THEORY OF INTEREST AND LIFE CONTINGENCIES WITH PENSION APPLICATIONS

A Problem-Solving Approach

Third Edition

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CHAPTER SEVEN

LIFE TABLES AND POPULATION PROBLEMS

7.1 INTRODUCTION

In Chapter 6 we saw how to combine the theory of interest with elementary probability theory to obtain the present value of contingent payments. In practical situations the following question is crucial: how do we determine the appropriate probabilities to be used in these calculations?

The answer is that we must have data to guide us. We must know what percentage of borrowers do not repay their loans, and we must be able to identify high-risk borrowers and either refuse to lend them money at all, or lend them money at higher rates of interest than we use for low-risk customers.

In almost all of the examples we study in the remainder of this text, the probabilities required are those of surviving to certain ages or of dying before certain ages. Data required to calculate these probabilities is collected empirically and is published in *life tables*. In this chapter we will introduce the basic notation underlying life tables, and see how to calculate required probabilities. Section 7.2 will consider the life tables as presenting survival data for a given fixed initial population. In Section 7.4 we will see how the same table can also be interpreted as giving information about a stationary population.

Historically, attempts were made by de Moivre, Gompertz, Makeham and others to find an analytic function $S(x)$ which would describe survival from age 0 to age x. In Section 7.3 we will examine some of these possibilities.

Finally, we briefly examine in Section 7.6 a few of the basic ideas of multiple decrement theory.

7.2 LIFE TABLES

TABLE 7.1

Age	ℓ_x	d_x	$1000q_x$
0	1,000,000	1580	1.58
1	998,420	680	68
2	997,740	485	.49
3	997,255	435	.44

In Table 7.1 we have presented an excerpt from a typical life table. In such a table the column ℓ_x denotes the number of lives which have survived to age x. For this to make sense, we have to assume a starting population ℓ_0. In our case $\ell_0 = 1{,}000{,}000$, but any value would have sufficed. As we shall see, it is the ratio of entries in the table, not the individual numbers, which is important. In particular, the ratio $\frac{\ell_x}{\ell_0}$ represents the probability of surviving from birth to age x. As a general function of x, it is called the *survival function* and is denoted $S(x)$.

Note that in Table 7.1, $\ell_1 = 998{,}420$. This means that 1580 lives have died in the first year of life, and this is the entry d_0. In general, d_x denotes the number of lives, out of those aged x, which do not survive to age $x + 1$. Thus

$$d_x = \ell_x - \ell_{x+1}. \tag{7.1}$$

q_x denotes the probability that a life aged x will not survive to age $x + 1$. Thus

$$q_x = \frac{d_x}{\ell_x}. \tag{7.2}$$

In our case, the final column tells us that $q_2 = \frac{.49}{1000} = .00049$, and $\frac{485}{997{,}740} = .00049$ as well.

Using a life table we can compute numerous probabilities concerning survival. In Example 7.1, we use Table 7.1 to assist us. In Examples 7.2 and 7.3 we will determine expressions which could be converted to numerical answers if we had access to a complete life table.

Example 7.1

Use Table 7.1 to find each of the following:

(a) The probability that a newborn will live to age 3.

(b) The probability that a newborn will die between age 1 and age 3.

Solution

(a) This equals $\frac{\ell_3}{\ell_0} = \frac{997{,}255}{1{,}000{,}000} = .997255$.

(b) The number of deaths between ages 1 and 3 is $\ell_1 - \ell_3$. Thus the probability is $\frac{\ell_1 - \ell_3}{\ell_0} = \frac{1165}{1{,}000{,}000} = .001165$. □

Example 7.2

Find an expression for each of the following:

(a) The probability that an 18-year-old lives to age 65.

(b) The probability that a 25-year-old dies between ages 40 and 45.

(c) The probability that a 25-year-old does *not* die between ages 40 and 45.

(d) The probability that a 30-year-old dies before age 60.

Solution

(a) This equals $\frac{\ell_{65}}{\ell_{18}}$.

(b) Since the number of people dying between age 40 and age 45 is $\ell_{40} - \ell_{45}$, this is $\frac{\ell_{40} - \ell_{45}}{\ell_{25}}$.

(c) This is the complement of (b), so the answer is given by $1 - \frac{\ell_{40} - \ell_{45}}{\ell_{25}}$. Alternatively, we could obtain this as the sum of $\frac{\ell_{25} - \ell_{40}}{\ell_{25}}$, the probability of dying before age 40, and $\frac{\ell_{45}}{\ell_{25}}$, the probability of dying after age 45.

(d) This is $\frac{\ell_{30} - \ell_{60}}{\ell_{30}}$. □

Example 7.3

There are four persons, now aged 40, 50, 60 and 70. Find an expression for the probability that both the 40-year-old and the 50-year-old will die within the five-year period starting ten years from now, but neither the 60-year-old nor the 70-year-old will die during that five-year period.

Solution

Working out each probability separately and multiplying the results, we obtain $\left(\frac{\ell_{50}-\ell_{55}}{\ell_{40}}\right)\left(\frac{\ell_{60}-\ell_{65}}{\ell_{50}}\right)\left(1-\frac{\ell_{70}-\ell_{75}}{\ell_{60}}\right)\left(1-\frac{\ell_{80}-\ell_{85}}{\ell_{70}}\right)$. □

This is all well and good, but how was such a life table constructed in the first place? The numbers ℓ_x and d_x do not represent actual numbers of real people, so where did they come from? The answer is that q_x was estimated from observations of mortality data from a suitable study sample, and these values of q_x, together with the arbitrary starting value ℓ_0, determine the whole table.

