

**You should bring a calculator to the test to be able to do problems such as Problem 12**

- (1) The following questions refer to Figure 5 on p. 103 of the notes. Assume that it is given that the graph represents a function  $y = f(x)$  which is increasing on  $[0, 2.5]$  and decreasing on  $[2.5, \infty)$ . Let  $a_n = f(n)$  and  $s = \sum_1^\infty a_n$ .

- (a) Find a specific value of  $n$ ,  $a$  and  $b$  such that the following inequality is guaranteed to hold. Choose both  $n$  and  $b$  as large as possible and  $a$  as small as possible, consistent with the information provided. Justify your answer with a diagram.

$$a_1 + a_2 + \cdots + a_n > \int_a^b f(x) dx$$

**Solution:** Interpret  $a_1$  and  $a_2$  as the areas of rectangles of width one having their right edges extending from the  $x$ -axis to the graph of  $y = f(x)$  at  $x = 1$  and  $x = 2$  respectively. Since  $f$  increases over  $[0, 2]$  the top of these rectangles are above the curve. Hence, we may choose  $a = 0$ ,  $b = 2$ , and  $n = 2$ .

- (b) Find a specific value of  $n$ ,  $a$  and  $b$  such that the following inequality is guaranteed to hold. Choose both  $n$  and  $a$  as large as possible and  $b$  as small as possible, consistent with the information provided. Justify your answer with a diagram.

$$a_1 + a_2 + \cdots + a_n < \int_a^b f(x) dx$$

**Solution:** Interpret  $a_1$  as the area of a rectangle of width one having its left edge extending from the  $x$ -axis to the graph of  $y = f(x)$  at  $x = 1$ . Since  $f$  increases over  $[1, 2]$  the top of this rectangle is below the curve over  $[1, 2]$ . Hence, we may choose  $a = 1$ ,  $b = 2$ , and  $n = 1$ .

REMARK. The graph seems to indicate that the rectangle of width one having its left edge extending from the

$x$ -axis to the graph of  $y = f(x)$  at  $x = 2$  lies below the curve. However, there is nothing in the given that *guarantees* that this is true. Since  $f$  is decreasing over  $[2.5, 3]$  there is no way of telling whether the right edge of this rectangle is really below the curve.

- (2) State the “official” definition of “ $\lim_{x \rightarrow a} f(x) = L$ .”

**Solution:** See p. 143 of the notes.

- (3) Prove, using well labeled diagrams, the following version of Theorem 2 in the text. Be careful to bring in the role of the Bounded Increasing Theorem and the decreasing nature of  $f$  into your proof. (The proof is given at the end of this review sheet.)

**THEOREM (2’).** *Suppose  $a_n > 0$  for all  $n$  and  $f(x)$  is an integrable, decreasing function on  $[0, \infty)$  such that  $a_n = f(n)$  for all  $n \in \mathbb{N}$ . Then  $s = \sum_1^\infty a_n$  exists if*

$$\int_0^\infty f(x) dx < \infty$$

- (4) Prove, using well labeled diagrams, the following version of Theorem 4 in the text. Be careful to bring in the role of the decreasing nature of  $f$  into your proof. (The proof is given at the end of this review sheet.)

**THEOREM (4’).** *Suppose  $a_n > 0$  for all  $n$  and  $f(x)$  is an integrable, decreasing function on  $[0, \infty)$  such that  $a_n = f(n)$  for all  $n \in \mathbb{N}$ . Then*

$$s_n \geq \int_1^{n+1} f(x) dx$$

- (5) **Use the integral test** to prove that the following series converges. Then write a sum that expresses  $s$  to within  $\pm 10^{-5}$ . *Hint:* To evaluate the integral, make the substitution  $u = x^4 + 5$ .

$$s = \sum_1^\infty \frac{4n^3}{(n^4 + 5)^2}$$

According to Theorem 2, the sum will converge if  $\int_1^\infty \frac{4x^3}{(x^4+5)^2} dx < \infty$ . Letting  $u = x^4 + 5$ ,  $du = 4x^3 dx$

$$\begin{aligned} \int_1^\infty \frac{4x^3}{(x^4+5)^2} dx &= \int_6^\infty u^{-2} du \\ &= \frac{1}{6} < \infty \end{aligned}$$

proving convergence. *It is required that you name the theorem being used. A list of theorems will be provided. A copy of this list is on the course web page.*

