \text{span}\{X_1, X_2, X_3, X_4, X_5, X_6\}$. Your basis will consist of certain of the $X_i$. *4pts*

From R, we can conclude that
$$\begin{cases} x_1 = -2x_2 - 4x_4 + 3x_6 \\ x_2 = x_2 \\ x_3 = -2x_4 - 4x_6 \\ x_4 = x_4 \\ x_5 = -5x_6 \\ x_6 = x_6 \end{cases}$$

By theorem 3
So the pivot columns are $X_1, X_3, X_5$ and they should be the basis of span$\{X_1, X_2, X_3, X_4, X_5, X_6\}$
So the basis is $\{X_1, X_3, X_5\}$

(b) Express each of the other $X_i$ as a linear combination of the basis elements. *4pts*

$$\begin{cases} x_1 = -2x_2 - 4x_4 + 3x_6 \\ x_2 = x_2 \\ x_3 = -2x_4 - 4x_6 \\ x_4 = x_4 \\ x_5 = -5x_6 \\ x_6 = x_6 \end{cases}$$

① If $x_2 = 1$, $x_4 = x_6 = 0$
$x_1 = -2$, $x_3 = x_5 = 0$
$\therefore -2X_1 + X_2 = 0 \Rightarrow \underline{X_2 = 2X_1}$ ✓

② If $x_4 = 1$, $x_2 = x_6 = 0$
$x_1 = -4$, $x_3 = -2$, $x_5 = 0$
$\therefore -4X_1 - 2X_3 + X_4 = 0 \Rightarrow \underline{X_4 = 4X_1 + 2X_3}$ ✓

③ If $x_6 = 1$, $x_2 = x_4 = 0$
$x_1 = 3$, $x_3 = -4$, $x_5 = -5$
$\therefore 3X_1 - 4X_3 - 5X_5 + X_6 = 0$
$\Rightarrow \underline{X_6 = -3X_1 + 4X_3 + 5X_5}$ ✓

(c) What is the dimension of $\mathcal{W}$? *2pts*

Since the basis of W contains three linearly independent elements $X_1, X_3, X_5$,
W is three dimensional. ✓

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3. Below is a matrix $A$, its reduced form $R$, and the reduced form $S$ for $A^t$. Use this information together with some theorems from the class to find two different bases for the row space of $A$. The first basis should consist of rows of $A$ and the second basis should have the property that each basis element has a one in a position where all of the other basis elements have a 0. *Clearly indicate in each part how in each part you know that the given set is a basis. It is not sufficient to say that a certain set forms a basis without some justification.*

$$A = \begin{bmatrix} 2 & 4 & 1 & 3 \\ 1 & 3 & -1 & -4 \\ -1 & 2 & 1 & -1 \\ -1 & 1 & -1 & -6 \end{bmatrix} \quad A^t = \begin{bmatrix} 2 & 1 & -1 & -1 \\ 4 & 3 & 2 & 1 \\ 1 & -1 & 1 & -1 \\ 3 & -4 & -1 & -6 \end{bmatrix}$$

$$R = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad S = \begin{bmatrix} 1 & 0 & 0 & -2/3 \\ 0 & 1 & 0 & \frac{13}{15} \\ 0 & 0 & 1 & \frac{8}{15} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Basis One is: *5pts*

Suppose the columns of $A^t = X_1, X_2, X_3, X_4$

From S, we can indicate that $X_1 = \frac{2}{3}X_4$, $X_2 = -\frac{13}{15}X_4$, $X_3 = \frac{-8}{15}X_4$

So $X_1, X_2, X_3$ are pivot matrixes, $X_4$ are free matrices. This means $\{X_1, X_2, X_3\}$ are basis of the column space of $A^t$.

The column space of $A^t$, however, is the row space of A, and $X_1, X_2, X_3, X_4$ are the rows of Matrix A. So $\{X_1, X_2, X_3\}$ is a basis of the row space of A.

which means $\{[2\ 4\ 1\ 3], [1\ 3\ -1, -4], [-1\ 2\ 1\ -1]\}$ is a basis of the row space of A.