This is done as follows. Start with ℓ_0. We know that $q_0\ell_0 = d_0$. Now we can find $\ell_1 = \ell_0 - d_0$. Then $q_1\ell_1 = d_1$, $\ell_2 = \ell_1 - d_1$, and so on. In general, we continue with the basic identities

$$q_x\ell_x = d_x \tag{7.3}$$

and

$$\ell_{x+1} = \ell_x - d_x. \tag{7.4}$$

Example 7.4

A scientist studies the mortality patterns of Golden-Winged Warblers. She establishes the following probabilities of deaths: $q_0 = .40$, $q_1 = .20$, $q_2 = .30$, $q_3 = .70$ and $q_4 = 1$. Starting with $\ell_0 = 100$, construct a life table.

Solution

Age	ℓ_x	d_x	q_x
0	100	40	.40
1	60	12	.20
2	48	14	.30
3	34	24	.70
4	10	10	1.0

□

Before continuing let us introduce a bit more notation. We let p_x represent the probability that an individual just turning age x will survive to age $x+1$. Hence

$$p_x = \frac{\ell_{x+1}}{\ell_x} = 1 - q_x. \tag{7.5}$$

More generally,

$_np_x$ = the probability that an individual just turning age x will survive to age $x+n$.

$_nq_x$ = the probability that an individual just turning age x will not survive to age $x+n$.

Thus

$$_np_x = \frac{\ell_{x+n}}{\ell_x} = 1 - {}_nq_x. \tag{7.6}$$

The reader should rewrite the answers to Examples 7.2 and 7.3 using this new notation. In the special case of $x = 0$, we have $_np_0 = S(n)$, the survival function defined on page 128.

Example 7.5

Explain both mathematically and verbally why each of the following is true.

(a) $\ell_x - \ell_{x+n} = d_x + d_{x+1} + \cdots + d_{x+n-1}$.

(b) $_{m+n}p_x = {}_mp_x \cdot {}_np_{x+m}$.

Solution

(a) Mathematically,

$$d_x + d_{x+1} + \cdots + d_{x+n-1} = (\ell_x - \ell_{x+1}) + (\ell_{x+1} - \ell_{x+2}) + \cdots + (\ell_{x+n-1} - \ell_{x+n}) = \ell_x - \ell_{x+n}.$$

Verbally, $\ell_x - \ell_{x+n}$ is the number of people alive at age x but dead at age $x+n$, (i.e., the number of people who die between age x and age $x+n$). But $d_x + \cdots + d_{x+n-1}$ is just the sum of the numbers of people dying at various ages between age x and age $x+n-1$ (inclusive), which is the same as above.

(b) Mathematically,

$$_{m+n}p_x = \frac{\ell_{x+m+n}}{\ell_x} = \frac{\ell_{x+m}}{\ell_x} \cdot \frac{\ell_{x+m+n}}{\ell_{x+m}} = {}_mp_x \cdot {}_np_{x+m}.$$

Verbally, $_{m+n}p_x$ is the probability that a person aged x lives $m+n$ years. To do this, he has to first live m years and then, at age $x+m$, live n more years. Hence $_{m+n}p_x$ is the product of the probabilities of these two events, namely $_mp_x \cdot {}_np_{x+m}$. □

Example 7.6

30% of those who die between ages 25 and 75 die before age 50. The probability of a person aged 25 dying before age 50 is 20%. Find ${}_{25}p_{50}$.

Solution

We want to find ${}_{25}p_{50} = \frac{\ell_{75}}{\ell_{50}}$. We are given $.30(\ell_{25} - \ell_{75}) = \ell_{25} - \ell_{50}$, and that $\frac{\ell_{25} - \ell_{50}}{\ell_{25}} = .20$. The second relation says that $.80\ell_{25} = \ell_{50}$, or $\ell_{25} = 1.25\ell_{50}$. When substituted in the first expression, this gives $.30(1.25\ell_{50} - \ell_{75}) = 1.25\ell_{50} - \ell_{50}$. Thus $.125\ell_{50} = .30\ell_{75}$, and finally $\frac{\ell_{75}}{\ell_{50}} = \frac{.125}{.3} = .4167$. □

Finally, let us remark that the expressions ${}_np_x$ and ${}_nq_x$ have only been defined thus far for integral values of n. What should we do in other cases? Say, for example, we want to find ${}_{1/4}p_{20}$, the probability that a person aged 20 lives to age $20\frac{1}{4}$. This information is not obtainable directly from a life table, but we can obtain a good approximation by assuming that deaths occur uniformly over a given year. In that case we would expect that $\frac{1}{4} \cdot d_{20}$ individuals die during the first $\frac{1}{4}$ of the year, leaving $\ell_{20} - \frac{1}{4} \cdot d_{20}$ alive. Hence an approximate value for ${}_{1/4}p_{20}$ is

$$
\begin{aligned}
{}_{1/4}p_{20} &= \frac{\ell_{20} - \frac{1}{4} \cdot d_{20}}{\ell_{20}} \\
&= \frac{\ell_{20} - \frac{1}{4}(\ell_{20} - \ell_{21})}{\ell_{20}} \\
&= \frac{\frac{3}{4}\ell_{20} + \frac{1}{4}\ell_{21}}{\ell_{20}}.
\end{aligned}
$$

In other words, we have used linear interpolation between ℓ_{20} and ℓ_{21} in the life table. It is possible to use more sophisticated finite difference formulae, but linear interpolation seems to be sufficiently accurate for most purposes.