According to Theorem 1,

$$\begin{aligned} s - s_n &< \int_n^\infty \frac{4x^3}{(x^4+5)^2} dx \\ &= \int_{n^4+5}^\infty u^{-2} du \\ &= \frac{1}{n^4+5} \end{aligned}$$

This will be less than  $10^{-5}$  provided

$$\frac{1}{n^4+5} < 10^{-5}$$

$$n^4+5 > 10^5$$

$$n > (10^5 - 5)^{1/4} = 17.78257181$$

Hence

$$s = \sum_1^{18} \frac{4n^3}{(n^4+5)^2} \pm 10^{-5}.$$

(6) Prove, using  $M$ , that the following series diverges.

$$\sum_1^\infty \frac{1}{n^{1/3}}$$

**Scratch work:** According to Theorem 2'

$$\begin{aligned} s_n &\geq \int_1^{n+1} x^{-1/3} dx \\ &= \frac{3}{2} ((1+n)^{2/3} - 1) \end{aligned}$$

Then  $s_n$  is greater than  $M$  if:

$$\begin{aligned}\frac{3}{2}((1+n)^{2/3} - 1) &> M \\ (1+n)^{2/3} &> \frac{2}{3}M + 1 \\ n &> \left(\frac{2}{3}M + 1\right)^{3/2} - 1\end{aligned}$$

**Proof:** Let  $M > 0$  be given. Let  $N = \left(\frac{2}{3}M + 1\right)^{3/2} - 1$ . From the scratch work, for  $n > N$ ,  $s_n > M$ , proving that  $\lim_{n \rightarrow \infty} s_n = \infty$ .

- (7) Classify each of the following series as either (a) conditionally convergent or (b) absolutely convergent or (c) divergent. Prove your answers.

$$(a) \quad \sum_1^{\infty} \frac{(-1)^n(\sqrt{3n+2})}{n^3 + 5n + 5}$$

$$(b) \quad \sum_1^{\infty} \frac{(-1)^n 7^n}{(7.1)^n + n}$$

$$(c) \quad \sum_1^{\infty} \frac{(-1)^n n^{27^n}}{8^n + n}$$

$$(d) \quad \sum_1^{\infty} \frac{(-1)^n n^2}{\sqrt{n^4 + 3}}$$

$$(e) \quad \sum_1^{\infty} \frac{(-1)^n(3n^2 + 1)}{\sqrt{n^5 + 3}}$$

### Solutions

- (a) Absolutely convergent.

$$\begin{aligned}|a_n| &= \frac{\sqrt{3n+2}}{n^3 + 5n + 5} \\ &\leq \frac{\sqrt{3n+2n}}{n^3} \\ &= \frac{\sqrt{5}}{n^{2.5}}\end{aligned}$$

Then

$$\sum_1^{\infty} \frac{\sqrt{3n+2}}{n^3+5n+5} < \sum_1^{\infty} \frac{\sqrt{5}}{n^{2.5}} < \infty$$

since  $\sum \frac{1}{n^p} < \infty$  for  $p > 1$ , proving absolute convergence.

(b) Absolutely convergent.

$$\begin{aligned} |a_n| &= \frac{7^n}{(7.1)^n + n} \\ &\leq \frac{7^n}{(7.1)^n} \\ &= \left(\frac{7}{7.1}\right)^n \end{aligned}$$

Then

$$\sum_1^{\infty} \frac{7^n}{(7.1)^n + n} < \sum_1^{\infty} \left(\frac{7}{7.1}\right)^n < \infty$$

proving absolute convergence.

(c) Absolutely convergent.

$$|a_n| = \frac{n^2 7^n}{8^n + n} \leq \frac{n^2 7^n}{8^n}$$

There is an  $N > 0$  such that for all  $n > N$ ,

$$n^2 7^n < (7.5)^n.$$

Then

$$\sum_{N+1}^{\infty} \frac{n^2 7^n}{8^n + n} < \sum_{N+1}^{\infty} \left(\frac{7.5}{8}\right)^n < \infty$$

since  $\sum x^n < \infty$  for  $|x| < 1$ . It follows that

$$\sum_1^{\infty} \frac{n^2 7^n}{8^n + n} < \infty$$

proving absolute convergence.