(b) Basis Two is: *5pts*

Since $R = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, By non-zero ~~reduced~~ echelon theory, the non zero rows of the reduced echelon Matrix are the basis of the row space. So the basis of the row space is the first three rows of R, which is $\{[1\ 0\ 0\ 2], [0\ 1\ 0\ -1], [0\ 0\ 1\ 3]\}$

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4. Let $A$ be the matrix in Problem 3. **Without doing any row reduction** prove that $X = [6, 9, 17, 54]$ belongs to the row space of $A$. *Hint:* Use one of the bases found in Problem 3. *4pts*

From Problem 3(b), We've alread figured out that the basis of row space of A is $\{[1\ 0\ 0\ 2], [0\ 1\ 0\ -1], [0\ 0\ 1\ 3]\}$, where $[1\ 0\ 0\ 2] = X_1$, $[0\ 1\ 0\ -1] = X_2$, $[0\ 0\ 1\ 3] = X_3$.

So if $X = aX_1 + bX_2 + cX_3$, it belongs to the row space of A.

So $[a, b, c, 2a-b+3c] = [6, 9, 17, 54]$

when $a=6$, $b=9$, $c=17$,

$2a-b+3c = 2\times6-9+17\times3 = 54$

So this equation works. and $X = 6X_1 + 9X_2 + 17X_3$

Therefore, X is a linear combination of the elements of the basis for A's row space.

X belongs to the row space of A.

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5. Find a basis for the nullspace of the matrix $A$ below. *10pts*

$$A = \begin{bmatrix} -2 & 1 & 3 & 3 \\ 0 & 1 & 1 & 7 \\ -4 & -1 & 3 & -15 \\ -14 & 4 & 18 & 0 \end{bmatrix}$$

Suppose $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$, $A \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$

Then $\begin{bmatrix} -2 & 1 & 3 & 3 & 0 \\ 0 & 1 & 1 & 7 & 0 \\ -4 & -1 & 3 & -15 & 0 \\ -14 & 4 & 18 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 & 2 & 0 \\ 0 & 1 & 1 & 7 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{cases} x_1 = x_3 - 2x_4 \\ x_2 = -x_3 - 7x_4 \\ x_3 = x_3 \\ x_4 = x_4 \end{cases}$

$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \underset{M}{\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}} s + \underset{N}{\begin{bmatrix} -2 \\ -7 \\ 0 \\ 1 \end{bmatrix}} t$

From Reduced echelon form, we know rank of $A = 2$

So nullity $= 4 - 2 = 2$. Thus the nullspace is 2 dimensional.

Since two elements M, N belong to the nullspace. They can be the basis as long as they are linearly independent.

And for $\begin{bmatrix} -1 & -2 & 0 \\ -1 & -7 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, this proves M and N are linearly independent. By theorem 3, M and N are the basis of the nullspace. In other words,

$\{[1, -1, 1, 0]^t, [-2, -7, 0, 1]^t\}$ is the basis of nullspace.

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6. $A$ is a $7 \times 5$ matrix. You are given that

(i) The rank of $A$ is 2.

(i) $AX_1 = AX_2 = AX_3 = \mathbf{0}$ where

$$X_1 = \begin{bmatrix} 2 \\ 1 \\ 4 \\ 2 \\ 5 \end{bmatrix} \quad X_2 = \begin{bmatrix} 5 \\ 4 \\ 3 \\ 2 \\ 1 \end{bmatrix} \quad X_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$$

(a) Prove that $\{X_1, X_2, X_3\}$ is a basis for the nullspace of $A$. *5pts*

① the rank of the nullspace = 5 - rank of A = 5-2=3.

② $AX_1 = AX_2 = AX_3 = 0 \Rightarrow X_1, X_2, X_3$ belong to the null space of A.

③ $X_1, X_2, X_3$ are linearly independent because by test for linealy independence

Assume $aX_1 + bX_2 + cX_3 = 0$

$$\begin{bmatrix} 2a+5b+c \\ a+4b+2c \\ 4a+3b+3c \\ 2a+2b+4c \\ 5a+b+5c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 5 & 1 & 0 \\ 1 & 4 & 2 & 0 \\ 4 & 3 & 3 & 0 \\ 2 & 2 & 4 & 0 \\ 5 & 1 & 5 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So $a=b=c=0$. Thus $X_1, X_2, X_3$ are linearly independent.

