(1) The following vectors X_1 and Y_1 are eigenvectors for a certain 3×3 matrix A corresponding to the eigenvalues $2 - i$ and -4 respectively. Find the general solution to the system $X' = AX$ **in real form**. No complex numbers allowed!

5 pts.

$$x_1 = \begin{bmatrix} i+1 \\ i \\ -2i \end{bmatrix}, \quad y_1 = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}.$$

$X' = AX$ so:

when $r_1 = -4$: we have form:

$$Y_1(t) = e^{r_1 t} \cdot y_1 = e^{-4t}\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$$

when $r_2 = 2 - i$, we have: $e^{(2-i)t} X_1$

$$e^{(2-i)t} = e^{2t - it} = e^{2t}(\cos(-t) + i(\sin(-t))) = e^{2t}(\cos t - i\sin t)$$

$$\text{So: } e^{(2-i)t} X_1 = e^{2t}(\cos t - i\sin t)\begin{bmatrix} i+1 \\ i \\ -2i \end{bmatrix} = \begin{bmatrix} e^{2t}(\cos t - i\sin t)(i+1) \\ e^{2t}(\cos t - i\sin t)i \\ -e^{2t}(\cos t - i\sin t)(2i) \end{bmatrix}$$

$$\text{since: } (\cos t - i\sin t)(i+1) = i\cos t - \underbrace{i^2}_{-1}\sin t + \cos t - i\sin t$$

$$= (\cos t + \sin t) + i(\cos t - \sin t)$$

$$(\cos t - i\sin t)i = i\cos t - \underbrace{i^2}_{-1}\sin t = \sin t + i\cos t$$

$$\Rightarrow e^{(2-i)t} X_1 = e^{2t}\begin{bmatrix} (\cos t + \sin t) + i(\cos t - \sin t) \\ \sin t + i\cos t \\ -2\sin t + i(-2\cos t) \end{bmatrix} = \underbrace{e^{2t}\begin{bmatrix} \cos t + \sin t \\ \sin t \\ -2\sin t \end{bmatrix}}_{X_1(t)} + i \cdot \underbrace{e^{2t}\begin{bmatrix} \cos t - \sin t \\ \cos t \\ -2\cos t \end{bmatrix}}_{X_2(t)}$$

So general solution:

$$Y = C_1 X_1(t) + C_2 X_2(t) + C_3 Y_1(t) = C_1 e^{2t}\begin{bmatrix} \cos t + \sin t \\ \sin t \\ -2\sin t \end{bmatrix} + C_2 e^{2t}\begin{bmatrix} \cos t - \sin t \\ \cos t \\ -2\cos t \end{bmatrix} + C_3 e^{-4t}\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$$

(2) The characteristic polynomial for the following matrix is $p(r) = -(r-4)^2(r-5)$ and the vector Y_1 is an eigenvector corresponding to $r = 5$. Find the general solution to the system $X' = AX$. 10 pts.

$$A = \begin{bmatrix} 7 & 0 & 1 \\ -4 & 2 & 0 \\ -6 & -3 & 4 \end{bmatrix} \quad Y_1 = \begin{bmatrix} -3 \\ 4 \\ 6 \end{bmatrix} \quad (r=5)$$

$r=4$

let: $B = A - 4I = \begin{bmatrix} 3 & 0 & 1 \\ -4 & -2 & 0 \\ -6 & -3 & 0 \end{bmatrix} \Rightarrow B^2 = \begin{bmatrix} 3 & 0 & 1 \\ -4 & -2 & 0 \\ -6 & -3 & 0 \end{bmatrix}\begin{bmatrix} 3 & 0 & 1 \\ -4 & -2 & 0 \\ -6 & -3 & 0 \end{bmatrix} = \begin{bmatrix} 9-6 & -3 & 3 \\ -12+8 & 4 & -4 \\ -18+12 & 6 & -6 \end{bmatrix} = \begin{bmatrix} 3 & -3 & 3 \\ -4 & 4 & -4 \\ -6 & 6 & -6 \end{bmatrix}$

$0 = B^2 X = \begin{bmatrix} 3 & -3 & 3 \\ -4 & 4 & -4 \\ -6 & 6 & -6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \Rightarrow x - y + z = 0 \Rightarrow y = x + z$ ($x = s$, $z = t$)

