

MA 366 Spring 2011 Assignments

For Wednesday 1/12:

Read 1.2–1.3.

Do: p. 25: 7, 9, 16

p. 360, The answer to Exercise 8 is given in the back of the text. Substitute the given functions x_1 and x_2 into the the system to show that they do solve the system. Show that they also satisfy the desired initial conditions stated in the exercise.

For Friday 1/14:

Read p. 31-39

Do:Exercises : p. 25: 13, 14

p. 360, The answer to Exercise 10 is given in the back of the text. Substitute the given functions x_1 and x_2 into the the system to show that they do solve the system. Show that they also satisfy the desired initial conditions stated in the exercise.

Exercises p. 39: 1(use dfield), 4(use dfield), 13,15, 16

Note: Note: In the dfield problems to determine the behavior as t goes to infinity you need to also draw several of the solutions as the text did in figure 2.1.4 on p. 38 and describe what you see. In part (c) you are asked to prove that your observation is correct.

For Wednesday 1/19:

Read p. 21, first paragraph, p. 42-47

p. 39: 14,19, 24 (use dfield)

p. 24: The differential equation in exercise 4 is not linear. Change one single number to make the equation linear. Do not just change one of the coefficients to zero!

p. 47, 1, 7, 10 (use dfield), 23. (Hint: Set $y' = 0$ and solve for x . You don't need to compute y' to do this! How can you tell that it is a minimum and not a maximum or a point of inflection?)

For Friday 1/21:

In all of these problems, do not do the computer part!

Read Example 1, p. 52. Then do Exercise 4, p. 60. Only find the concentration prior to the point of overflow. *Do not do the parts described in the final two sentences of the problem.*

p. 48, 22 (This is similar to 23, but now you want $y' = \infty$.), 24 (Prove that your value really is a maximum.), 30 (a)-(e)

p. 62, 16.

For Wednesday 1/26

In all of these problems, do not do the computer part!

p. 50, 31(a), (b)

p. 202, 1 (See the discussion relating to Equation (12) on p. 195. You will need the technique used in this problem for 5(c) below.)

p. 60, 5, 20

Note in some versions of the text the rate in and rate out are given as 3 gal/min. Please use 2 gal/min instead.

For Friday , 1/28

p. 100 1, 2, 3, 21

In all of these problems, do not do the computer part!

p. 50, 32(a), (b)

p. 60, 21(a), (b)

p. 77, 27(b) *Hint* You need to express $\frac{dy}{dx}$ in terms of v and $\frac{dv}{dx}$.

1. At time $t = 0$, a tank of water contains 500gal of water with 50g of salt dissolved in it. Water flows into the tank at 30gal/hr and the well stirred mixture flows out at the same rate. Assume that the concentration of the salt coming in at time t is $10e^{-t}$ lb/gal. Write a differential equation and initial value condition which could be solved to find a formula for the amount $Q(t)$ of salt in the tank. You need not solve the equation.
2. Let the data be exactly the same as in the preceding problem except now water flows out of the tank at 40gal/hr. Write a differential equation and initial value condition which could be solved to find a formula for the amount $Q(t)$ of salt in the tank at any time prior to the time

the tank become empty. You need not solve the equation.

For Wednesday 2/2

p. 77: 28 p. 100: 7, 11, 15, 22

1. An object of mass 1 kg. is thrown upward with an initial velocity of 20 m/s from a building 30 m high. There is a force of resistance of $9.8v^2$ where the velocity v is measured in m/s. (1) Find a formula for the velocity $v(t)$ for the object when it is rising. Then (2) Find the time t_o at which it reaches its maximum height. Finally, (3) Find a formula for $v(t)$ for $t > t_o$ which is valid until the object hits the ground. You may leave this answer in implicit form.

Ans: Going up: $v(t) = \tan(1.52 - 9.8t)$, $t_o = 0.155$, Going down: $\frac{1+v}{1-v} = e^{-19.6(t-0.155)}$.

Remark. The numbers in this problem are not meant to be realistic. They are chosen to make the calculations easier.

2. The statement of the extra Exercises 1 and 2 from the assignment for Friday , 1/28, have been corrected. (“concentration” was changed to “amount.”) Redo these exercises with the corrected statement.
3. For the following differential equation (a) find the general solution (b) find a value of y_o for which the initial value problem $y(0) = y_o$ has two solutions. (c) Explain, using the Existence and Uniqueness Theorem (Theorem 2.8.1 on p.112), why there is only one possible choice of y_o in (b) (d) Solve the initial value problem $y(0) = 1$. There is only one solution!