Example 7.7

Using Table 7.1 and assuming a uniform distribution of deaths over each year, find each of the following:

(a) ${}_{4/3}p_1$

(b) The probability that a newborn will survive the first year, but die in the first two months thereafter.

Solution

(a) By linear interpolation we obtain

$$_{4/3}p_1 = \frac{\ell_2 - \frac{1}{3}\cdot d_2}{\ell_1} = \frac{\ell_2 - \frac{1}{3}(\ell_2 - \ell_3)}{\ell_1} = \frac{\frac{2}{3}\ell_2 + \frac{1}{3}\ell_3}{\ell_1} = .999157.$$

(b) The number dying in the period described is $\frac{1}{6}\cdot d_1$. Hence, the answer is $\frac{\frac{1}{6}\cdot d_1}{\ell_0} = .00011333.$ □

7.3 ANALYTIC FORMULAE FOR ℓ_x

Calculations of the type described in the previous section are often straightforward if we assume a simple analytic formula for ℓ_x. Here is an example.

Example 7.8

Given $\ell_x = 1000\left(1 - \frac{x}{105}\right)$, determine each of the following:

(a) ℓ_0 (b) ℓ_{35} (c) q_{20} (d) $_{15}p_{35}$ (e) $_{15}q_{25}$

(f) The probability that a 30-year-old dies between ages 55 and 60.

(g) The probability that a 30-year-old dies after age 70.

(h) The probability that a 15-year-old reaches age 110.

(i) The probability that, given a 20-year-old and a 30-year-old, one but not both of these individuals reaches age 70.

Solution

(a) $\ell_0 = 1000\left(1 - \frac{0}{105}\right) = 1000.$

(b) $\ell_{35} = 1000\left(1 - \frac{35}{105}\right) = 667$

(note the answer must be an integer).

(c) $q_{20} = \frac{\ell_{20} - \ell_{21}}{\ell_{20}} = 1 - .98824 = .01176.$

(d) $_{15}p_{35} = \frac{\ell_{50}}{\ell_{35}} = .78571.$

(e) $_{15}q_{25} = 1 - {}_{15}p_{25} = 1 - \frac{\ell_{40}}{\ell_{25}} = .1875.$

(f) $\frac{\ell_{55} - \ell_{60}}{\ell_{30}} = .0667.$

(g) This is equal to the probability that a 30-year-old reaches age 70, which is $_{40}p_{30} = \frac{\ell_{70}}{\ell_{30}} = .4667.$

EXERCISES

7.1 Introduction; 7.2 Life Tables

7-1. For a certain type of insect, we find that $q_0 = .70$, $q_1 = .30$, $q_2 = .40$ and $q_3 = 1.0$. Starting with $\ell_0 = 1000$, construct a life table.

7-2. Write expressions for each of the following:
- (a) The probability that a 20-year-old lives 25 years.
- (b) The probability that a 20-year-old reaches age 25.
- (c) The probability that a 20-year-old dies between ages 25 and 26.
- (d) The probability that a 20-year-old lives for at least 40 years.
- (e) The probability that a pair of 20-year-olds do not both survive to age 60.

7-3. 80% of people age 25 survive to age 60. 40% of people who die between age 25 and age 60 do so before age 45. Find the probability that a 45-year-old will die before reaching age 60.

7-4. Four persons are all aged 30.
- (a) Find an expression for the probability that any three of them will survive to age 60, with the other dying between age 50 and age 55.
- (b) Find an expression for the probability that at least 2 of the persons will survive for at most 30 years.

7-5. Explain, both mathematically and verbally, why the following are true.
- (a) $\ell_x = d_x + d_{x+1} + d_{x+2} + \cdots$
- (b) $\ell_{x+n} = \ell_x \cdot p_x \cdot p_{x+1} \cdot \cdots \cdot p_{x+n-1}$
- (c) ${}_{m+n}p_x = {}_np_x \cdot {}_mp_{x+n}$
- (d) $q_x + p_x \cdot q_{x+1} + {}_2p_x \cdot q_{x+2} + \cdots = 1$

7-6. Complete the missing entries in the following table.

x	ℓ_x	d_x	p_x	q_x
0	1000	100		
1				
2	750		.80	
3				
4	300			.60
5				
6	0			

7-7. If $p_x = .95$ for all x, find each of the following:
(a) p_{20} (b) $_2q_{30}$
(c) The probability that a 20-year-old dies at age 50 last birthday.
(d) The probability that a 20-year-old dies between age 50 and age 55.

7-8. Find an expression for the probability that a 30-year-old will die in the second half of the year following her 35th birthday.

7-9. Derive each of the following approximations, where $0 < t < 1$.
(a) $\ell_{x+t} = \ell_x - t \cdot d_x$
(b) $\ell_{x+t} = (1-t) \cdot \ell_x + t \cdot \ell_{x+1}$
(c) $_tp_x = 1 - t \cdot q_x$

7-10. Let $_{n|m}q_x$ denote the probability that a person aged x will die between ages $x+n$ and $x+n+m$. (When $m = 1$ it is omitted in this notation.)
(a) Show that $_{n|m}q_x = \dfrac{\ell_{x+n} - \ell_{x+n+m}}{\ell_x}$.
(b) Show that $_{n|m}q_x = {}_np_x - {}_{n+m}p_x$. Explain this identity verbally.
(c) Show that $_{n|m}q_x = {}_np_x(1 - {}_mp_{x+n})$. Explain this identity verbally.