$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} s \\ s+t \\ t \end{bmatrix} = s\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ ($\downarrow X_1$, $\downarrow X_2$)

$\lambda = 4$

$X_1(t) = e^{At}X_1 = e^{(B+4I)t}X_1 = e^{4t}(I+Bt)X_1 = e^{4t}(I+Bt)\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$X_2(t) = e^{At}X_2 = e^{(B+4I)t}X_2 = e^{4t}(I+Bt)X_2 = e^{4t}(I+Bt)\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

when: $\lambda = 5$

$Y_1(t) = e^{5t}\begin{bmatrix} -3 \\ 4 \\ 6 \end{bmatrix}$

general solution:

$Y = C_1X_1(t) + C_2X_2(t) + C_3Y_1(t)$

$= C_1e^{4t}(I+Bt)\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + C_2e^{4t}(I+Bt)\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + C_3e^{5t}\begin{bmatrix} -3 \\ 4 \\ 6 \end{bmatrix}$

5 pts.

(3) Given that X_1, X_2 and X_3 are eigenvectors for the following matrix, find the general solution to $X' = AX$. *Hint:* To find the eigenvalue, compute AX_i.

$$A = \begin{bmatrix} 2 & 6 & -3 \\ -2 & -10 & 5 \\ -6 & -12 & 5 \end{bmatrix} \quad X_1 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} \quad X_2 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} \quad X_3 = \begin{bmatrix} -1 \\ 3 \\ 5 \end{bmatrix}$$

$$AX_1 = \begin{bmatrix} 2 & 6 & -3 \\ -2 & -10 & 5 \\ -6 & -12 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} = \begin{bmatrix} 2-6+6 \\ -2+10-10 \\ -6+12-10 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ -4 \end{bmatrix} = 2\begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} = \lambda_1 \overset{X_1}{\begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}} \Rightarrow \lambda_1 = 2$$

$$AX_2 = \begin{bmatrix} 2 & 6 & -3 \\ -2 & -10 & 5 \\ -6 & -12 & 5 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -2+12-6 \\ 2-20+10 \\ 6-24+10 \end{bmatrix} = \begin{bmatrix} 4 \\ -8 \\ -8 \end{bmatrix} = -4\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} = \lambda_2 \overset{X_2}{\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}} \Rightarrow \lambda_2 = -4$$

$$AX_3 = \begin{bmatrix} 2 & 6 & -3 \\ -2 & -10 & 5 \\ -6 & -12 & 5 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} -2+18-15 \\ 2-30+25 \\ 6-36+25 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ -5 \end{bmatrix} = -1\cdot\begin{bmatrix} -1 \\ 3 \\ 5 \end{bmatrix} = \lambda_3 \begin{bmatrix} -1 \\ 3 \\ 5 \end{bmatrix} \Rightarrow \lambda_3 = -1$$

So: $X_1(t) = e^{2t}\begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$, $X_2(t) = e^{-4t}\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$, $X_3(t) = e^{-t}\begin{bmatrix} -1 \\ 3 \\ 5 \end{bmatrix}$

general solution: $y = C_1X_1(t) + C_2X_2(t) + C_3X_3(t)$

$$= C_1e^{2t}\begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} + C_2e^{-4t}\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} + C_3e^{-t}\begin{bmatrix} -1 \\ 3 \\ 5 \end{bmatrix}$$

(4) The following matrix A has characteristic polynomial $p(r) = -(r-5)^3$. 10 pts.

(a) Find e^{tA}.

(b) Find the general solution to $X' = AX$.

$$A = \begin{bmatrix} 5 & 2 & 1 \\ 0 & 5 & 0 \\ 0 & 1 & 5 \end{bmatrix}$$

[a] Since $p(r) = -(r-5)^3 \Rightarrow$ when eigenvalue $\lambda = 5$: if we let $A-5I=B$, then:

$B^3X = 0$.