Hint Completing the square helps in solving for y in terms of x .

$$(3 + 2y)y' = 3x^2 - 1$$

4. For the following differential equation (a) find the general solution (b) Find a value of t_o for which the initial value problem $y(t_o) = 1$ has no solution. (c) Explain, using the Existence and Uniqueness Theorem (Theorem 2.8.1 on p.112), why there is only one possible choice of t_o in (b)

$$ty' + 2y = t^2e^{-t}$$

5. For the following differential equation (a) find the general solution (b) Find a value of y_0 for which the initial value problem $y(3) = y_0$ has more than one solution. (c) Explain, using the Existence and Uniqueness Theorem (Theorem 2.8.1 on p.112), why there is only one possible choice of y_0 in (b)

$$y' = (2y - 3)^{3/5}(x - 4)^2$$

For Wednesday, 2/16

Read §2.5, p. 78-81, §3.1, p. 137-143, §3.3, p.157-163

Do: p.89, 8, 10

p. 144, 5,6,7, 9, 12 (Do not graph),

p. 163, 1, 3, 7, 18

For Friday, 2/18/09

In all of these problems, do not do the computer part and do not graph

p. 163, 11, 19, 34(a), (b)

p. 144, 17,21,

p. 171, 2, 6

1. Consider the differential equation

$$y^{(4)} + 4y''' + 3y'' - 4y' - 4y = 0. \quad (1)$$

Let $L(y)$ be the corresponding linear operator on functions.

- (a) Show that

$$L(e^{rt}) = (r^2 - 1)(r + 2)^2 e^{rt}.$$

- (b) Use the fact that

$$L\left(\frac{\partial y}{\partial r}\right) = \frac{\partial}{\partial r} L(y)$$

to show that

$$L(te^{rt}) = (2r(r + 2)^2 + 2(r^2 - 1)(r + 2) + t(r^2 - 1)(r + 2)^2) e^{rt}.$$

- (c) Use the above formula to show that $L(te^{-2t}) = 0$.

(d) Verify by direct substitution into equation (1) that $L(te^{-2t}) = 0$.

For Wednesday, 2/23/09

In all of these problems, do not do the computer part and do not graph

Read §3.5, p. 174-183,

§3.1: p. 144, 23

§3.3: p. 165, 36

§3.4 p. 171, 12

§3.5 p. 183, 1, 2,3,5

For Friday, 2/25

In all of these problems, do not do the computer part and do not graph

Read The section on variation of parameters. (§3.6)

§3.5: p. 183, 8, 29

§3.6: p. 189, 5, 17 *Note:* In 17 you must divide the differential equation by x^2 before applying the technique.

In the problems on p. 189 DO NOT use the text's formulas (26) and (27). Instead, explicitly write the equations

$$\begin{aligned}u_1' y_1 + u_2' y_2 &= 0 \\ u_1' y_1' + u_2' y_2' &= g(t)\end{aligned}$$

as I did in class where y_1 and y_2 and g are EXPLICIT functions. (The y_i are the fundamental set of solutions and the g is the function on the right side of the differential equation.) Then express this system in terms of matrices and use Cramer's rule to find u_1' and u_2' . Finally, integrate to find u_1 , u_2 , and the general solution to the equation, which is

$$y = u_1 y_1 + u_2 y_2.$$

For Wednesday, 3/2

In all of these problems, do not do the computer part and do not graph

§3.1 p. 144, 24

§3.3 p. 165, 39

§3.5 p. 183, 30

§3.7 p. 202, 5 *Note:* The information in the first sentence of the statement of the problem is needed to find the spring constant k . See Example 1, p. 194.

p. 202, 9: Do only the part up to “Plot u versus t .” (You may use a computer.) Also plot on the same graph $y = \left(\sqrt{2^2 + (5/\sqrt{6})^2} \right) e^{-10t}$ and $y = - \left(\sqrt{2^2 + (5/\sqrt{6})^2} \right) e^{-10t}$.

§3.6 p. 189, 10

DO NOT use the text’s formulas (26) and (27). Instead, explicitly write the equations

$$\begin{aligned}u_1' y_1 + u_2' y_2 &= 0 \\u_1' y_1' + u_2' y_2' &= g(t)\end{aligned}$$

as I did in class where y_1 and y_2 and g are EXPLICIT functions. (The y_i are the fundamental set of solutions and the g is the function on the right side of the differential equation.) Then express this system in terms of matrices and use Cramer’s rule to find u_1' and u_2' . Finally, integrate to find u_1 , u_2 , and the general solution to the equation, which is

$$y = u_1 y_1 + u_2 y_2.$$

For Friday, 3/4

§3.6: p. 189, 11 (DO NOT use the text’s formulas (26) and (27).)