By theorem 3, ①②③ can prove that $\{X_1, X_2, X_3\}$ is a basis for the nullspace of A.

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(b) Prove that $AZ \neq \mathbf{0}$ where $Z = [1, 1, 2, 3, 1]^t$. *5pts*

If $AZ \neq 0$, then $Z$ is not in the nullspace.

Assume $Z$ is in the nullspace, then $Z$ should be a linearly combination of the basis. $X_1, X_2, X_3$.

Suppose $aX_1 + bX_2 + cX_3 = Z$

$$\begin{bmatrix} 2a+5b+c= \\ a+4b+2c \\ 4a+3b+3c \\ 2a+2b+4c= \\ 5a+b+5c= \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 5 & 1 & 1 \\ 1 & 4 & 2 & 1 \\ 4 & 3 & 3 & 2 \\ 2 & 2 & 4 & 3 \\ 5 & 1 & 5 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} (*)$$

For (*), $0 \neq 1$ makes no sense. So this system has no solution, which indicates $Z$ can not be written as a linear combination of $X_1, X_2, X_3$.

Thus $Z$ is not in the nullspace and $AZ \neq 0$.

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7. Let

$$A = \begin{bmatrix} 2 & 1 & 4\pi & 2\sqrt{7} & -2 & 5 \\ 1 & 1 & 3\sqrt{3} & 2 & 1 & 1 \\ 1 & 0 & 4\pi - 3\sqrt{3} & 2\sqrt{7}-2 & -3 & 4 \\ 5 & 3 & 8\pi + 3\sqrt{3} & 4\sqrt{7}+2 & -3 & 11 \end{bmatrix}$$

Without doing any row reduction, answer the following questions. Use theorems from linear algebra to justify your answers. **No row reduction is allowed!**

(a) What is the rank of $A$? *2pts*

Let the rows of A be $X_1, X_2, X_3, X_4$. Then we can find out that $X_1$ and $X_2$ are linearly independent, $X_3 = X_1 - X_2$, $X_4 = 2X_1 + X_2$. So the basis of the row space of A is $\{X_1, X_2\}$. Thus for the row reduced form of A, ~~they~~ the last two rows will be changed to 0 since the original rows are linear combination of $X_1, X_2$. The first two rows will be non-zero rows since the original rows are linearly independent. So there are 2 non-zero rows in reduced echelon form of A. So rank of A is 2.

(b) What is the dimension of the nullspace of $A$? *2pts*

Dimension of nullspace = 6 - rank of A = 6 - 2 = 4

(c) What is the dimension of the nullspace of $A^t$? *2pts*

rank of A = rank of $A^t$ = 2.

Since $A^t$ has 4 columns,

dimension of nullspace of $A^t$ = 4 - rank of $A^t$ = 4 - 2 = 2

(d) Find a basis for the row space of $A$. *2pts*

Let the rows of A be $X_1, X_2, X_3, X_4$.
We can see that $X_3 = X_1 - X_2$, $X_4 = 2X_1 + X_2$.
Also, $X_1, X_2$ are linearly independent. If they are dependent, since the first element of $X_1$ is twice of that of $X_2$. The second of $X_2$ should also be twice that of $X_2$. However, it's not. So $X_1, X_2$ are linearly independent.

Since $X_1, X_2$ belong to the row space of A, and they are linearly independent, and row space of A is two dimensional (known from the rank of A). $\{X_1, X_2\}$ is the basis for the row space of A. In other words, $\{[2, 1, 4\pi, 2\sqrt{7}, -2, 5], [1, 1, 3\sqrt{3}, 2, 1, 4]\}$ is the basis for the row space of A.