So we have: $e^{tA} = e^{t(B+5I)}$

$= e^{5t}\left(I + Bt + \frac{B^2t^2}{2!} + \frac{B^3t^3}{3!} + \cdots\right)$ ($B^3 = 0$)

$= e^{5t}\left(I + Bt + \frac{B^2t^2}{2}\right)$

$$B = A-5I = \begin{bmatrix} 5 & 2 & 1 \\ 0 & 5 & 0 \\ 0 & 1 & 5 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \Rightarrow Bt = \begin{bmatrix} 0 & 2t & t \\ 0 & 0 & 0 \\ 0 & t & 0 \end{bmatrix}$$

$$B^2 = \begin{bmatrix} 0 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \frac{B^2t^2}{2} = \begin{bmatrix} 0 & \frac{t^2}{2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow e^{tA} = e^{5t}\begin{bmatrix} 1 & 2t+\frac{t^2}{2} & t \\ 0 & 1 & 0 \\ 0 & t & 1 \end{bmatrix}$$

[b] general solution for $X' = AX$, let $X = \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix}$, we have:

$$e^{tA}X = e^{5t}\begin{bmatrix} 1 & 2t+\frac{t^2}{2} & t \\ 0 & 1 & 0 \\ 0 & t & 1 \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} = e^{5t}C_1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + e^{5t}C_2\begin{bmatrix} 2t+\frac{t^2}{2} \\ 1 \\ t \end{bmatrix} + e^{5t}C_3\begin{bmatrix} t \\ 0 \\ 1 \end{bmatrix}$$

(5) A certain 6×6 matrix A has characteristic polynomial $p(r) = -(r-2)^2(r-5)^4$. Let X be a generalized eigenvector for A corresponding to $r = 5$. Give a formula for $e^{tA}X$ that does not require summing an infinite series. Your formula should use as few matrix products as possible relative to the given information.

5 pts.

when $r=5$: let $A-5I=B$, then since $p(r)=-(r-2)^2(r-5)^4$, we have: $B^4X=0$

So: $e^{tA}X = e^{t(B+5I)}X = e^{5t}\left(I+Bt+\frac{B^2t^2}{2!}+\frac{B^3t^3}{3!}+\frac{B^4t^4}{4!}+\cdots\right)X$

$= e^{5t}\left(I+Bt+\frac{B^2t^2}{2!}+\frac{B^3t^3}{3!}\right)X$

(6) Find all singular points for the following differential equation and state which are regular. **Don't forget to justify your answers!**

6 pts.

$$x^2(2x-5)^3y'' + x^3(2x-5)y' + (x-1)y = 0$$

$$\Rightarrow \quad y'' + \frac{x^3(2x-5)}{x^2(2x-5)^3}y' + \frac{x-1}{x^2(2x-5)^3}y = 0$$

let $x^2(2x-5)^3=0 \Rightarrow x=0$ or $x=\frac{5}{2}$ ← singular points.

$x=0$ $\quad x^2\underbrace{(2x-5)^3}_{p(x)}y'' + x\underbrace{(x^2(2x-5))}_{q(x)}y' + \underbrace{(x-1)}_{r(x)}y=0$

if $x=0$ is a regular singular point, $p(x), q(x), r(x)$ should be analytic and $p(0)\neq 0$, since $p(0)=(2(0)-5)^3 \neq 0$, so $x=0$ is a regular singular point.

$x=\frac{5}{2}$ $\quad \underbrace{(2x-5)^2\left[x^2(2x-5)\right]}_{p(x)}y'' + (2x-5)\cdot\underbrace{x^3}_{q(x)}y' + \underbrace{(x-1)}_{r(x)}y=0$

if $x=\frac{5}{2}$ is a regular singular point, then $p(x), q(x), r(x)$ should all be analytic, and $p(\frac{5}{2})\neq 0$, since $p(x)=x^2(2x-5)$, $p(\frac{5}{2})=(\frac{5}{2})^2(2\cdot\frac{5}{2}-5)=0$

so $x=\frac{5}{2}$ is not a regular singular point

(7) Substitute $y = \sum_{n=0}^{\infty} a_n x^n$ into the differential equation

$$4y'' + (x^2 + 4)y' + y = 0$$

and simplify until you obtain an expression of the form

$$\sum_{n=?}^{\infty} ?x^n + \sum_{n=?}^{\infty} ?x^n + \sum_{n=?}^{\infty} ?x^n = 0$$

where the exponent of x in each sum is n and the question marks are explicit expressions. (You do not need to use exactly 3 summation signs.) **Do not simplify further!** 10 pts.