7-11. You are given the following probabilities:
   (i) That two persons age 35 and 45 will both live for 10 years equals .80.
   (ii) That a person age 60 will die within 5 years, whereas another person age 55 will live for 5 years is .05.
   (iii) That a person age 35 will live 30 years is .60.

Find the probability that a person age 35 will die between ages 55 and 60.

7-12. The probability that a person age 10 will survive to age 30 is .80. Sixty per cent of the deaths between ages 10 and 40 occur after age 30. The probability that three lives aged 30, 50 and 70 will all survive for 20 years is .20. Find ${}_{50}p_{40}$.

7.3 Analytic Formulae for ℓ_x

7-13. Given $\ell_x = 1000\left(1 - \frac{x}{120}\right)$, determine each of the following:
   (a) ℓ_0 (b) ℓ_{120} (c) d_{33} (d) ${}_{20}p_{30}$ (e) ${}_{30}q_{20}$
   (f) The probability that a 25-year-old lives for at least 20 years and at most 25 years.
   (g) The probability that three 25-year-olds all survive to age 80.

7-14. For the ℓ_x given in Question 13, calculate general formulae for p_x, q_x and μ_x. Then sketch graphs of all four functions.

7-15. Prove that in the general case of de Moivre's formula, given by (7.7), we have $\mu_x = \frac{1}{\omega - x}$.

7-16. Obtain an expression for μ_x if $\ell_x = ks^x w^{x^2} g^{c^x}$. (This is called Makeham's second formula.)

7-17. Show that Gompertz' formula for ℓ_x implies $p_x = g^{c^x(c-1)}$.

7-18. Show that Makeham's formula for ℓ_x implies $p_x = sg^{c^x(c-1)}$.

7-19. (a) Starting with Gompertz' formula $\ell_x = kg^{c^x}$, verify that the force of mortality is $\mu_x = Bc^x$ for suitable B.

(b) Starting with Makeham's formula $\ell_x = ks^x g^{c^x}$, verify that $\mu_x = A + Bc^x$ for suitable A and B.

7-20. Show that, under de Moivre's formula, $_{n|}q_x$ is independent of n. (See Question 7-10 for the definition of this notation.)

7-21. If $\ell_x = 100{,}000\left(\frac{c-x}{c+x}\right)$ and $\ell_{35} = 44{,}000$, find each of the following:

(a) The value of c.

(b) The terminal age in the life table.

(c) The probability of surviving from birth to age 50.

(d) The probability that a person aged 15 will die between age 40 and age 50.

7-22. If $\ell_x = 250\,(64 - .80x)^{1/3}$, find each of the following:

(a) $_{70}p_0$

(b) μ_{70}

(c) The terminal age of the population.

7-23. If $\mu_x = .0017$ for $20 \leq x \leq 30$, find each of the following:

(a) p_{20} (b) $_5p_{20}$ (c) q_{23} (d) $_5q_{23}$ (e) $_{4|}q_{23}$ (f) $_{4|3}q_{23}$

7-24. Show that if uniform distribution of deaths over year of age x is assumed, then $_tp_x\mu_{x+t} = q_x$ for all $0 < t < 1$.

7-25. During the first 12 months of life, infants in a developing country are subject to a force of mortality given by $\mu_x = \frac{1}{3+x}$, where x is measured in months. Calculate the probability that a newborn will survive for 4 months but not for 7 months.

7-26. (a) Show that $_np_x = e^{-\int_x^{x+n}\mu_r\,dr} = e^{-\int_0^n \mu_{x+t}\,dt}$.

(b) Use (7.8a) to derive the formula $_nq_x = \displaystyle\int_0^n {}_tp_x\mu_{x+t}\,dt$.

7-27. If x is fixed and t is a variable, show that $\mu_{x+t} = \dfrac{-D_t(_tp_x)}{_tp_x}$.

CHAPTER EIGHT
LIFE ANNUITIES

8.1 BASIC CONCEPTS

In Section 6.3 we saw how to calculate the present value of contingent payments, and in Chapter 7 we learned how probabilities concerning survival can be calculated from life tables (or, occasionally, from an analytic formula). In the next two chapters we will combine these ideas to solve problems involving payments which are contingent on either survival or death.

Example 8.1

Yuanlin is 38 years old. If he reaches age 65, he will receive a single payment of 50,000. If $i = .12$, find an expression for the value of this payment to Yuanlin today.

Solution

The probability of survival to age 65 is ${}_{27}p_{38}$. Hence the answer is $50{,}000({}_{27}p_{38})(1.12)^{-27}$. □

To obtain a numerical answer to Example 8.1 we could consult life tables. If, for example, $\ell_{38} = 8327$ and $\ell_{65} = 5411$, then we would have ${}_{27}p_{38} = \frac{5411}{8327} = .64981$, so the value would be equal to $50{,}000(.64981)(1.12)^{-27} = 1523.60$. On the other hand, if we assumed in Example 8.1 that $\ell_x = \ell_0\left[1 - \frac{x}{105}\right]$, then ${}_{27}p_{38} = \frac{\ell_{65}}{\ell_{38}} = \frac{40}{67}$, and the value would be $50{,}000\left(\frac{40}{67}\right)(1.12)^{-27} = 1399.81$.

