(1) (a) $y'' - 6y' - 7y = 0$

$r^2 - 6r - 7 = 0 \quad (r-7)(r+1) = 0$

$r_1 = 7 \quad r_2 = -1 \qquad y_1 = e^{7t} \quad y_2 = e^{-t}$

$\Rightarrow y = c_1 e^{7t} + c_2 e^{-t}$

(b) $y'' - 6y' + 9y = 0$

$r^2 - 6r + 9 = 0 \quad (r-3)^2 = 0 \quad r = 3$

$y_1 = e^{3t} \quad y_2 = te^{3t}$

$\Rightarrow y = c_1 e^{3t} + c_2 t e^{3t}$

(c) $y'' - 6y' + 25y = 0$

$r^2 - 6r + 25 = 0 \quad r = \frac{6 \pm \sqrt{36-100}}{2} = \frac{6 \pm 8i}{2} = 3 \pm 4i$

$\Rightarrow y = c_1 e^{3t} \cos 4t + c_2 e^{3t} \sin 4t$

(d) $t^2 y'' - 3ty' + 5y = 0$

Let $x = \ln t \quad t = e^x$

$\frac{dy}{dt} = t^{-1}\frac{dy}{dx} \qquad \frac{d^2y}{dt^2} = \frac{dy}{dt}\frac{d}{dt} = t^{-1}(-t^{-1}\frac{dy}{dx}) + t^{-2}\frac{d^2y}{dx^2}$

$= -t^{-2}\frac{dy}{dx} + t^{-2}\frac{d^2y}{dx^2}$

$\Rightarrow t^2(-t^{-2}\frac{dy}{dx} + t^{-2}\frac{d^2y}{dx^2}) - 3t(t^{-1}\frac{dy}{dx}) + 5y = 0$

$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 5y = 0$

$r^2 - 4r + 5 = 0 \quad r = \frac{4 \pm \sqrt{16-20}}{2} = 2 \pm i$

$y = c_1 e^{2x} \cos x + c_2 e^{2x} \sin x$

$x = \ln t \quad \Rightarrow y = c_1 t^2 \cos(\ln t) + c_2 t^2 \sin(\ln t)$

homogeneous part

(2) $y'' + 5y' + 6y = 0$

$r^2 + 5r + 6 = 0 \qquad (r+2)(r+3) = 0 \qquad r_1 = -2 \quad r_2 = -3$

$y_1 = e^{-2t} \qquad y_2 = e^{-3t}$

$y'' + 5y' + 6y = 2e^t + 3$

i) $y'' + 5y' + 6y = 3 \qquad$ Guess $y = A$

$\Rightarrow 0 + 0 + 6A = 3 \qquad A = \frac{1}{2}$

ii) $y'' + 5y' + 6y = 2e^t \qquad$ Guess $y = Be^t \quad y' = Be^t \quad y'' = Be^t$

$\Rightarrow Be^t + 5Be^t + 6Be^t = 2e^t \qquad 12Be^t = 2e^t \qquad B = \frac{1}{6}$

-0

$\Rightarrow y = c_1 e^{-2t} + c_2 e^{-3t} + \frac{1}{2} + \frac{1}{6}e^t$

(3) (a) $y_0 = A\cos(2t) + B\sin(2t)$

(b) $y_0 = At\cos(2t) + Bt\sin(2t)$

(c) $y_0 = At^3 + Bt^2 + Ct + D$

(d) $y_0 = At^2e^{2t} + Bte^{2t}$

(e) $y_0 = Ate^{2t}\sin t + Bte^{2t}\cos t$

(4) (a) $\begin{cases} au_1' + bu_2' = c \\ du_1' + eu_2' = f \end{cases} \qquad a = y_1 = e^x \quad b = y_2 = x \quad c = 0$

$d = y_1' = e^x \quad e = y_2' = 1$

$f = g(x) = (1-x)e^x$

(b) $W = \det\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} = \det\begin{bmatrix} e^x & x \\ e^x & 1 \end{bmatrix} = e^x - e^x x = (1-x)e^x$

$u_1' = \det\begin{bmatrix} 0 & x \\ (1-x)e^x & 1 \end{bmatrix} / W = \frac{-x(1-x)e^x}{(1-x)e^x} = -x$

$u_1 = -\frac{x^2}{2} + C_1$

$$u_2' = \det\begin{bmatrix} e^x & 0 \\ e^x & (1-x)e^x \end{bmatrix} / W = \frac{e^x \cdot e^x (1-x)}{e^x (1-x)} = e^x$$

$$u_2 = e^x + c_2$$

(c) $$y = y_1 u_1 + y_2 u_2 = e^x\left(-\frac{x^2}{2} + c_1\right) + x\left(e^x + c_2\right)$$

$$= -\frac{x^2}{2}e^x + xe^x + c_1 e^x + c_2 \cdot x$$

(5) (a) $mu'' + \gamma u' + ku = F(t)$

$m = 2 \quad \gamma = 0 \quad k = 8 \quad F(t) = 0$

$2u'' + 8u = 0 \qquad u(0) = 0 \qquad u'(0) = 4 (m/s)$

(b) $2u'' + 8u = 0$

$2r^2 + 8 = 0 \qquad r^2 + 4 = 0 \quad r = \pm 2i$

$u = c_1 \cos 2t + c_2 \sin 2t$

$u(0) = c_1 = 0 \qquad u'(0) = 2c_2 \cos(2 \cdot 0) = 2c_2 = 4 \quad c_2 = 2$

$u = 2\sin 2t$

(c) $2u'' + \gamma u' + 8u = 0 \qquad u(0) = 0 \quad u'(0) = 4 (m/s)$

(d) $2r^2 + \gamma r + 8 = 0$

$$r = \frac{-\gamma \pm \sqrt{\gamma^2 - 64}}{4}$$

According to the graph. there could be overdamped or criticallydamped situations.

Therefore $\gamma^2 - 64 \geq 0 \qquad \gamma \geq 8$

(6) (a) $2u'' + 2u' + 8u = 3\cos t$

$u'' + u' + 4u = \frac{3}{2}\cos t$

Guess $u = A\cos t + B\sin t$

$u' = -A\sin t + B\cos t \quad u'' = -A\cos t - B\sin t$

$\Rightarrow \quad -A\cos t - B\sin t - A\sin t + B\cos t + 4A\cos t + 4B\sin t = \frac{3}{2}\cos t$

$= \quad (-A + B + 4A)\cos t + (-B - A + 4B)\sin t = \frac{3}{2}\cos t$

$\Rightarrow \quad \begin{array}{l} 3A + B = \frac{3}{2} \\ 3B - A = 0 \end{array} \qquad \begin{array}{l} A = \frac{9}{20} \\ B = \frac{3}{20} \end{array}$

Steady state part : $u = \frac{9}{20}\cos t + \frac{3}{20}\sin t$

(b) Amplitude $\sqrt{A^2 + B^2} = \sqrt{(\frac{9}{20})^2 + (\frac{3}{20})^2} = \frac{3\sqrt{10}}{20}$ (cm)