(d) Divergent:

$$\lim_{n \rightarrow \infty} \frac{n^2}{\sqrt{n^4 + 3}} = 1.$$

Hence

$$\lim_{n \rightarrow \infty} \frac{(-1)^n n^2}{\sqrt{n^4 + 3}}$$

does not exist. Hence the sum cannot converge since if  $\sum a_n$  converges, then  $\lim_{n \rightarrow \infty} a_n = 0$

(e) Conditionally convergent:

$$\begin{aligned} |a_n| &= \frac{3n^2 + 1}{\sqrt{n^5 + 3}} \\ &\geq \frac{3n^2}{\sqrt{n^5 + 3n^5}} \\ &= \frac{3}{2} \frac{1}{n^{0.5}} \end{aligned}$$

Hence

$$\sum_1^{\infty} |a_n| \geq \frac{3}{2} \sum_1^{\infty} \frac{1}{n^{0.5}} = \infty$$

since  $\sum \frac{1}{n^p} = \infty$  for  $p \leq 1$ , proving absolute divergence.

However, this series converges by the alternating series test since

$$\lim_{n \rightarrow \infty} \frac{3n^2 + 1}{\sqrt{n^5 + 3}} = 0.$$

- (8) For each of the series  $\sum_1^{\infty} a_n$  in (a)-(e) above, what is the set of  $x$  for which  $\sum_1^{\infty} a_n x^n$  converges? (No proof is required, although some explanation should be provided.)

**Solution:**

(a) Answer:  $-1 \leq x \leq 1$ .

Explanation:  $|a_n x^n|$  grows like a multiple of  $|x|^n / n^{2.5}$ . If  $|x| \leq 1$  the sum converges because

$$|x|^n / n^{2.5} \leq 1 / n^{2.5}.$$

If  $|x| > 1$ , the sum diverges because  $\lim_{n \rightarrow \infty} |x|^n / n^{2.5} = \infty$ .

(b) Answer:  $-\frac{7.1}{7} < x < \frac{7.1}{7}$ .

Explanation:  $|a_n x^n|$  grows like a multiple of

$$\left(\frac{7}{7.1}\right)^n |x|^n = \left(\frac{7|x|}{7.1}\right)^n$$

which converges if  $\frac{7|x|}{7.1} < 1$  and diverges if  $\frac{7|x|}{7.1} \geq 1$  since in this case,  $\lim_{n \rightarrow \infty} a_n x^n \neq 0$ .

(c) Answer:  $-\frac{8}{7} < x < \frac{8}{7}$ .

Explanation: Similar to (b)

(d) Answer:  $-1 < x < 1$ .

Explanation: Similar to (a) except now  $|a_n x^n|$  grows like  $|x|^n$ .

(e) Answer:  $-1 < x \leq 1$ .

Explanation: Similar to (a) except now  $|a_n x^n|$  grows like  $|x|^n/n^5$ . We get convergence (but not absolute convergence) at  $x = 1$  as explained in the solution to problem 7.

(9) Prove that the following series converges.

$$\sum_1^{\infty} \frac{n(\ln n)^2}{n^3 + 3}$$

**Solution:** There is an  $N$  such that for  $n > N$ ,  $\ln n < n^{1/4}$ . Hence,

$$\begin{aligned} \sum_{N+1}^{\infty} \frac{n(\ln n)^2}{n^3 + 3} &< \sum_{N+1}^{\infty} \frac{n(n^{1/2})}{n^3} \\ &= \sum_{N+1}^{\infty} \frac{1}{n^{1.5}} < \infty \end{aligned}$$

Thus

$$\sum_1^{\infty} \frac{n(\ln n)^2}{n^3 + 3} < \infty$$

since this sum differs from the above sum only by a finite number of terms.

(10) Let

$$s = \sum_1^{\infty} \frac{n^2 + \sqrt{\ln n + 1}}{n^4 + 3n + 7}$$

(a) Prove that this series converges.

(b) Write a sum which computes  $s$  to within  $\pm 10^{-3}$ .