Example 8.1 is an illustration of what is called a *pure endowment*, and a formula for the general case is easy to find. Assume that a unit of money is to be paid t years from now to an individual currently aged x, if the individual survives to that time. The value of this payment at the present time is equal to

$$ {}_tE_x = ({}_tp_x)(1+i)^{-t} = v^t\,{}_tp_x. \tag{8.1}$$

This important expression will be used to move payment values from one time point to another in the rest of this text. The present value of a pure endowment is also called the *net single premium* for the pure endowment.

A more common type of situation is called a *life annuity*. Example 6.14 was one illustration of a life annuity, and here is another.

Example 8.2

Aretha is 27 years old. Beginning one year from today, she will receive 10,000 annually for as long as she is alive. Find an expression for the present value of this series of payments assuming $i = .09$.

Solution

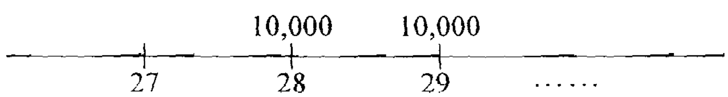

FIGURE 8.1

We can view this as a series of pure endowments of the type described in the previous example. Thus the answer is

$$10{,}000(p_{27})(1.09)^{-1} + 10{,}000({}_2p_{27})(1.09)^{-2} + \cdots$$
$$= \sum_{k=1}^{\infty} (10{,}000)({}_kp_{27})(1.09)^{-k}.$$

Although this appears to be an infinite sum, in practice it will be finite since ${}_kp_{27} = 0$ eventually. □

Now, however, we have a serious problem. If ${}_kp_{27}$ is obtained from a life table for each k, it appears that there is no nice way of calculating this sum. Unlike our examples in Chapter 3, where the terms formed a geometric sequence, we would here have to resort to adding terms one at a time.

To get around this difficulty life tables have columns in which these terms have already been added together. In other words, there are life tables constructed in conjunction with a number of commonly encountered rates of interest, and for various values of x and i it will be possible to look up such sums in the tables. The name given to these sums is *commutation functions*, and we will study them carefully in the next section.

A general formula for a life annuity is easily written down. As with interest-only annuities, we will assume constant payments of 1 per year for as long as the individual is alive, with the first payment due at the end if the year, as illustrated in Figure 8.2.

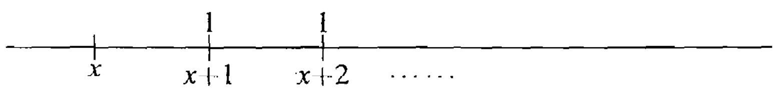

FIGURE 8.2

The symbol for the present value of these payments to a life aged x is a_x, and the formula is

$$\begin{aligned} a_x &= (1+i)^{-1}p_x + (1+i)^{-2}\,{}_2p_x + \cdots \\ &= \sum_{t=1}^{\infty} (1+i)^{-t}\,{}_tp_x \\ &= \sum_{t=1}^{\infty} v^t\,{}_tp_x. \end{aligned} \tag{8.2}$$

Again we remark that the present value is also called the *net single premium* for the annuity. (In some texts the phrase *actuarial present value* is also used; in this text we will continue to use the simpler *present value*.)

If we have a formula for ℓ_x or p_x, we might be able to sum the above series algebraically.

Example 8.3

Consider Aretha's life annuity in Example 8.2. Find the net single premium for this annuity in each of the following cases:

(a) $p_x = .95$ for each x

(b) $\ell_x = \ell_0\left[1 - \frac{x}{105}\right]$

Solution

(a) Since ${}_kp_x = (.95)^k$ for all k, the required value is

$$\begin{aligned}
10{,}000a_{27} &= \sum_{k=1}^{\infty} (10{,}000)(.95)^k(1.09)^{-k} \\
&= \sum_{k=1}^{\infty} (10{,}000)\left[\frac{.95}{1.09}\right]^k \\
&= \frac{10{,}000\left(\frac{.95}{1.09}\right)}{1-\frac{.95}{1.09}} = 67{,}857.14.
\end{aligned}$$

(b) In this case, ${}_kp_x = \dfrac{\ell_{x+k}}{\ell_x} = \dfrac{105-x-k}{105-x}$ for all k, so we have $10{,}000\left[\left(\frac{77}{78}\right)(1.09)^{-1} + \left(\frac{76}{78}\right)(1.09)^{-2} + \cdots\right] = \frac{10{,}000}{78}(Da)_{\overline{77}|.09}$, since payments will be 0 after 77 years. From Formula (3.29), we have $\dfrac{10{,}000}{78}\left[\dfrac{77-a_{\overline{77}|}}{.09}\right] = 93{,}879.59.$ □

Now let us introduce a bit more notation. Perhaps a life annuity has payments which will end after a certain period.

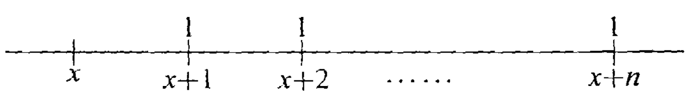

FIGURE 8.3

A *temporary* life annuity which will only continue for a maximum of n years is denoted $a_{x:\overline{n}|}$, and the formula is

$$\begin{aligned}
a_{x:\overline{n}|} &= \sum_{t=1}^{n} (1+i)^{-t}\,{}_tp_x \\
&= \sum_{t=1}^{n} v^t\,{}_tp_x. \qquad (8.3)
\end{aligned}$$

An n-year *deferred* life annuity is one in which the first payment to a person now aged x does not occur until age $x+n+1$.

