(1) State the “official” definition of “\( \lim_{x \to a} f(x) = L \).”

(2) Suppose that \( f(x) \) and \( g(x) \) are both continuous at \( x = a \). Prove that \( h(x) = f(x)g(x) \) is also continuous at \( x = a \). You may use the product theorem for limits of functions from Chapter 10.

**Solution:**
\[
\lim_{x \to a} h(x) = \lim_{x \to a} (f(x)g(x)) \\
= \left( \lim_{x \to a} f(x) \right) \left( \lim_{x \to a} g(x) \right) \text{ Product Theorem} \\
= f(a)g(a) = h(a) \text{ Continuity of } f \text{ and } g
\]

Since \( \lim_{x \to a} h(x) = h(a) \), \( h(x) \) is continuous at \( x = a \).

(3) Use a \( \delta \)-\( \epsilon \) argument to prove the following limit statements:

(a) \( \lim_{x \to 2} \frac{3x}{x+1} = 2 \)

(b) \( \lim_{x \to 2} \frac{1}{x^2} = \frac{1}{4} \)

(c) \( \lim_{x \to 1} (x^2 + 3) = 4 \)

(d) \( \lim_{x \to 1} \frac{1}{x^2 + 3} = \frac{1}{4} \)

(e) \( \lim_{x \to 0} \frac{1}{1-x} = 1 \)

(f) \( \lim_{x \to 1} \frac{1}{\sqrt{3} + x} = \frac{1}{2} \)

**Solutions:**

(a):

**Scratch Work:** Let \( \epsilon > 0 \) be given. We want
\[
\left| \frac{3x}{x+1} - 2 \right| < \epsilon \\
\left| \frac{x-2}{x+1} \right| = \frac{1}{|x+1|} < \epsilon
\]
Assume that $x = 2 \pm 1$ so that $1 < x < 3$ and $2 < x + 1 < 3$. Then

\[ |x - 2| \left| \frac{1}{x + 1} \right| < \frac{1}{2} |x - 2| \]

This will be $< \epsilon$ if $|x - 2| < 2\epsilon$.

Our Proof: Let $\epsilon > 0$ be given and let $\delta = \min\{1, 2\epsilon\}$. Assume that $0 < |x - 2| < \delta$. Then from the scratch work

\[ \left| \frac{3x}{x + 1} - 2 \right| < \epsilon, \]

proving the limit statement.

(b):

Scratch Work: Let $\epsilon > 0$ be given. We want

\[ \left| \frac{1}{x^2} - \frac{1}{4} \right| = \epsilon \]
\[ \left| x^2 - 4 \right| = \left| x - 2 \right| \left| x + 2 \right| < \epsilon \]

Assume that $x = 2 \pm 1$. Then

\[ 1 < x < 3 \]
\[ 1 < x^2 < 9 \]
\[ 4 < 4x^2 < 36 \]
\[ 3 < x + 2 < 5 \]

Then

\[ |x - 2| \left| \frac{x + 2}{4x^2} \right| < \frac{5}{4} |x - 2| \]

This will be $< \epsilon$ if $|x - 2| < \frac{4}{5} \epsilon$.

Our Proof: Let $\epsilon > 0$ be given and let $\delta = \min\{1, \frac{1}{5} \epsilon\}$. Assume that $0 < |x - 2| < \delta$. Then from the scratch work

\[ \left| \frac{1}{x^2} - \frac{1}{4} \right| < \epsilon, \]

proving the limit statement.
(c):

**Scratch Work:** Let $\epsilon > 0$ be given. We want

$$\begin{align*}
| (x^2 + 3) - 4 | &< \epsilon \\
| (x - 1)(x + 1) | & = |x - 1||x + 1| < \epsilon
\end{align*}$$

Assume that $x = 1 \pm 1$ so that $0 < x < 2$. Then

$$1 < x + 1 < 3$$

Then

$$|x - 1||x + 1| < 3|x - 1|$$

This will be $< \epsilon$ if $|x - 1| < \frac{1}{3}\epsilon$.

**Our Proof:** Let $\epsilon > 0$ be given and let $\delta = \min\{1, \frac{1}{3}\epsilon\}$. Assume that $0 < |x - 1| < \delta$. Then from the scratch work

$$|(x^2 + 3) - 4| < \epsilon,$$

proving the limit statement.

