① $\delta = 0.03$ $\quad A_x = \sum_{k=0}^{\infty} {}_kp_x\, q_{x+k}\, v^{k+1} = \sum_{k=0}^{\infty} v^{k+1} \frac{l_{x+k} - l_{x+k+1}}{l_x}$

a) $A_2 = 1 \sum_{k=0}^{1} v^{k+1} \frac{l_{x+k} - l_{x+k+1}}{l_x} + 2 \sum^{2} v^{k+1} \frac{l_{x+k} - l_{x+k+1}}{l_x} + 3 \sum_{k=3}^{\infty} v^{k+1} \frac{l_{x+k} - l_{x+k+1}}{l_x}$

$= e^{-.03}\left(\frac{75-25}{75}\right) + e^{-.06}\left(\frac{25-10}{75}\right) + 2(e^{-.09})\left(\frac{10-0}{75}\right) + 3(0)$

$= \boxed{1.07903}$

b) $e_x = E[K(x)] = \sum_{k=1}^{\infty} {}_kp_x = \sum_{k=1}^{\infty} \frac{l_{x+k}}{l_x}$

$e_2 = \sum_{k=1}^{\infty} {}_kp_2 = \sum_{k=1}^{\infty} \frac{l_{2+k}}{75} = \frac{25}{75} + \frac{10}{75} + \frac{0}{75} = \frac{35}{75} = \boxed{0.4667}$

other Answer: $e_2 = \sum k({}_kp_2\, q_{k+2}) = 0\left(\frac{75-25}{75}\right) + 1\cdot\left(\frac{25-10}{75}\right) + 2\left(\frac{10-0}{75}\right) = \frac{35}{75}$

② $\delta = 0.05$ $\quad l_x = \frac{1}{1+x^2} = (1+x^2)^{-1}$ $\quad l'_x = -(1+x^2)^{-2}(2x) = -\frac{2x}{(1+x^2)^2}$

a) $\mu_x = \frac{-l'_x}{l_x} = -\frac{2x}{(1+x^2)^2} \Big/ \frac{1}{(1+x^2)} = -\frac{2x}{(1+x^2)^2} \cdot \frac{(1+x^2)}{1} = \boxed{+\frac{2x}{(1+x^2)}}$

b) $\bar{A}^{1}_{25:\overline{15}|} = \int_0^{15} v^t\, {}_tp_x \mu_x(t)\, dt$ $\quad {}_tp_x \mu_x(t)\, dt = \frac{l'_{x+t}}{l_x}$

$= \int_0^{15} e^{-.05t} \left(\frac{-2(x+t)}{(1+(x+t)^2)^2} \Big/ \frac{1}{(1+x^2)} \right) dt$

where $x = 25$ (-2)

c) $\mathring{e}_x = E[T(x)] = \int_0^{\infty} {}_tp_x\, dt$

$\mathring{e}_{25} = \int_0^{\infty} {}_tp_{25}\, dt = \int_0^{\infty} \frac{l_{25+t}}{l_{25}}\, dt = \int_0^{\infty} \frac{1}{1+(25+t)^2} \Big/ \frac{1}{1+25^2}\, dt$

$= 626 \int_0^{\infty} \frac{1}{1+(25+t)^2}\, dt$

d) $\bar{a}_x = \int_0^{\infty} \frac{l_{x+t}}{l_x} v^t\, dt$

$\bar{a}_{25} = \int_0^{\infty} \frac{l_{25+t}}{l_{25}} e^{-.05t}\, dt = \int_0^{\infty} e^{-.05t} \left(\frac{1}{1+(25+t)^2} \Big/ \frac{1}{1+25^2} \right) dt$

$= 626 \int_0^{\infty} e^{-.05t} \left(\frac{1}{1+(25+t)^2} \right) dt$

e) $P(e^{T(25)} \geq 20)$ $\qquad l_x = \frac{1}{1+x^2}$

$P(e^{T(25)} \geq 20) = P(\ln e^{T(25)} \geq \ln 20)$

$= P(T(25) \geq \ln 20) = \frac{l_{25+\ln 20}}{l_{25}} = \left(\frac{1}{1+(25+\ln 20)^2}\right) / \left(\frac{1}{1+25^2}\right)$