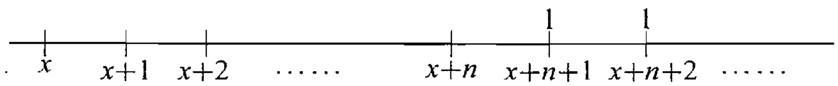

FIGURE 8.4

This is denoted by ${}_{n|}a_x$, and we have

$$
\begin{aligned}
{}_{n|}a_x &= \sum_{t=1}^{\infty} (1+i)^{-n-t}\,{}_{n+t}p_x \\
&= \sum_{t=n+1}^{\infty} v^t\,{}_{t}p_x. \qquad (8.4)
\end{aligned}
$$

We note that ${}_{n|}a_x$ can be thought of as omitting the first n payments from a_x, so we have

$$
{}_{n|}a_x = a_x - a_{x:\overline{n}|}. \qquad (8.5)
$$

As with interest-only annuities in Chapter 3, there are *annuities-due* in life contingencies as well as annuities-immediate. The notation is analogous: $\ddot{a}_x$ denotes a life annuity whose first payment occurs immediately.

Time diagrams for $\ddot{a}_x$, $\ddot{a}_{x:\overline{n}|}$, and ${}_{n|}\ddot{a}_x$ are shown below.

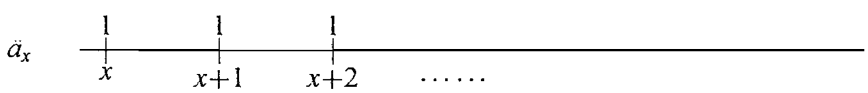

FIGURE 8.4a

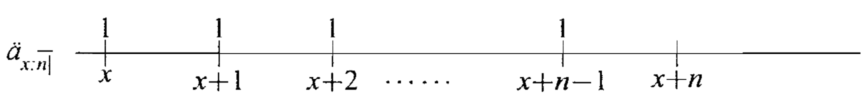

FIGURE 8.4b

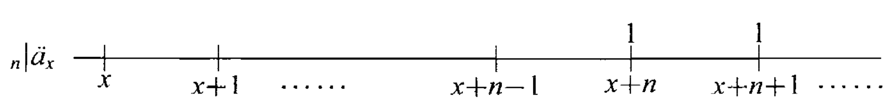

FIGURE 8.4c

Hence we have

$$\ddot{a}_x = 1 + a_x, \tag{8.6}$$

$$\ddot{a}_{x:\overline{n}|} = 1 + a_{x:\overline{n-1}|}, \tag{8.7}$$

and

$$_{n|}\ddot{a}_x = {}_{n-1|}a_x. \tag{8.8}$$

However, the reader should be careful! Because of the uncertainty of payments in life annuities, other formulae do not carry over directly. For instance, the identity $\ddot{a}_{\overline{n}|} = (1+i)\,a_{\overline{n}|}$ does *not* extend to life annuities $\ddot{a}_{x:\overline{n}|} = (1+i)\,a_{x:\overline{n}|}$. To obtain the correct analogue, we note that

$$\begin{aligned}
\ddot{a}_{x:\overline{n}|} &= 1 + \sum_{t=1}^{n-1} (1+i)^{-t}\,{}_tp_x \\
&= (1+i)\left[(1+i)^{-1} + \sum_{t=1}^{n-1} (1+i)^{-t-1}\,{}_tp_x\right]
\end{aligned}$$

Therefore

$$\begin{aligned}
p_{x-1}\ddot{a}_{x:\overline{n}|} &= (1+i)\left[(1+i)^{-1}p_{x-1} + \sum_{t=1}^{n-1} (1+i)^{-t-1}\,{}_tp_x\,p_{x-1}\right] \\
&= (1+i)\left[(1+i)^{-1}p_{x-1} + \sum_{t=1}^{n-1} (1+i)^{-t-1}\,{}_{t+1}p_{x-1}\right] \\
&= (1+i)\left[\sum_{t=1}^{n}(1+i)^{-t}\,{}_tp_{x-1}\right] \\
&= (1+i)\,a_{x-1:\overline{n}|}.
\end{aligned}$$

This is usually written as

$$a_{x-1:\overline{n}|} = v\,p_{x-1}\,\ddot{a}_{x:\overline{n}|}. \tag{8.9}$$

The reader should try to give a verbal explanation for this identity. Other relationships similar to (8.9) are presented in the exercises.

EXERCISES

8.1 Basic Concepts

8-1. Find the net single premium for a 28-year pure endowment of 20,000 sold to a male aged 36, in each of the following cases. Assume $i = .12$.

(a) $\ell_{36} = 9618$, $\ell_{64} = 7100$

(b) $p_x = .96$ for all x

(c) $p_x = .98$ if $0 \leq x < 40$, $p_x = .95$ if $40 \leq x < 70$

(d) $\ell_x = \ell_0 \left[1 - \frac{x}{110}\right]$

(e) $\ell_x = \sqrt{100 - x}$

8-2. Henri, 11 years old, wins first prize in the Parisian Lottery. He can have 1,000,000 francs if alive at age 21, or X francs today. Find X if $i = .13$ and ${}_{10}p_{11} = .975$.

8-3. Elaine, aged 30, purchases a contract which provides for three payments of 2000 each at ages 40, 50 and 55, if she is alive. Given $\ell_x = 110 - x$ and $i = .09$, find the net single premium for this contract.

8-4. Find the net single premium for a life annuity of 5000 per year, with the first payment due in one year, sold to a 30-year-old in each of the following cases.