(d):

**Scratch Work:** Let $\epsilon > 0$ be given. We want

$$\begin{align*}
| \frac{1}{x^2 + 3} - \frac{1}{4} | &< \epsilon \\
| \frac{1 - x^2}{4(x^2 + 3)} | & = |x - 1| \frac{|1 + x|}{4(x^2 + 3)} < \epsilon
\end{align*}$$

Assume that $x = 1 \pm 1$ so that $0 < x < 2$. Then

$$0 < x < 2$$

$$1 < 1 + x < 3$$

$$0 < x^2 < 4$$

$$3 < x^2 + 3 < 7$$

$$12 < 4(x^2 + 3) < 28$$

Hence

$$|x - 1| \frac{|1 + x|}{4(x^2 + 3)} < \frac{3}{12} |x - 1|$$
This will be \( < \epsilon \) if \(|x - 1| < 4\epsilon\).

**Our Proof:** Let \( \epsilon > 0 \) be given and let \( \delta = \min\{1, 4\epsilon\} \). Assume that \( 0 < |x - 1| < \delta \). Then from the scratch work

\[
\left| \frac{1}{x^2 + 3} - \frac{1}{4} \right| < \epsilon,
\]

proving the limit statement.

(e):

**Scratch Work:** Let \( \epsilon > 0 \) be given. We want

\[
\left| \frac{x}{1 - x} - 1 \right| < \epsilon
\]

\[
\left| \frac{x}{1 - x} \right| = |x| \left| \frac{1}{1 - x} \right| < \epsilon
\]

*If we assume that \( x = 0 \pm 1 \) we get*

\[-1 < x < 1 \]
\[1 > -x > -1 \]
\[2 > 1 - x > 0 \]

*We cannot have 0 in the denominator. Hence, we assume instead that \( x = 0 \pm .5 \). Then*

\[-.5 < x < .5 \]
\[.5 > x > -.5 \]
\[-.5 < -x > .5 \]
\[.5 > 1 - x > 1.5 \]

*Hence*

\[
|x| \left| \frac{1}{1 - x} \right| < \frac{1}{.5} |x|
\]

This will be \( < \epsilon \) if \(|x| < .5\epsilon\).
Our Proof: Let $\epsilon > 0$ be given and let $\delta = \min\{.5, .5\epsilon\}$. Assume that $0 < |x - 0| < \delta$. Then from the scratch work

$$\left| \frac{1}{1 - x} - 1 \right| < \epsilon,$$

proving the limit statement.

(f):

Scratch Work: Let $\epsilon > 0$ be given. We want

$$\left| \frac{1}{\sqrt{3 + x}} - \frac{1}{2} \right| < \epsilon \quad \frac{2 - \sqrt{3 + x}}{2\sqrt{3 + x}} < \epsilon \quad \frac{|2 - \sqrt{3 + x}| |2 + \sqrt{3 + x}|}{(2\sqrt{3 + x}) |2 + \sqrt{3 + x}|} < \epsilon \quad \left| 1 - x \right| \frac{1}{(2\sqrt{3 + x}) |2 + \sqrt{3 + x}|} < \epsilon$$

Assume that $x = 1 \pm 1$ so that

$0 < x < 2$
$3 < 3 + x < 5$
$\sqrt{3} < \sqrt{3 + x} < \sqrt{5}$
$2 + \sqrt{3} < 2 + \sqrt{3 + x} < 2 + \sqrt{5}$
$2\sqrt{3} < 2\sqrt{3 + x} < 2\sqrt{5}$

Then

$$\left| 1 - x \right| \frac{1}{(2\sqrt{3 + x}) |2 + \sqrt{3 + x}|} < \left| 1 - x \right| \frac{1}{(2\sqrt{3})(2 + \sqrt{3})}$$

This will be $< \epsilon$ if $|x - 1| < (2\sqrt{3})(2 + \sqrt{3})\epsilon$.

Our Proof: Let $\epsilon > 0$ be given and let $\delta = \min\{1, (2\sqrt{3})(2 + \sqrt{3})\epsilon\}$. Assume that $0 < |x - 1| < \delta$. Then from the scratch
work
\[ \left| \frac{1}{\sqrt{3 + x}} - \frac{1}{2} \right| < \epsilon, \]
proving the limit statement.

