This equation can also be solved as an exact equation.

(a) Find all solutions to the following initial value problems. $y$ should be expressed as an explicit function of $x$.

\[(1+y)y' - 3x^2 - 1 = 0, \quad y(0) = -2\]

\[
\begin{align*}
(1+y)y' &= 3x^2 + 1 \\
\int \left( y + \frac{y^2}{2} \right) &= \int (3x^2 + 1) \\
y + \frac{y^2}{2} &= x^3 + x + C \\
y^2 + 2xy + 1 &= (x^3 + x + C)^{x^2} \\
(y+1)^3 &= 2(x^3 + 2x + C) \\
\Rightarrow y &= -1 \pm \sqrt[2]{x^3 + x + C} \\
\therefore C &= 1 \\
\therefore y &= -1 - \sqrt[2]{x^3 + 2x + 1}
\end{align*}
\]

(b) \[(x + 1)y' - 3y - (x + 1)^3 = 0, \quad y(-2) = 1\]

\[
\begin{align*}
(x + 1)y' - 3y &= (x + 1)^3 \\
\frac{y'}{y} - \frac{3}{x+1} y &= (x + 1)^3 \\
\Rightarrow -\int \frac{3}{x+1} &= -3 \ln|x+1| \\
\int \frac{y}{(x+1)^3} &= \frac{1}{(x+1)^2} \\
\Rightarrow y &= \frac{1}{(x+1)^2} + C(x + 1)^3 \\
\therefore y(-2) &= 1 \\
\therefore C &= -2
\end{align*}
\]
(2) Find all solutions to the following initial value problems. 
may be expressed as an implicit function of $x$.

(a) 
\[(xe^{xy} + 1)y' + (ye^{xy} + 2x) = 0, \quad y(0) = 0.\]

\[
\begin{align*}
\frac{d}{dy}(ye^{xy} + 2x) & = e^{xy} + ye^{xy} \\
\frac{d}{dx}(xe^{xy} + 1) & = e^{xy} + ye^{xy}
\end{align*}
\]

\[
\frac{dy}{dx} = ye^{xy} + 2x, \quad \psi = e^{xy} + ye^{xy} + C
\]

\[
\frac{dy}{dx} = xe^{xy} + 1, \quad e^{xy} + x^2 + y^2 = C
\]

\[
1 - 0 + 0 < C \quad 1 = C
\]

\[
\frac{d}{dy}(e^{xy} + x^2 + y^2) = 1
\]

(b) 
\[(y^3 + x^4)y' - (y^4 + x^3y + x^4) = 0, \quad y(-1) = 1. \Rightarrow y = 1\]

Divide by $x^4$

\[
\left(\frac{y^3}{x^3} + 1\right)y' - \left(\frac{y^4}{x^4} + \frac{y}{x} + 1\right) = 0
\]

\[
\begin{align*}
(\frac{y^3}{x^3} + 1) & = 0 \\
& = \frac{y^3}{x^3} + 1
\end{align*}
\]

\[
\begin{align*}
\frac{1}{u^2} \left(\frac{y}{x}\right)^u + \frac{y}{x} & = \ln|x| + C \\
\frac{1}{u} \left(-1\right)^u & = \ln |x| + C \\
\frac{1}{u} & = C
\end{align*}
\]

\[
\frac{1}{u} \left(\frac{y}{x}\right)^u + \frac{y}{x} = \ln |x| - \frac{3}{u}
\]

\[
\frac{1}{u} \left(\frac{y}{x}\right)^u + \frac{y}{x} = \ln |x| \cdot \frac{2}{u}
\]
(3) This question refers to the initial value problem

\[(1 + y)y' - 3x^2 - 1 = 0, \quad y(0) = b.\]

(a) Find a value of \(b\) such that the existence or the uniqueness (or both) of the solution is not guaranteed by the fundamental existence and uniqueness theorems. Justify your answer using these theorems.

\[
\begin{align*}
(1 + y)y' &= 3x^2 + 1, \\
y' &= f(x, y) = \frac{3x^2 + 1}{y + 1}
\end{align*}
\]

\(b = y(0) = -1\) is not guaranteed because \(y'(x, y)\) is discontinuous at this point, so existence and uniqueness are not guaranteed at that point.