§3.7: p. 202 6,

10: Do only the part up to “Plot u versus t .” (You may use a computer.)
13

Additional problem:

1. In Exercise 5 on p. 202, for which coefficients of damping would the motion be over damped? Critically damped? Under damped?

p. 215, 1 (See Friday's class notes), 5, 6, 7(a), 8(a), (b) (The steady state solution is the part that is not multiplied by e^{-at} for some $a > 0$. The transient part is the part that **is** multiplied by e^{-at} .), 9, 15, 17

For Wednesday, 3/30

p. 259: 2 (a),(b) (See Example 2, p. 255) The solution y_1 is found by setting $a_0 = 1$ and $a_1 = 0$ in the recursion relation. The solution y_2 is found by setting $a_0 = 0$ and $a_1 = 1$ in the recursion relation.

For Friday, 4/1

p. 259: 7(a), (b), 17(a), (b)

1. Repeat the instructions for 7(a), (b) on p. 259 for the equation

$$x^2 y'' + (x^3 - 2)y = 0.$$

For Wednesday, 4/6

p. 259: Re-do Exercise 7(a), (b) with $x_0 = 1$,
p. 259: 6 (a), (b) 10(a), (b)

1. Find a formula for the Wronskian $W(x)$ for the solutions y_1 and y_2 in Exercise 6, p. 259. Use the following theorem:

Theorem 1. *Suppose that y_1 and y_2 satisfy $L(y_1) = L(y_2) = 0$ where*

$$L(y) = y'' + py' + qy.$$

Then the Wronskian W of y_1 and y_2 satisfies

$$W' + pW = 0.$$

2. Determine the largest r such that $f(x)$ has a series expansion

$$f(x) = \sum_0^{\infty} a_n x^n$$

which is valid for all x , $|x| < r$ for the following functions $f(x)$. (See Examples 1, 2 on p. 263-264.) *You are not asked to find the expansion.*

(a) $f(x) = \frac{x^3}{x^2-x+12}$. Ans: $r = 2\sqrt{3}$.

(b) $f(x) = \frac{x^3}{x^2+4x+8}$. Ans: $r = 2\sqrt{2}$.

p. 265: 9, Problem 6. Use the following theorem which follows from Theorem 5.3.1 on p. 262 of the text:

Theorem 2. *Suppose that y satisfies*

$$y'' + p(x)y' + q(x)y = 0$$

where $p(x)$ has a power series expansion that converges for $|x| < r_1$ and $q(x)$ has a power series expansion that converges for $|x| < r_2$. Then $y(x)$ has a power series expansion which converges at least for all $|x| < \min\{r_1, r_2\}$.

For Friday, 4/8

p. 265, 7. Note: $x^3 + 1 = (x + 1)(x^2 - x + 1)$.

1. Let

$$L(y) = y'' + py' + qy$$

where p and q are functions.

(a) Suppose that $y_2 = uy_1$ where u and y_1 are functions. Show that

$$L(uy_1) = u''y_1 + (2y_1' + py_1)u' + uL(y_1).$$

(I essentially did this Wednesday. Look at the notes from the last part of class.)

(b) Suppose that $L(y_1) = 0$. Let u be a function and set $v = u'$. Show that $L(uy_1) = 0$ if and only if

$$v' + (2(\ln y_1)' + p)v = 0.$$

(c) Show that the function v from part 1b is given by the formula

$$v = Cy_1^{-2}e^{-\int p dx}. \tag{2}$$

(d) Use the formula just derived to do problem 36 on p. 174 of the text. Specifically, $y_2 = y_1u$ where $u = \int v$ and v is computed from formula (2).

For Wednesday, 4/13

At the end of Monday's class I started to state the stated the following definition:

Definition 1. We say that a differential equation has regular singularities at $x = 0$ if it is equivalent to an equation of the form

$$x^2p(x)y'' + xq(x)y' + s(x)y = 0 \quad (3)$$

where p , q , and s are analytic at $x = 0$ (i.e. they have power series expansions in powers of x which converge in an interval containing $x = 0$.) and where $p(0) \neq 0$.