(4) Assume that \( \lim_{x \to a} f(x) = 2 \). Use a \( \delta-\epsilon \) argument to prove:

\[ \begin{align*}
(a) & \quad \lim_{x \to a} \frac{2}{f(x) + 2} = \frac{1}{2} \\
(b) & \quad \lim_{x \to a} \sqrt{f(x)} + 2 = 2 \\
(c) & \quad \lim_{x \to a} f(x)^2 = 4 \\
(d) & \quad \lim_{x \to a} \frac{3f(x)}{f(x) + 1} = 2
\end{align*} \]

Solutions:

(a):

**Scratch work:** Let \( \epsilon > 0 \) be given. We want
\[ \left| \frac{2}{f(x) + 2} - \frac{1}{2} \right| < \epsilon \]
\[ \frac{|2 - f(x)|}{2|f(x) + 2|} < \epsilon \]
\[ |f(x) - 2| \frac{1}{2|f(x) + 2|} < \epsilon \]

The term on the left is our “gold” since it becomes small as \( x \) approaches \( a \). The other term is our “trash” which we will bound. Specifically, we reason that for all \( x \) sufficiently close to \( a \), \( f(x) = 2 \pm 1 \). Thus, for such \( x \),
\[ 1 < f(x) < 3 \]
\[ 3 < f(x) + 2 < 5 \]
\[ 6 < 2|f(x) + 2| < 10 \]

Hence
\[ |f(x) - 2| \frac{1}{2|f(x) + 2|} < \frac{1}{6} |f(x) - 2| \]
This is $< \epsilon$ if $|f(x) - 2| < 6\epsilon$, which is true for all $x$ sufficiently close to $a$.

**Proof:** Let $\epsilon > 0$ be given and choose $\delta_1 > 0$ so that

$$|f(x) - 2| < 1$$

for $0 < |x - a| < \delta_1$.

Choose $\delta_2 > 0$ such that

$$|f(x) - 2| < 2\epsilon$$

for $0 < |x - a| < \delta_2$. Let $\delta = \min\{\delta_1, \delta_2\}$. From the scratch work, $0 < |x - a| < \delta$ implies that

$$\left| \frac{2}{f(x) + 2} - \frac{1}{2} \right| < \epsilon$$

proving the limit statement.

(b):

**Scratch work:** Let $\epsilon > 0$ be given. We want

$$|\sqrt{f(x) + 2} - 2| < \epsilon$$

$$\frac{|(\sqrt{f(x) + 2} - 2)(\sqrt{f(x) + 2} + 2)|}{\sqrt{f(x) + 2} + 2} < \epsilon$$

$$\frac{|f(x) - 2|}{\sqrt{f(x) + 2} + 2} < \epsilon$$

The numerator is our “gold” since it becomes small as $x$ approaches $a$. The other term is our “trash” which we will bound. Specifically, we reason that for all $x$ sufficiently close to $a$, $f(x) = 2 \pm 1$. Thus, for such $x$,

$$1 < f(x) < 3$$

$$3 < f(x) + 2 < 5$$

$$\sqrt{3} < \sqrt{f(x) + 2} < \sqrt{5}$$

$$2 + \sqrt{3} < \sqrt{f(x) + 2} + 2 < 2 + \sqrt{5}$$
Hence

\[
\frac{|f(x) - 2|}{\sqrt{f(x) + 2} + 2} < \frac{1}{2 + \sqrt{3}} |f(x) - 2|
\]

This is \( < \epsilon \) if \( |f(x) - 2| < (2 + \sqrt{3})\epsilon \), which is true for all \( x \) sufficiently close to \( a \).

**Proof:** Let \( \epsilon > 0 \) be given and choose \( \delta_1 > 0 \) so that

\[ |f(x) - 2| < 1 \]

for \( 0 < |x - a| < \delta_1 \).

Choose \( \delta_2 > 0 \) such that

\[ |f(x) - 2| < (2 + \sqrt{3})\epsilon \]

for \( 0 < |x - a| < \delta_2 \). Let \( \delta = \min\{\delta_1, \delta_2\} \). From the scratch work, \( 0 < |x - a| < \delta \) implies that

\[ |\sqrt{f(x) + 2} - 2| < \epsilon \]

proving the limit statement.