$\Rightarrow y' = \sum_{n=0}^{\infty} n a_n x^{n-1}$

$x^2 y' = \sum_{n=0}^{\infty} n a_n x^{n+1} = \sum_{n-1=0}^{\infty} (n-1) a_{n-1} x^n = \sum_{n=1}^{\infty} (n-1) a_{n-1} x^n$

$4y' = \sum_{n=0}^{\infty} 4n a_n x^{n-1} = \sum_{n+1=0}^{\infty} 4(n+1) a_{n+1} x^n = \sum_{n=0}^{\infty} 4(n+1) a_{n+1} x^n$

$\Rightarrow y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n+2=0}^{\infty} (n+2)(n+1) a_{n+2} x^n = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$

(since when $n=-2$, $n=-1$, $\sum (n+2)(n+1) a_{n+2} x^n = 0$)

$\Rightarrow \underbrace{\sum_{n=0}^{\infty} 4(n+2)(n+1) a_{n+2} x^n}_{4y''} + \underbrace{\sum_{n=1}^{\infty} (n-1) a_{n-1} x^n}_{x^2 y'} + \underbrace{\sum_{n=0}^{\infty} 4(n+1) a_{n+1} x^n}_{4y'} + \underbrace{\sum_{n=0}^{\infty} a_n x^n}_{y} = 0$

10 pts. (8) In attempting to solve a certain differential equation, we substituted $y = \sum_{n=0}^{\infty} a_n x^n$ into the differential equation and simplified, obtaining

$$\sum_{n=0}^{\infty} 3n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} -3n(n-1)a_n x^n + \sum_{n=0}^{\infty} -3na_n x^n + \sum_{n=0}^{\infty} -a_n x^n = 0.$$

(a) Continue the solution process to obtain the recursion relation.

$$\sum_{n=0}^{\infty} 3n(n-1)a_n x^{n-2} = \sum_{n+2=0}^{\infty} 3(n+2)(n+1)a_{n+2}x^n = 3(-2+2)(-2+1)a_0x^{-2} + 3(-1+2)(-1+1)a_1x^{-1} + \sum_{n=0}^{\infty} 3(n+2)(n+1)a_{n+2}$$

$$\Rightarrow \sum_{n=0}^{\infty} \left[3(n+2)(n+1)a_{n+2} - 3n(n-1)a_n - 3na_n - a_n\right]x^n = 0$$

$$3(n+2)(n+1)a_{n+2} = \left[3n(n-1)+3n+1\right]a_n = (3n^2-3n+3n+1)a_n = (3n^2+1)a_n$$

$$\Rightarrow a_{n+2} = \frac{3n^2+1}{3(n+2)(n+1)}a_n$$

(b) Find the first three **non-zero** terms of the power series expansion for the solution y_1 satisfying $y_1(0) = 0, y_1'(0) = 2$.

$y_1 = a_0 + a_1x^1 + a_2x^2 + \cdots \Rightarrow y_1(0) = a_0 = 0$

$y_1' = a_1 + 2a_2x + 3a_3x^2 + \cdots \Rightarrow y_1'(0) = \boxed{a_1 = 2}$

$n=0$: $a_2 = \frac{1}{3(2)(1)}a_0 = 0$

$n=1$: $a_3 = \frac{4}{3(3)(2)}a_1 = \frac{4\times 2}{3\times 3\times 2} = \boxed{\frac{4}{9}}$

$n=2$: $a_4 = \frac{3(2)^2+1}{3(4)(3)}a_2 = 0$

$n=3$: $a_5 = \frac{3(3^2)+1}{3(5)(4)}a_3 = \frac{7}{15}a_3 = \frac{7}{15}\cdot\frac{4}{9} = \boxed{\frac{28}{135}} \doteq 0.207407407$

(9) The following differential equation has a regular singularity at $x = 0$. 9 pts.

$$x^2(x^3 + 2x + 1)y'' + x(4x^3 + x + 6)y' + (x + 6)y = 0.$$

Since $x=0$ is the regular singular pt

(a) Give the approximating Euler equation.