In this case the equation

$$x^2p(0)y'' + xq(0)y' + s(0)y = 0 \quad (4)$$

is referred to as the **approximating Euler** equation. The equation

$$p(0)r(r - 1) + q(0)r + s(0) = 0$$

is the **indicial equation** and its roots are the **exponents at the singularity**.

Note that if we set $P(x) = x^2p(x)$, $Q(x) = xq(x)$ and $R(x) = r(x)$ then

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{xQ(x)}{P(x)} &= \lim_{x \rightarrow 0} \frac{q(x)}{p(x)} \\ &= \frac{q(0)}{p(0)} \\ \lim_{x \rightarrow 0} \frac{x^2R(x)}{P(x)} &= \lim_{x \rightarrow 0} \frac{r(x)}{p(x)} \\ &= \frac{r(0)}{p(0)} \end{aligned}$$

This leads to the Books definition:

Definition 2 (Book). We say that a differential equation has regular singularities at $x = 0$ if it is equivalent to an equation of the form

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

where P , Q , and R are analytic at $x = 0$ (i.e. they have power series expansions in powers of x which converge in an interval containing $x = 0$.) and where the limits

$$p_o = \lim_{x \rightarrow 0} \frac{xQ(x)}{P(x)}$$

$$q_o = \lim_{x \rightarrow 0} \frac{x^2R(x)}{P(x)}$$

both exist.

In this case the equation

$$x^2y'' + xp_oy' + q_oy = 0$$

is referred to as the **approximating Euler equation**. The equation

$$r(r - 1) + p_or + q_o = 0$$

is the **indicial equation** and its roots are the **exponents at the singularity**.

It is a theorem that both definitions are equivalent—any equation that has regular singularities under the book's definition has regular singularities under my definition and vice versa. The book's definition gives a slightly different indicial equation than my definition. From the above discussion

$$p_o = \frac{q(0)}{p(0)}, \quad q_o = \frac{r(0)}{p(0)}.$$

Hence the book's indicial equation is

$$r(r - 1) + \frac{q(0)}{p(0)}r + \frac{r(0)}{p(0)} = 0$$

which is equivalent with mine.

I prefer my definition because you don't need limits to tell if an equation has regular singularities: if it has an approximating Euler equation, then it has regular singularities. For example the equation

$$(x^5 + 2x^3 + x^2)y'' + (3x^2 + 7x)y' + (x^2 + 3x + 5)y = 0$$

may be written

$$x^2(x^3 + 2x + 1)y'' + x(3x + 7)y' + (x^2 + 3x + 5)y = 0.$$

Hence in (3)

$$p(x) = x^3 + 2x + 1, \quad q(x) = 3x + 7, \quad \text{and } r(x) = x^2 + 3x + 5.$$

Since $p(x)$, $q(x)$, and $r(x)$ are polynomials they are analytic at $x = 0$. Also $p(0) \neq 0$. Hence the equation has regular singularities at $x = 0$.

From formula (4) above the approximating Euler equation is

$$x^2y'' + 7xy' + 5y = 0.$$

The equation

$$(x^5 + 2x^3 + x^2)y'' + (3x^2 + 7)y' + (x^2 + 3x + 5)y = 0$$

does not have regular singularities. To see this we first write the equation as

$$x^2(x^3 + 2x + 1)y'' + x(3x + \frac{7}{x})y' + (x^2 + 3x + 5)y = 0.$$

We cannot let $q(x) = 3x + \frac{7}{x}$ since this is not defined at $x = 0$. To avoid this we multiply both sides of our equation by x writing our equation as

$$x^2(x^4 + 2x^2 + x)y'' + x(3x^2 + 7)y' + (x^3 + 3x^2 + 5x)y = 0.$$

But now $p(x) = x^4 + 2x^2 + x$ so $p(0) = 0$, which is not allowed.

The Assignment:

1. If we write Problem 8 on p. 282 as in equation (3) above, what are $p(x)$, $q(x)$, and $r(x)$? What is the approximating Euler equation?
2. Use my definition of regular singularities to do parts (a) and (b) of Problem 8 on p. 282.
3. Use the book's definition of regular singularities to do parts (a) and (b) of Problem 8 on p. 282.
4. Do parts (c) and (d) of Problem 8 on p. 282.
5. Use formula (2) from last Friday's assignment to do problem 37 on p. 174 of the text.

6. Let

$$L(y) = y'' + py' + qy$$

where p and q are functions.