(b) Attempt to solve the given initial value problem for the value of \(b\) you found in part (a). (Note that you considered the same differential equation in Problem (1a).) Does existence actually fail in this case? Does uniqueness actually fail? Explain.

From (1a)

\[
\begin{align*}
y^2 + y - x^3 - x &= C, \\
y(0) &= -1 \\
\frac{1}{2} - 1 &= C \\
&= -\frac{1}{2} = C \\
y^2 + y - x^3 - x &= -\frac{1}{2} \\
y^2 + 2y + 1 &= 2x^3 + 2x \\
(y + 1)^2 &= 2x^3 + 2x \\
y + 1 &= \pm\sqrt{2x^3 + 2x} \\
y &= \pm\sqrt{2x^3 + 2x} - 1
\end{align*}
\]

\(y = \sqrt{2x^3 + 2x} - 1\) or \(y = -\sqrt{2x^3 + 2x} - 1\)

\(y = -1 \Rightarrow y = -1\) or \(y = 0 \Rightarrow y = 0\)

Both solutions are valid.

\(\Rightarrow\) Existence does not fail, Uniqueness does fail.

\(\Rightarrow\) Existence does not fail, Uniqueness does fail.
(4) Consider the following initial value problem. **Without solving it** answer the following questions:

\[ y' = \frac{(x - 1)^2}{1 + ey}, \quad y(1) = 2. \]

(a) Is there just one solution to this problem? Explain.

\[ y' = f(x, y) = \frac{(x - 1)^2}{1 + ey}, \quad y(1) = 2 \]

\[ \frac{df}{dy} = \frac{(x - 1)^2}{(1 + ey)^2} \quad \frac{dy}{dx} = \frac{e^y(x - 1)^2}{1 + ey} \]

The **existence + uniqueness theorem is not violated**, so uniqueness is guaranteed for all \( y \).

(b) On the basis of the fundamental existence and uniqueness theorem what can you say about the set of values \( x \) for which \( y(x) \) is defined?

\[ y(x) \text{ is defined for all } x \]

(c) Assume that the solution exists for all \( x \). Find all values of \( x \), if any, at which \( y \) has a relative max or min. Which produce a max and which a min? Prove your answer. Remember, you must do this without finding \( y \).

\[ y' = \frac{(x - 1)^2}{1 + ey} = 0 \quad \text{at } x = 1 \text{ or inflection point} \]

So, \( y \) has relative max or min at \( y = 1 \)

\[ x < 1: \quad y' = \frac{(-)}{(+) = (-)} \]

\[ x > 1: \quad y' = \frac{(-)}{(+) = (-)} \]

\( y \) has negative slope before and after \( x = 1 \), so it is **neither a maximum nor a minimum**. There are no points at which \( y \) has a relative max or min.
(5) At time \( t = 0 \), a tank of water contains 400 gal of water with 3 lb of salt dissolved in it. Water containing 10 lb/gal of salt flows into the tank at 30 gal/hr and the well-stirred mixture flows out at the same rate. Pure water also evaporates from the tank at the rate of 0.2 gal/hr. Write a differential equation and initial value condition that could be solved to find a formula for the concentration \( Q(t) \) of salt in the tank. You need not solve the equation.

\[ Q(t) = \text{Concentration of salt in tank at time } t = \frac{1}{2} \text{ lb gal}^{-1} \]

\[ Q(0) = \frac{3}{400} \text{ lb gal}^{-1} \]