$= \boxed{0.797695}$

③ a) $Var(Z)$ where $Z = v^{T(40)}$ $\quad$ Assume UDD $\quad i = 0.06$

$Var(Z) = {}^2\bar{A}_{40} - (\bar{A}_{40})^2 = \frac{i'}{\delta'}\,{}^2A_{40} - \left(\frac{i}{\delta}A_{40}\right)^2$ $\qquad \delta = \ln(1.06) = 0.058269$

${}^2A_{40} = 0.04863 \quad A_{40} = 0.16132$ $\qquad i' = e^{2\delta} - 1 = 0.1236$

$Var(Z) = \frac{0.1236}{.116538}(.04863) - \left(\frac{.06}{.058269}(.16132)\right)^2$ $\qquad \delta' = 2\delta = 0.116538$

$= \boxed{0.02398}$ ✓

b) $Var(Y)$ where $Y = \bar{a}_{\overline{T(40)}|} = \frac{1 - v^{T(40)}}{\delta}$ $\quad$ Assume UDD

$Var(a_{\overline{T}|}) = \frac{1}{\delta^2}\left({}^2\bar{A}_x - (\bar{A}_x)^2\right)$

$Var(a_{\overline{T(40)}|}) = \frac{1}{(.058269)^2}(0.02398) = \boxed{7.062776}$

c) $500 A^1_{40:\overline{10}|}$

$A_x = A^1_{x:\overline{m}|} + v^m\, {}_mp_x\, A_{x+m}$

$A_{40} = 0.16132$

$v^m = v^{10} = e^{-.058269(10)} = 0.558394$

${}_mp_x = \frac{l_{x+m}}{l_x} = \frac{l_{50}}{l_{40}} = \frac{8,950,901}{9,313,166} = 0.9611018$

$A_{50} = 0.24905$

$0.16132 = A^1_{40:\overline{10}|} + (.558394)(.9611018)(0.24905)$

$A^1_{40:\overline{10}|} = 0.027661$

$500 A^1_{40:\overline{10}|} = \boxed{13.830735}$

d) $\ddot{a}_{40:\overline{10|}} = \dfrac{1 - A_{40:\overline{10|}}}{d}$

$A_{40:\overline{10|}} = A^1_{40:\overline{10|}} + v^{10}{}_{10}p_x = 0.027661 + (.558394)(0.961101$

$A_{40:\overline{10|}} = 0.5643345$

$d = \dfrac{i}{1+i} = \dfrac{0.06}{1.06} = 0.056604$

$\ddot{a}_{40:\overline{10|}} = \dfrac{1 - 0.5643345}{0.056604} = \boxed{7.696757}$

(4) $Z_i = v^{T_i(40)}$

$S = (500)\left[\sum_{n=1}^{100} v^{T_i(40)}\right]$

$E(S) = (500)\left[\sum_{n=1}^{100} E(v^{T_i(40)})\right] = 100\bar{A}_{40}(500)$

$\bar{A}_{40} = \dfrac{i}{\delta} A_{40} = \dfrac{0.06}{0.058269}(0.16132) = 0.166112$ Assume UDD

$E(S) = 100(0.166112)(500) = 8305.617$

$Var(S) = \sum_{n=1}^{100} Var(500Z_i) = 500^2 \sum_{n=1}^{100} Var(Z_i) = 100(500)^2 Var(Z_i)$

$Var(Z_i) = {}^2\bar{A}_{40} - (\bar{A}_{40})^2 = 0.02398$ Assume UDD

$Var(S) = 100(500)^2(0.02398) = 599{,}500$

$\sigma = \sqrt{Var(S)} = 774.2739$

$P(S \le x) = 0.975 \rightarrow P\left(\dfrac{S - 8305.617}{774.2739} \le \dfrac{x - 8305.617}{774.2739}\right) \le .975$

$\dfrac{S - 8305.617}{774.2739} = 1.960 \rightarrow \boxed{S = 9823.1938}$