(a) $p_x = .96$ for each x, and $i = .09$.

(b) $\ell_x = 1000\left[1 - \frac{x}{115}\right]$, and $i = .13$.

8-5. Do Question 4 if the first payment is deferred until age 40.

8-6. Do Question 4 if the maximum number of payments will be 40.

8-7. Find an expression for the present value of a life annuity which will pay 500 at the end of every two years if $i = .13$ and the annuity is sold to a person aged 45.

8-8. Do Question 7 if $p_x = .96$ for all x.

8-9. Do Question 7 if $\ell_x = \ell_0\left[1 - \frac{x}{115}\right]$.

8-10. For a given population, $\ell_x = 120 - x$. Given that $i = .07$, find the net single premium at age 60 for a deferred life annuity with annual payments of 1000 commencing at age 70 if at most twenty payments will be made.

8-11. Derive each of the following identities.

(a) $a_x = v\,p_x\,\ddot{a}_{x+1}$

(b) $\ddot{a}_x = 1 + v\,p_x\,\ddot{a}_{x+1}$

(c) $\ddot{a}_{x:\overline{n}|} = a_{x:\overline{n}|} + 1 - v^n\,{}_np_x$

(d) ${}_{n|}a_x = v^n\,{}_np_x\,a_{x+n}$

(e) $a_{x:\overline{n+m}|} = a_{x:\overline{m}|} + v^m\,{}_mp_x\,a_{x+m:\overline{n}|}$

(f) $p_{x-1} \cdot \ddot{a}_x = (1+i)\,a_{x-1}$

8-12. Give verbal explanations for each of the identities in Question 11.

8-13. Julio's mortality for $1 \le t \le 4$ is governed by ${}_tp_x = .3(4-t)$, and Harold's mortality for $1 \le t \le 5$ is governed by ${}_tp_x = .25(5-t)$. If $i = .07$, find the value at time 0 of an annuity which pays 1000 at the end of each year as long as *both* Julio and Harold are alive.

8-14. Repeat Question 13 if the annuity is paid as long as *either* Julio or Harold is still alive.

8-15. Assume in Question 13 that Noreen's mortality is also governed by ${}_tp_x = .3(4-t)$.

(a) Find the value at $t = 0$ of a life annuity paying 1000 at the end of each year as long as *at least two* of Julio, Harold and Noreen survive.

(b) Do part (a) if the annuity is paid only if *at most two* of Julio, Harold and Noreen survive.

8-21. Harold, aged 60, purchases a life annuity which will provide annual payments of 1000 commencing at age 61. For the year beginning at age 60 only, Harold is subject to a higher risk of death, namely $q_{60}+.10$, where q_{60} is from the standard life table. Given $N_{60}=4650$, $N_{61}=3950$, $N_{62}=3350$ and $i=.07$, find the net single premium for this annuity.

8-22. A select-and-ultimate disabled life table has a select period of two years. Select probabilities are related to ultimate probabilities by the rules $p_{[x]}=\frac{1}{3}\cdot p_x$ and $p_{[x]+1}=\frac{1}{2}\cdot p_{x+1}$. Given $\ddot{a}_{25}=17$, $D_{25}=2000$, and $D_{26}=1800$, find $a_{[25]}$.

8-23. Assume $i=.08$ and that we are dealing with a four-year select period. We know that $q_{[30]}=.40$, $p_{[30]+1}=.80$, $q_{[30]+2}=.10$ and $q_{[30]+3}=.10$. Also $D_{34}=1000$ and $D_{35}=920$. Find the probability that a person entering the select group at age 30 (a) will remain in the population for 5 years; (b) will remain in the population for at most 3 years.

8.3 Annuities Payable m^{thly}

8-24. (a) How much money must be invested to provide John, aged 50, with monthly payments of 400 for life if $a_{50}=16.5$? The first payment will occur in exactly one month.

(b) Repeat part (a) if the payment is 1200 every three months, the first payment occurring in exactly three months.

(c) Repeat part (a) if the first 400 payment occurs immediately.

(d) Can you solve part (a) if the first payment occurs in exactly 13 months? If not, what additional information is required?

8-25. Derive each of the following approximate formulae:

(a) $_{n|}\ddot{a}_x^{(m)} = {}_{n|}\ddot{a}_x - \left(\frac{m-1}{2m}\right){}_nE_x$

(b) $\ddot{a}_{x:\overline{n}|}^{(m)} = \ddot{a}_{x:\overline{n}|} - \left(\frac{m-1}{2m}\right)(1-{}_nE_x)$

8-26. Explain verbally why the formula $\ddot{a}_{x:\overline{n}|}^{(m)} = \frac{1}{m}+\ddot{a}_{x:\overline{n}|}$ is incorrect.

8-27. Derive each of the following:

(a) $$a_x^{(m)} = \frac{N_{x+1} + \left(\frac{m-1}{2m}\right)D_x}{D_x} = \frac{N_x^{(m)}}{D_x} - \frac{1}{m}.$$

(b) $${}_{n|}\ddot{a}_x^{(m)} = \frac{N_{x+n}^{(m)}}{D_x}$$

8-28. Express $\ddot{a}^{(m)}_{x:\overline{n}|}$ in terms of commutation functions. Give two answers, one involving $N_y^{(m)}$ terms, and one using regular N_y terms.

