REVIEW FOR TEST 2

The test will cover sections 2.3, 3.1, 3.2, 3.3, 3.4, 5.1. 5.2, 5.3 (Cramer's Rule only). There will be no eigenvectors or coordinates on the exam. The coverage of determinants will emphasize computations. (But I consider problems such as number 5 on p. 287 as computational.) *You should learn the proof of the question asked in (15) below.*

As on the last test, there will be a table of matrices and their reduced forms at the end of the test.

You should study all of your past homeworks. I suggest in particular that you look at (5) and (6) in 3.1, (10) and (11) in Section 3.3, (12), (13) in Section 3.4, (5) in Section 5.2. Additionally, you should consider the problems below.

(1) For the transformation $T : \mathbf{R}^3 \to \mathbf{R}^2$ defined below, use the definition of linearity to prove that it is linear and find a matrix A such that $T(X) = AX$.

$$T([x, y, z]^t) = [2x + 3y - z, 2y + z]^t.$$

(2) Find a basis for the image and nullspace for the transformation in the previous problem.

(3) Let A be a 3×4 matrix. Suppose that multiplication by A transforms $[1, 2, 1, 1]^t$ onto $[2, 3, 2]^t$ and $[0, 1, -3, 2]^t$ onto $[1, 0, 1]^t$. What does multiplication by A transform $[2, 7, -7, 8]^t$ onto?

(4) Find a basis for the space spanned by the 6 vectors below by using the concept of row space. $X_1 = [1, 3, 0, 4]^t$, $X_2 = [2, 4, 1, 7]^t$, $X_3 = [1, 1, 1, 3]^t$, $X_4 = [-2, -12, 3, -11]^t$, $X_5 = [3, 7, 2, 12]^t$, $X_6 = [4, 2, 2, 8]^t$.

(5) Find a basis for the image of the transformation defined by the matrix A below. What is the dimension of the nullspace? Is the equation $AX = B$ always solvable, regardless of B? If this equation is solvable, is there only one solution? Explain.

$$A = \begin{bmatrix} 1 & 3 & 0 & 4 \\ 2 & 4 & 1 & 7 \\ 1 & 1 & 1 & 3 \\ -2 & -12 & 3 & -11 \\ 3 & 7 & 2 & 12 \\ 4 & 2 & 2 & 8 \end{bmatrix}$$

(6) For the matrix A below, answer the following questions *without doing any row reduction at all.* Justify your conclusions carefully.

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 3\pi & -e & 6\pi - e \\ 7 & 13 & 27 \\ \sqrt{2} + \sqrt{3} & -2\sqrt{2} & 2\sqrt{3} \\ 1 & -7 & -5 \\ -17 & 5 & -29 \end{bmatrix}$$

(a) What is the rank of A? Prove your answer.
(b) What is the dimension of the nullspace of A?
(c) What is the dimension of the image of A?
(c) Will the equation $AX = B$ be solvable for all B in $\mathbf{R}^6$. Explain.
(d) Will the equation $AX = B$ have at most one solution. Explain.
(e) Find *two* different bases for the row space of A. Use some theorems from linear algebra to justify your answer.
(f) I claim that the nullspace of A is the solution set of the system of equations below. Explain.

$$\begin{aligned} x + 2y + \ \ 4z &= 0 \\ -17x + 5y - 29z &= 0 \end{aligned}$$

(7) Find a 3×4 matrix A which transforms $\mathbf{R}^4$ onto the subspace of $\mathbf{R}^3$ spanned by $[2, 1, 3]^t$ and $[1, 1, 1]^t$. *Build your matrix in such a way that exactly one of its entries is 0.*
(8) Suppose that A is a 4×9 matrix whose nullspace has dimension 6. Let $B = A^t$. What is the dimension of the nullspace of B? What is the dimension of image of the transformation defined by B? Is the equation $BX = Y$ solvable for all Y in $\mathbf{R}^3$. Does the equation $BX = Y$ have at most one solution? (All answers must be explained.) Answer the same questions for A.
(9) Suppose that $C = AB$ where A is a 4×3 matrix and B is 3×4. What are the possible values for the dimension of the nullspace of C? Can C be invertible? Explain.
(10) For the given matrix A, find a 3×3, rank 2 matrix B such that $AB = 0$. Prove that any matrix B satisfying $AB = 0$ can have rank at most 2. [*Hint:* The columns of B belong to the nullspace of A.]

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \end{bmatrix}$$

(11) Demonstrate that you understand how inverses may be used to solve systems of equations by using inverses to solve the system below. Other methods will not be accepted.

$$\begin{aligned} x + \ \ 3y + \ \ z &= 1 \\ 3x + 4y + 5z &= 7 \\ 2x + 5y + 7z &= 2 \end{aligned}$$

(12) **Use Cramer's rule** to express the value of y in the preceding problem to express y as a ratio of 2-determinants. Then evaluate the determinants.
(13) Let A be a 3×3 invertible matrix such that $A[3, 2, -4]^t = [2, 1, 3]^t$ and $A[-4, 7, 5]^t = [1, 1, 1]^t$. Find $A^{-1}[4, 3, 5]^t$.
(14) Suppose that A is an $n \times n$ matrix such that $A^tA = I$. Prove that $\det A = \pm 1$.
(15) Let A and B be matrices such that AB is defined. Prove that rank AB is less than or equal to both the rank of A and the rank of B.