**(c):**

**Scratch work:** Let \( \epsilon > 0 \) be given. We want

\[
|f(x)^2 - 4| < \epsilon \\
|f(x) - 2| |f(x) + 2| < \epsilon
\]

The term on the left is our “gold” since it becomes small as \( x \) approaches \( a \). The other term is our “trash” which we will bound. Specifically, we reason that for all \( x \) sufficiently close to \( a \), \( f(x) = 2 \pm 1 \). Thus, for such \( x \),

\[
1 < f(x) < 3 \\
3 < f(x) + 2 < 5
\]

Hence

\[
|f(x) - 2| |f(x) + 2| < 5 |f(x) - 2|
\]

This is \( < \epsilon \) if \( |f(x) - 2| < \epsilon/5 \), which is true for all \( x \) sufficiently close to \( a \).
Proof: Let $\epsilon > 0$ be given and choose $\delta_1 > 0$ so that

$$|f(x) - 2| < 1$$

for $0 < |x - a| < \delta_1$.

Choose $\delta_2 > 0$ such that

$$|f(x) - 2| < \epsilon/5$$

for $0 < |x - a| < \delta_2$. Let $\delta = \min\{\delta_1, \delta_2\}$. From the scratch work, $0 < |x - a| < \delta$ implies that

$$|f(x)^2 - 4| < \epsilon$$

proving the limit statement.

(d):

Scratch work: Let $\epsilon > 0$ be given. We want

$$\left| \frac{3f(x)}{f(x) + 1} - 2 \right| < \epsilon$$

Choose $\delta > 0$ such that

$$\frac{|f(x) - 2|}{|f(x) + 1|} < \epsilon$$

$$|f(x) - 2| \frac{1}{|f(x) + 1|} < \epsilon$$

The term on the left is our “gold” since it becomes small as $x$ approaches $a$. The other term is our “trash” which we will bound. Specifically, we reason that for all $x$ sufficiently close to $a$, $f(x) = 2 \pm 1$. Thus, for such $x$,

$$1 < f(x) < 3$$
$$2 < f(x) + 1 < 4$$

Hence

$$|f(x) - 2| \frac{1}{|f(x) + 1|} < \frac{1}{2} |f(x) - 2|$$

This is $< \epsilon$ if $|f(x) - 2| < 2\epsilon$, which is true for all $x$ sufficiently close to $a$.

Proof: Let $\epsilon > 0$ be given and choose $\delta_1 > 0$ so that

$$|f(x) - 2| < 1$$
for $0 < |x - a| < \delta_1$.

Choose $\delta_2 > 0$ such that

$$|f(x) - 2| < 2\epsilon$$

for $0 < |x - a| < \delta_2$. Let $\delta = \min\{\delta_1, \delta_2\}$. From the scratch work, $0 < |x - a| < \delta$ implies that

$$\left| \frac{3f(x)}{f(x) + 1} - 2 \right| < \epsilon$$

proving the limit statement.

(5) Find a value of $a$ for which the following function is continuous at $x = 2$.

$$f(x) = \begin{cases} 
  x + a & x < 2 \\
  x^2 & x \geq 2
\end{cases}$$

**Solution:**

$$f(2) = 4$$

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} x^2 = 4$$

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} x + a = 2 + a$$

Hence, $f(x)$ will be continuous at $x = 2$ if and only if $2 + a = 4$; hence $a = 2$.

(6) Prove that there is a point $(x, y)$ at which the graphs of $y = e^x$ and $y = 2\cos x$ cross. What theorem are you using?

**Solution:**

The graphs cross at $(x, y)$ if and only if $e^x = 2\cos x$ which is the same as $e^x - 2\cos x = 0$. Let

$$g(x) = e^x - 2\cos x.$$  

Then $g(0) = e^0 - 2\cos 0 = -1$ while $g(\pi/2) = e^{\pi/2} - 2\cos \pi/2 = e^{\pi/2}$. Since $g(0) < 0$ and $g(\pi/2) > 0$, it follows from the Intermediate Value Theorem that there is an $x$ between 0 and $\pi/2$ such that $g(x) = 0$, proving that the graphs cross.
(7) Suppose that \( f \) is continuous at every \( x \) in \([0, 1]\) and that for all \( x \) in this interval, \( 0 \leq f(x) \leq 1 \). Prove that there is an \( x \in [0, 1] \) such that \( f(x) = x^2 \).