**Solution:**

(a) For  $n \geq 1$

$$\begin{aligned} \frac{n^2 + \sqrt{\ln n + 1}}{n^4 + 3n + 7} &\leq \frac{n^2 + \sqrt{n^4 + n^4}}{n^4} \\ &\leq \frac{n^2 + \sqrt{2}n^2}{n^4} \\ &\leq \frac{3}{n^2} \end{aligned}$$

Thus, the series converges since  $\sum_1^\infty \frac{3}{n^2} < \infty$ .

(b) The number of terms it takes for  $t = \sum_1^\infty \frac{3}{n^2}$  to approximate its sum to within  $\pm 10^{-3}$  will also work for our series. To find this we compute

$$t - t_n < \int_n^\infty \frac{3}{x^2} dx = \frac{3}{n} < 10^{-3}.$$

This is true if  $n > 3000$  Hence

$$s = \sum_1^{3001} \frac{n^2 + \sqrt{\ln n + 1}}{n^4 + 3n + 7} \pm 10^{-3}.$$

(11) Prove that  $Z = 1/(3\pi + 5)$  is irrational. You may assume that  $\pi$  is irrational. You MAY NOT use Proposition 1 from Chapter 9.

**Solution:** Assume that  $Z$  is rational. Then  $Z = \frac{p}{q}$  where  $p$  and  $q$  are integers with  $q \neq 0$ . It is also true that  $p \neq 0$  since  $Z \neq 0$ . Then

$$\begin{aligned} \frac{p}{q} &= \frac{1}{3\pi + 5} \\ \frac{q}{p} &= 3\pi + 5 \\ \frac{q - 5p}{p} &= 3\pi \\ \frac{q - 5p}{3p} &= \pi \end{aligned}$$

Since  $q - 5p$  and  $3p$  are both integers, we conclude that  $\pi$  is rational, which is nonsense. Hence  $Z$  must be irrational.

(12) Find

- (a) an *explicit* irrational number  $Z$  satisfying  $17/13 < Z < 18/13$ . You need not prove that  $Z$  is irrational.

**Solution:**

- (a)  $18/13 - 17/13 = 1/13$ . Choose  $n$  so that

$$\begin{aligned}\frac{\sqrt{2}}{n} &< \frac{1}{13} \\ \frac{n}{\sqrt{2}} &> 13 \\ n &> 13\sqrt{2}\end{aligned}$$

For example, choose  $n = 26$ . Then  $\frac{17}{13} + \frac{\sqrt{2}}{26}$  works.

- (b) an *explicit* rational number  $Z$  satisfying  $\pi < Z < 22/7$ .  
 (b)  $22/7 = 3.14285\dots$  and  $\pi = 3.14159\dots$ . Hence  $Z = 3.143$  works.
- (13) Find an explicit one-to-one correspondence between the sets  $A$  and  $B$  where:
- (a)  $A = (-1, 3)$  and  $B = (0, 1)$ . *Prove your answer.*
- (b)  $A$  is the set of even natural numbers and  $B$  is the set of odd natural numbers.
- (c)  $A$  is the set of natural numbers which are multiples of 2 and  $B$  is the set of natural numbers which are multiples of 3.

**Solutions:**

(a) Let  $f(x) = ax + b$ . Choose  $a$  and  $b$  so that  $f(-1) = 0$ ; hence  $a = b$ . Also we want  $f(3) = 1$ ; hence  $a = b = \frac{1}{4}$ . So  $f(x) = \frac{1}{4}(x + 1)$ .

To prove our answer we must prove that (a)  $f$  maps  $(-1, 3)$  into  $(0, 1)$ , (b)  $f$  maps  $(-1, 3)$  onto  $(0, 1)$ , and (c)  $f$  is one-to-one.

**Into:** Assume  $-1 < x < 3$ . Then

$$\begin{aligned}0 &< x + 1 < 4 \\ 0 &< \frac{1}{4}(x + 1) < 1\end{aligned}$$

Hence  $f$  maps  $(-1, 3)$  into  $(0, 1)$ .