Compare to $x^2 \cdot p(x)\, y'' + x \cdot q(x)\, y' + r(x)\, y = 0$

we have: $p(x) = x^3+2x+1$, $q(x) = 4x^3+x+6$, $r(x) = x+6$

$\Rightarrow$ $p(0) = 1$, $q(0) = 6$, $r(0) = 6$

Approximating Euler Equation has form: $x^2 \cdot p(0)\, y'' + x\, q(0)\, y' + r(0)\, y = 0$

$\Rightarrow$ $x^2y'' + 6xy' + 6y = 0$

(b) Give the indicial equation.

Let $y = x^r$, then we have: $r(r-1)x^{r-2} \cdot x^2 + 6r x^{r-1} \cdot x + 6x^r = 0$

$(r^2 - r + 6r + 6)\, x^r = 0$

$(r^2 + 5r + 6)\, x^r = 0$

$\Rightarrow r^2 + 5r + 6 = 0$ ← indicial equation

$(r+2)(r+3) = 0 \Rightarrow r_1 = -2,\ r_2 = -3$

(c) Use Theorem 5.6.1 on p. 289 of the text to describe the expected form of the solutions. **Do not find the coefficients of the series expansions!**

$r_1 = -2,\ r_2 = -3 \Rightarrow$ $r_1 > r_2$ ($r_1 = r_2 + 1$)

$$\begin{cases} y_1(t) = x^{r_1} \sum_{n=0}^{\infty} a_n x^n = x^{-2} \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n-2} \\ y_2(t) = a \ln x \cdot y_1(t) + x^{r_2} \sum_{n=0}^{\infty} b_n x^n = a(\ln x) \sum_{n=0}^{\infty} a_n x^{n-2} + x^{-3} \sum_{n=0}^{\infty} b_n x^n \end{cases}$$

$$= a(\ln x) \sum_{n=0}^{\infty} a_n x^{n-2} + \sum_{n=0}^{\infty} b_n x^{n-3}$$

(a could be 0 or 1)

(10) You are given that $y(x) = x^2$ is a solution of the following differential equation. Use the method of reduction of order to find a second independent solution. **Other methods will not receive credit!**

6 pts.

$$x^2y'' - 3xy' + 4y = 0.$$

Since $y_1(x) = x^2$

$$\Rightarrow \begin{cases} y_2 = u \cdot y_1 \\ u = \int v \\ v = y_1^{-2} \cdot e^{-\int p} \end{cases}$$

$$x^2y'' - 3xy' + 4y = 0 \Rightarrow y'' - \frac{3x}{x^2}y' + \frac{4}{x^2}y = 0$$

$$\Rightarrow y'' \underbrace{- \frac{3}{x}}_{p} y' + \underbrace{\frac{4}{x^2}}_{q} y = 0$$

So: $-\int p = -\int -\frac{3}{x} = \int \frac{3}{x} = 3\ln|x|$

$$e^{-\int p} = e^{3\ln|x|} = x^3$$

$$\Rightarrow v = y_1^{-2} e^{-\int p} = (x^2)^{-2} x^3 = x^{-4} \cdot x^3 = x^{-1}$$

$$\Rightarrow u = \int v = \int x^{-1} = \int \frac{1}{x} = \ln|x|$$

$$y_2 = u \cdot y_1 = \ln|x| \cdot x^2$$

(11) *Use the definition of the Laplace transform* to compute the Laplace transform $\mathcal{L}(f)$ of the function $f(t)$ defined below. **Other methods will not receive credit!** 4 pts.

$$f(t) = \begin{cases} e^{2t}, & 0 \le t < 3 \\ 4, & 3 \le t. \end{cases}$$

$$L(f) = \int_0^3 e^{2t} e^{-st}\, dt + \int_3^\infty 4e^{-st}\, dt$$

$$= \int_0^3 e^{(2-s)t}\, dt + 4\int_3^\infty e^{-st}\, dt$$

$$= \frac{1}{2-s} e^{(2-s)t}\Big|_0^3 + \frac{4}{-s} e^{-st}\Big|_3^\infty$$

$$= \frac{1}{2-s}\left(e^{3(2-s)} - e^0\right) + \frac{4}{-s}\left(e^{-\infty} - e^{-3s}\right)$$