**Solution:** Let 
\[ g(x) = f(x) - x^2. \]

Then 
\[ g(0) = f(0) - 0^2 = f(0) \geq 0. \]

If \( f(0) = 0 \) then \( f(0) = 0^2 \) so there is an \( x \) such that \( f(x) = x^2 \). Hence we may assume that \( g(0) > 0 \).

Also
\[ g(1) = f(1) - 1^2 = f(1) - 1 \leq 0. \]

If \( f(1) = 1 \) then \( f(1) = 1^2 \) so there is an \( x \) such that \( f(x) = x^2 \). Hence we may assume that \( g(1) < 0 \).

Since \( g(0) > 0 \) and \( g(1) < 0 \), it follows from the Intermediate Value Theorem that there is an \( x \) between 0 and 1 such that \( g(x) = 0 \); hence there is an \( x \) such that \( f(x) = x^2 \).

(8) Let \( f(x) = (x + 3)^{-2} \).

(a) Find constants \( m \) and \( M \) such that 
\[ m \leq f''(x) \leq M \]
for all \( x \in [0, 3] \). (You need not prove that your values really work.)

**Solution**

\[ f(x) = (x + 3)^{-2} \]

\[ f'(x) = -2(x + 3)^{-3} \]

\[ f''(x) = 6(x + 3)^{-4} \]

\[ = \frac{6}{(x + 3)^4} \]

For \( 0 \leq x \leq 3, 3 < x + 3 < 6 \). Hence
\[ \frac{6}{6^4} < \frac{6}{(x + 3)^4} < \frac{6}{3^4} \]
\[ \frac{1}{6^3} < \frac{6}{(x + 3)^4} < \frac{2}{3^3} \]
(b) Use the information from part 8a and integration to obtain constants $a_0, a_1, C$ and $D$ such that

$$Cx^2 \leq (x+3)^{-2} - (a_0 + a_1x) < Dx^2$$

holds for all $x \in [0, 3]$. Do not use Maclaurin’s theorem on p. 187. Your method, however, should parallel the proof of this theorem.

**Solution:**

We integrate the last inequality from 0 to $x$ twice:

$$\frac{x}{6^3} < -2(x+3)^{-3}\bigg|_0^x < \frac{2x}{3^3}$$

$$\frac{x}{6^3} < -2(x+3)^{-3} + 2(3)^{-3} < \frac{2x}{3^3}$$

$$\frac{x^2}{2 \cdot 6^3} < (x+3)^{-2} - 3^{-2} + 2(3)^{-3}x < \frac{x^2}{3^3}$$

$$\frac{x^2}{2 \cdot 6^3} < (x+3)^{-2} - (3^{-2} - 2(3)^{-3}x) < \frac{x^2}{3^3}$$

Hence $C = \frac{1}{2 \cdot 6^3}$, $D = \frac{1}{3^3}$, $a_0 = 3^{-2}$, $a_1 = -2(3)^{-3}$.

(c) According to the information in part 8b, what is the smallest value of $\epsilon$ for which

$$(.4 + 3)^{-2} = a_0 + a_1(.4) \pm \epsilon?$$

**Solution:** From the last inequality above with $x = .4$,

$$|(.4 + 3)^{-2} - (3^{-2} - 2(3)^{-3}.4)| < \frac{(\cdot4)^2}{3^3}.$$ 

Hence

$$(.4 + 3)^{-2} = 3^{-2} - 2(3)^{-3}.4 \pm \frac{(\cdot4)^2}{3^3}.$$ 

(9) Prove Theorem 3 on p. 164 of the notes:

**Theorem 3 (Sequence).** Let $f(x)$ be continuous at $a$ and let $x_n$ be a sequence such that $\lim_{n \to \infty} x_n = a$. Then

$$\lim_{n \to \infty} f(x_n) = f(a).$$