**Onto:**

Note that if  $y = 14(x + 1)$  then  $x = 4y - 1$ . Assume that  $0 < y < 1$ . Then

$$\begin{aligned} 0 &< 4y < 4 \\ -1 &< 4y - 1 < 3 \\ -1 &< x < 3 \end{aligned}$$

Hence,  $y = f(x)$  for some  $x \in (-1, 3)$ .

**One-to-one:** Suppose that  $f(x_1) = f(x_2)$ . Then

$$\begin{aligned} \frac{1}{4}(x_1 + 1) &= \frac{1}{4}(x_2 + 1) \\ x_1 + 1 &= x_2 + 1 \\ x_1 &= x_2 \end{aligned}$$

showing one-to-one.

(b) If  $n$  is an even number, then  $n - 1$  is odd. Hence let  $f(n) = n - 1$ .

(c) Let  $f(n) = \frac{3}{2}n$ .

- (14) (a) Demonstrate a way of listing the set all of the rational numbers in the interval  $(0, 1)$ . How does this prove that the set in question is countable.
- (b) Demonstrate a way of listing the set all of integers. How does this prove that the set in question is countable.
- (c) Show that it is impossible to list all numbers in the interval  $(0, 1)$ . How does this prove that the set in question is uncountable.

**Solutions:**

The solution to (a) is on p. 130 of the notes. This list defines a one-to-one correspondence between the set of rational numbers in  $(0, 1)$  and the set of natural numbers, proving the countability.

For (b), one possible list is:

$$\begin{array}{cccccccc} 0 & 1 & -1 & 2 & -2 & 3 & -3 & \dots \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & \dots \end{array}$$

This list defines a one-to-one correspondence between the set of integers and the set of natural numbers, proving the countability.

For (c), the proof is found on p. 128 of the notes, below formula (2). The material on p. 129 is not part of this proof. The inability to list the set of all numbers in  $(0, 1)$  shows that this set cannot be put into a one-to-one correspondence with the set of natural numbers, proving the uncountability.

### Various Results from the Text

**THEOREM (2').** *Suppose  $a_n > 0$  for all  $n$  and  $f(x)$  is an integrable, decreasing function on  $[0, \infty)$  such that  $a_n = f(n)$  for all  $n \in \mathbb{N}$ . Then  $s = \sum_1^\infty a_n$  exists if*

$$\int_0^\infty f(x) dx < \infty$$

*Proof* Each  $a_n$  is the length of a line segment drawn from the point  $(n, 0)$  on the  $x$ -axis to the graph of  $y = f(x)$  as in Figure 1.

FIGURE 1. Theorems 2' and 4'

The area of a rectangle of width one having this line segment as its right edge is  $a_n$ . (See Figure 2). This rectangle also lies entirely below the graph of  $y = f(x)$  since this graph is decreasing.

Since the left side of the first rectangle extends to  $x = 0$ ,

$$(1) \quad s_n = a_1 + a_2 + \cdots + a_n \leq \int_0^n f(x) dx \leq \int_0^\infty f(x) dx.$$

Finally, since the  $a_n$  are all positive,  $s_n$  is an increasing sequence. From the Bounded Increasing Theorem,  $\lim s_n$  either exists or equals  $\infty$ . Formula (1) proves that the limit is not  $\infty$ . Hence the limit exists, proving the convergence of the sum.  $\square$

FIGURE 2. Theorem 2'

THEOREM (4'). Suppose  $a_n > 0$  for all  $n$  and  $f(x)$  is an integrable, decreasing function on  $[0, \infty)$  such that  $a_n = f(n)$  for all  $n \in \mathbb{N}$ . Then

$$s_n \geq \int_1^{n+1} f(x) dx$$

*Proof* Each  $a_n$  is the length of a line segment drawn from the point  $(n, 0)$  on the  $x$ -axis to the graph of  $y = f(x)$  as in Figure 1. The area of a rectangle of width one having this line segment as its left edge is  $a_n$ . (See Figure 3). This rectangle also lies entirely above the graph of  $y = f(x)$  since this graph is decreasing.

FIGURE 3. Theorem 4'

Since the right side of the  $n$ th rectangle extends to  $x = n + 1$ ,

$$(2) \quad s_n = a_1 + a_2 + \cdots + a_n \geq \int_1^{n+1} f(x) dx.$$

proving Theorem 4'. □