$$= \frac{e^{3(2-s)} - 1}{2-s} + \frac{4}{s}\left(e^{-3s}\right)$$

4 pts. (12) Find the inverse Laplace transform of

$$F(s) = \frac{2s^2+4}{s(s^2+4)}. = \frac{A}{S} + \frac{BS+C}{S^2+4}$$

$\Rightarrow \quad 2S^2+4 = A(S^2+4) + (BS+C)S = AS^2 + 4A + BS^2 + CS = (A+B)S^2 + CS + 4A$

$$\begin{cases} 4 = 4A \\ 2 = A+B \\ 0 = C \end{cases} \Rightarrow \begin{cases} A=1 \\ B=1 \end{cases} \Rightarrow F(s) = \frac{1}{S} + \frac{S}{S^2+4} \quad \left(\frac{S}{S^2+a^2} \rightarrow \cos at,\ a=0\right)$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$
$$f(t) \qquad 1 \qquad \cos 2t$$

So: $f(t) = L^{-1}(F(s)) = 1 + \cos 2t$

4 pts. (13) Find the inverse Laplace transform of

$U_5(t)$

$$F(s) = \frac{e^{-5s}(2s^2+4)}{s(s^2+4)}.$$

Since $\frac{2S^2+4}{S(S^2+4)} \Rightarrow 1 + \cos 2t$

So: $f(t) = L^{-1}\left(\frac{e^{-5s}(2S^2+4)}{S(S^2+4)}\right) = U_5(t)\left(1 + \cos[2(t-5)]\right)$

(14) Find the inverse Laplace transform of 4 pts.

$$F(s) = \frac{e^{2s}}{(s+2)^5}.$$

$(c=-2)$ $u_{-2}(t)$

mistake in problem e^{-2s}

Since: $L(t^n e^{at}) = \frac{n!}{(s-a)^{n+1}}$

So: for $F(s) = \frac{1}{(s+2)^5} = \frac{4!}{(s-(-2))^{4+1}} \cdot \frac{1}{4!}$

we know $a=-2$, $n=4$, So:

$f(t) = \frac{1}{4!}(t^4 e^{-2t})$

but we still have a term: e^{2s} in $F(s)$, So:

$L^{-1}(F(s)) = f(t) = \frac{1}{4!} u_{-2}(t) \cdot (t+2)^4 \cdot e^{-2(t+2)}$

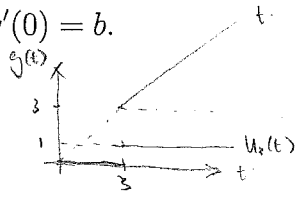
(15) Find the Laplace transform $Y(s)$ of the solution $y(t)$ to the following initial value problem in terms of a and b. **Do not find $y(t)$. All we want is $Y(s)$!**

8 pts.

$$y'' + 2y' - 4y = g(t), \quad y(0) = a,\ y'(0) = b.$$

where

$$g(t) = \begin{cases} 0, & 0 \le t < 3 \\ t, & 3 \le t \end{cases}$$



$\Rightarrow g(t) = u_3(t) \cdot t$

$Y'' = s^2 Y(s) - sy(0) - y'(0) = s^2 Y(s) - as - b$

$Y' = sY(s) - y(0) = sY(s) - a$

$Y = Y(s)$

$\Rightarrow y'' + 2y' - 4y = s^2 Y(s) - as - b + 2sY(s) - 2a - 4Y(s)$

$= (s^2 + 2s - 4)Y(s) - as - 2a - b$

$g(t) = u_3(t) \cdot t = (t-3)u_3(t) + 3u_3(t)$ ✓

$G(s) = \mathcal{L}(g(t)) = \mathcal{L}((t-3)u_3(t)) + \mathcal{L}(3u_3(t))$

$= e^{-3s} \cdot \frac{1}{s^2} + 3 \cdot \frac{e^{-3s}}{s} = \frac{e^{-3s}}{s^2} + \frac{3e^{-3s}}{s}$ ✓

$\Rightarrow (s^2 + 2s - 4)Y(s) - as - 2a - b = g(s) = \frac{e^{-3s}}{s^2} + \frac{3e^{-3s}}{s}$

$\Rightarrow Y(s) = \frac{e^{-3s} + 3se^{-3s}}{s^2(s^2 + 2s - 4)} + \frac{as + 2a + b}{s^2 + 2s - 4}$