- (a) Suppose that $y_2 = uy_1$ where u and y_1 are functions. It was shown in last Friday's homework that

$$L(uy_1) = u''y_1 + (2y_1' + py_1)u' + uL(y_1).$$

Suppose that $L(y_1) = 0$. Let u and r be functions and set $v = u'$. Show that $L(uy_1) = r$ if and only if

$$v' + (2(\ln y_1)' + p)v = y_1^{-1}r.$$

- (b) Show that the function v is given by the formula

$$v = y_1^{-2}e^{-\int p dx} \int \left(ry_1e^{\int p dx}\right) dx. \quad (5)$$

- (c) Use the formula just derived to do problem 17 on p. 189 of the text. This demonstrates that we can use reduction of order as a replacement for variation of parameters.

For Friday, 4/15

I have posted copies of the exercises from the remaining sections of the text that might be covered in the file "Some problems from the text." The solutions are contained in the file "Solutions to the problems from the text." If your text differs from the one I am using, you should do these problems. I have also (finally) posted solutions to Test 2.

1. Use my definition of regular singularities to do parts (a) and (b) of Problem 5 on p. 282.
2. Use the book's definition of regular singularities to do parts (a) and (b) of Problem 5 on p. 282.
3. Do parts (c) and (d) of Problem 2 on p. 282.

4. For the differential equation in Exercise 5 on p. 282, give the approximating Euler equation. Use it to find the indicial equation and its roots. Use Theorem 5.6.1 on p. 289 to describe the expected form of the solutions y_1 and y_2 . **Do not find the coefficients!** This is meant to be a *short* exercise.
5. Repeat Exercise 4 for Exercise 9 on p. 282.
6. Use formula (2) from Wednesday's assignment to do problem 35 on p. 174 of the text: Find a second solution to

$$ty'' - y' + 4t^3y = 0, y_1(t) = \sin(t^2).$$

7. Use formula (2) from last Friday's assignment to do problem 7 on p. 189 of the text: Find a particular solution to

$$y'' + 4y' + 4y = t^{-2}e^{-2t}.$$

p. 359: 1(See Example 1, p. 357), 3, 7(a). By "solve for x_2 ", they mean write $x_2 = x_1' + 2x_1$.

For Wednesday, 4/10

Read p.390-398. (See Example 5 on p. 381.)

p. 398, 2(a), (b), When you plot the trajectories be sure to plot the four trajectories that lie along the eigenvectors as well as a few others. The "keyboard input" option in pplane is useful for this.

p. 398, 11, 12, 15. Do not describe the behavior of the solutions in these problems.

1. For the differential equation in Exercise 10 on p. 282, give the approximating Euler equation. Use it to find the indicial equation and its roots. Use Theorem 5.6.1 on p. 289 to describe the expected form of the solutions y_1 and y_2 . **Do not find the coefficients!** This is meant to be a *short* exercise.

2. Repeat Problem 1 for Exercise 3 on p. 282.
3. Do Exercise 10(b) and 10(c) on p. 282.

For Friday, 4/22

Read, p. 401-402

Assignment:

p. 409, 1

1. Given that X_1 , X_2 and X_3 are eigenvectors for the following matrix
 - (a) Find the general solution to $X' = AX$. *Hint:* To find the eigenvalue, compute AX_i .
 - (b) Find the solution $X(t)$ satisfying

$$X(0) = \begin{bmatrix} -1 \\ 4 \\ 5 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 6 & -3 \\ -2 & -10 & 5 \\ -6 & -12 & 5 \end{bmatrix} \quad X_1 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} \quad X_2 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} \quad X_3 = \begin{bmatrix} -1 \\ 3 \\ 5 \end{bmatrix}$$

2. Find all eigenvalues and eigenvectors for the matrix in Exercise 7 on p. 428 of the text. Explain why the techniques studied so far are insufficient to solve this exercise.
3. Given that for the matrix A in Exercise 5 on p. 428, $\det(A - rI) = -(r + 1)(r - 2)^2$, find the eigenvalues and eigenvectors. Explain why the techniques studied so far are insufficient to solve this exercise.

For Wednesday, 4/27

Read the note on Exponential Series posted on the web. Do Exercises 1(b),(c) p. 4 of the notes. (These are nilpotent matrices.), p. 14: 7(c)(See Example 1 on p.8 of the notes.), 10(a) (Only find a fundamental set of solutions. Do not do the part about $X(0)$.) , 13 (The reference to the answers should be to Exercise 12, not Exercise 11.), 14, 15, 18.