(e) Find a basis for the column space of $A$. *2pts*

Because rank of $A^t$ = rank of $A = 2$. The column space should be two dimensional and is spanned by two linearly independent elements.
$M_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 5 \end{bmatrix}$ and $M_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 3 \end{bmatrix}$ belong to the column space. And they are linearly independent. (If they are linearly dependent, the second elements of $M_1$ should be twice that of $M_2$, as the first element of $M_1$ does). So $\{M_1, M_2\}$ is a basis for the column space of A. In other words, $\left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 3 \end{bmatrix} \right\}$ is a basis for the column space of A.

(f) Is the equation $AX = B$ always solvable? *2pts*

No. $AX = B$ is always solvable only when rank = number of rows of the matrix, which means the rows of the matrix are linearly independent. For matrix A, rank = 2, but it has 4 rows. $2 \neq 4$. So $AX = B$ is not always solvable.

10D

8. A $3 \times 3$ matrix $A$ is lower triangular if it has the form below. What is the dimension of the space of all such matrices? Prove your answer. *10pts*

$$A = \begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{bmatrix}$$

$$A = a\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + e\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} + f\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

① A is spanned by 6 elements showed above.

② These 6 elements are linearly independent because each has a 1 in the position where the others have 0.

For these two reasons, the dimension of the space is six.

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9. Suppose that $f(x)$ and $g(x)$ are elements of $C^\infty(\mathbb{R})$ such that $f(1) = 2, f(0) = -1$ and $g(1) = 1$, $g(0) = 3$. *Use the test for linear independence* to prove that the set $\{f(x), g(x)\}$ is linearly independent. *10pts*

Assume $af(x) + bg(x) = 0$

if $x=1$, $af(1)+bg(1)=0$
$\because f(1)=2, g(1)=1$
$\therefore 2a+b=0$

if $x=0$, $af(0)+bg(0)=0$
$\because f(0)=-1, g(0)=3$
$\therefore -a+3b=0$

$\therefore \begin{cases} 2a+b=0 \\ -a+3b=0 \end{cases} \Rightarrow \begin{cases} a=0 \\ b=0 \end{cases}$

✓

$\therefore$ For $af(x)+bg(x)=0$, a and b can only be zero.
So the set $\{f(x), g(x)\}$ is linearly independent.

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10. Suppose that $X_1$, $X_2$ and $X_3$ are linearly independent elements of some vector space $\mathcal{V}$. Let $Y_1 = X_1 + 2X_2$, $Y_2 = X_1 + X_2$ and $Y_3 = X_1 + X_2 + X_3$. Prove that $Y_1$, $Y_2$ and $Y_3$ are also independent. *10pts*

Assume $aY_1 + bY_2 + cY_3 = 0$

$a(X_1+2X_2) + b(X_1+X_2) + c(X_1+X_2+X_3) = 0$

$(a+b+c)X_1 + (2a+b+c)X_2 + cX_3 = 0$

Since $X_1$, $X_2$ and $X_3$ are linearly independent, $\begin{cases} a+b+c=0 \\ 2a+b+c=0 \\ c=0 \end{cases}$

$\rightarrow \begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ So $a=b=c=0$

So for equation $aY_1 + bY_2 + cY_3 = 0$, $a, b, c$ can only be zero.

Thus $Y_1$, $Y_2$ and $Y_3$ are linearly independent.

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11. Suppose that in Problem 10 it is known that the $X_i$ span $\mathcal{V}$. Prove that then the $Y_i$ also span $\mathcal{V}$. *4pts*

① Since $X_1, X_2, X_3$ span V. ∴ V should be three dimensional.

② Since $Y_1 = X_1 + 2X_2$, $Y_2 = X_1 + X_2$, $Y_3 = X_1 + X_2 + X_3$, they are linear combination of $X_1, X_2, X_3$. Thus $Y_1$, $Y_2$ and $Y_3$ belong to the space $\{X_1, X_2, X_3\}$ span, which is V.

③ And We have already proved that $Y_1, Y_2$ and $Y_3$ are linearly independent in Problem 10

For reasons ①, ②, ③, $\{Y_1, Y_2, Y_3\}$ is a basis of V, thus $Y_1, Y_2$ and $Y_3$ also span V.