Let \( R(t) = \text{lbs of salt in tank at time } t = 1 \text{ lb} \)

\[ R(0) = 3 \]

\[ R'(t) = \text{Rate in - Rate out} \]

\[ R'(t) = (\frac{30 \text{ gal}}{1 \text{ hr}})(\frac{10 \text{ lb}}{1 \text{ gal}}) - (\frac{30 \text{ gal}}{1 \text{ hr}})(\frac{R(t) \text{ lb}}{400 \text{ gal}} \times 0.02 \text{ gal/hr}) \]

\[ R'(t) = 300 - \frac{30R}{400-0.02t} \]

\[ Q(t) = \frac{\frac{R(t)}{1 \text{ gal}}}{\frac{R(t)}{1 \text{ gal}} + \frac{R(t) \text{ lb}}{400 \text{ gal}}} \]

\[ Q(t) = \frac{R(t)}{400-0.02t} \]

\[ Q'(t) = \frac{(400-0.02t)R'(t) - R(t)(-0.02)}{(400-0.02t)^2} \]

\[ Q'(t) = \frac{1}{400-0.02t} \frac{(300-0.02Q)(400-0.02t)}{400-0.02t} \cdot Q(400-0.02t)(-0.02) \]

\[ Q' = \frac{300 - 0.02Q + 0.02Q}{400 - 0.02t} \]

\[ Q(t) = \frac{3}{400} \text{ lb gal}^{-1} \]

\[ Q(0) = \frac{3}{400} \text{ lb gal}^{-1} \]
(6) Assume that an object of mass $m$ kg is thrown straight up from ground level with an initial velocity of 10 m/sec. Assume that air resistance produces a force of magnitude $F_{air} = 0.0001v^3$ where $v$ is the velocity.

(a) Write a differential equation and initial value condition that could be solved to find a formula for the velocity $v(t)$ during the period the object is rising. You need not solve the equation.

\[
F = ma = mg - 0.0001v^3 \\
v' = -g - \frac{0.0001v^3}{m}
\]

\[v(0) = 10\]

(b) Will the differential equation in part (a) change when the object is falling? If so, how? Explain.

No, the equation will not change.

When the object is rising, the force of air resistance should be negative (opposite direction). The velocity will be positive, so $0.0001v^3$ will be positive. Thus, the $0.0001v^3$ must be subtracted to make it a negative force. So $v' = -g - \frac{0.0001v^3}{m}$

When the object is falling, the force of air resistance should be positive (opposite direction). The velocity will be negative, so $0.0001v^3$ will be negative. Thus, the $0.0001v^3$ must be subtracted to make it a positive force $v' = -g - \frac{0.0001v^3}{m}$
(7) Consider the differential equation
\[ y' = t^2 + y^2. \]

(a) Show that the substitution \( v = ty \) converts this equation into an equation equivalent with
\[ tv' = t^2 + v + v^2. \]

\[
\begin{align*}
    v &= ty \\
y' &= \frac{v'}{t^2} + \frac{v'}{t} \\
    \frac{v'}{t} &= y \\
\frac{v'}{t^2} + \frac{v'}{t} &= t^2 + \frac{v'}{t^2} \\
    v + tv' &= t^2 + v^2
\end{align*}
\]

\[ tv' = t^2 + v + v^2 \checkmark \]

(b) Show that the substitution \( y = v^{1/2} \) converts this equation into an equation equivalent with
\[ y' = 2\sqrt{y}(t^2 + v). \]

\[
\begin{align*}
    y &= v^{1/2} \\
y' &= \frac{1}{2\sqrt{v}} v' \\
    \frac{1}{2\sqrt{v}} v' &= t^2 + (v^{1/2})^2 \\
    \frac{1}{2\sqrt{v}} v' &= t^2 + v \\
    v' &= 2\sqrt{v}(t^2 + v) \checkmark
\end{align*}
\]
(8) We are considering a differential equation $y' = f(y)$ where the graph of $f = f(y)$ is as indicated below.

(a) What are the equilibrium solutions?

The equilibrium solutions are $y = -\frac{1}{2}$ and $y = 1$.

(b) Draw an approximate graph showing the direction fields.

(c) Classify each equilibrium solution as either stable, unstable, or semi-stable.

Figure 1. $f(y)$ vs $y$