8-29. Marvin, aged 50, purchases an annuity of k per month, the first payment to be made immediately. For the first 60 months, payments will be guaranteed (i.e., will be made independent of Marvin's survival). After that, payments will continue for as long as Marvin is alive. Marvin pays 50,000 for the entire package. Let $i = .07$, $D_{50} = 5200$, $D_{55} = 4100$, and $N_{55} = 60{,}000$. Find k.

8-30. Repeat Question 29 if no payments are guaranteed. Assume, in addition, that $N_{50} = 83{,}500$.

8-31. Jeannette, aged 65, is about to retire. Her salary is 70,000 per year and, because of her long service and senior position, she will receive a pension of 50% of her final salary at the end of each year as long as she survives. Find the present value of these benefits if $N_{65} = 700$ and $D_{65} = 82$.

8-32. Repeat Question 31 if the yearly payment is to be divided into twelve equal monthly payments, the first occurring in one month.

8-33. Marilyn, aged 45, works for the same company as Jeannette (Question 31) and will have the same retirement benefit, beginning at age 65. Marilyn's current salary is 25,000 per year and her salary will increase at 7% each year for the next 20 years. If $i = .08$, and it is assumed that Marilyn will not die before age 65, find the present value to Marilyn of her future retirement benefits.

8-34. Repeat Question 33 if the yearly pension payment is to be divided into twelve equal monthly payments, the first occurring one month after retirement.

8-35. Repeat Questions 33 and 34 if death is possible before retirement, and we assume that ${}_{20}p_{45} = .84$.

CHAPTER NINE
LIFE INSURANCE

9.1 BASIC CONCEPTS

In the previous chapter we saw how techniques from the theory of interest can be combined with elementary probability theory to study life annuities, which are annuities contingent upon survival. Now we will see how the same ideas can be used to study life insurance, where the contingency of interest is that of dying at certain times in the future. We saw one example of this type of problem in Section 6.3. Here is another.

Example 9.1

Rose is 38 years old. She wishes to purchase a life insurance policy which will pay her estate 50,000 at the end of the year of her death. If $i = .12$, find an expression for the (actuarial) present value of this benefit.

Solution

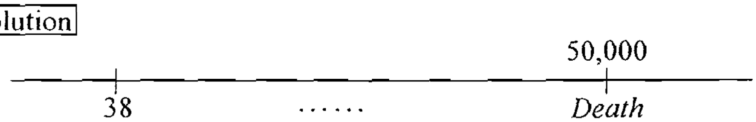

FIGURE 9.1

The expected value of this benefit payable $t+1$ years hence is the probability that Rose dies at age last birthday $38+t$ multiplied by 50,000, which is $({}_tp_{38})(q_{38+t})(50{,}000)$. The present value of the entire policy is the sum of the present values of these terms, which is $50{,}000\sum_{t=0}^{\infty}({}_tp_{38})(q_{38+t})(1.12)^{-t-1}$. Note that v^{t+1} is required since payment is at the end of the t^{th} year. □

To obtain a numerical answer to Example 9.1, we could consult life tables but, as with life annuities, it would be laborious to add all the terms together. We will see in Section 9.2 that commutation functions can be used to aid in the calculation. On the other hand, if a simple formula for p_x is assumed, it may be that this sum can be calculated. For example, if $p_x = .94$ for all x, then ${}_tp_{38} = (.94)^t$ from which we find $q_{38+t} = 1 - p_{38+t} = .06$, so our present value is

$$\begin{aligned} 50{,}000\sum_{t=0}^{\infty}(.94)^t(.06)(1.12)^{-t-1} &= 50{,}000\left(\frac{.06}{1.12}\right)\sum_{t=0}^{\infty}\left(\frac{.94}{1.12}\right)^t \\ &= 50{,}000\left(\frac{.06}{1.12}\right)\left(\frac{1}{1-\frac{.94}{1.12}}\right) \\ &= 16{,}666.67. \end{aligned}$$

Example 9.1 is an illustration of a *whole life policy*, a policy where a fixed amount, the *face value*, is paid to the insured's beneficiary at the end of the year of death, whenever that may be. The price of such a policy with face value of 1, to an insured aged x, is given by the symbol A_x. The formula is

$$A_x = \sum_{t=0}^{\infty} {}_tp_x\, q_{x+t} v^{t+1}. \tag{9.1}$$

Note that eventually ${}_tp_x = 0$, so this sum is actually finite. It will be assumed for the rest of Sections 9.1 and 9.2, as well as in the exercises for these sections, that insurances are payable at the end of the year of death.

Example 9.2

Michael is 50 years old and purchases a whole life policy with face value 100,000. If $\ell_x = 1000\left(1 - \frac{x}{105}\right)$ and $i = .08$, find the price of this policy.

Solution

The required price is $100{,}000A_{50} = 100{,}000\sum_{t=0}^{54} {}_tp_{50}\; q_{50+t}\; (1.08)^{-t-1}$. Note that our sum terminates at $t = 54$ because ${}_tp_{50} = 0$ for all larger values. We have ${}_tp_{50} = \frac{\ell_{50+t}}{\ell_{50}} = \frac{105-50-t}{105-50} = \frac{55-t}{55}$, and $q_{50+t} = 1 - p_{50+t} = 1 - \frac{105-51-t}{105-50-t}$. Hence the premium is

$$\begin{aligned} &100{,}000\sum_{t=0}^{54}\left(\frac{55-t}{55}\right)\left(\frac{1}{55-t}\right)(1.08)^{-t-1} \\ &\qquad = \left(\frac{100{,}000}{55}\right)\left(\frac{1}{1.08}\right)\left(\frac{1-\left(\frac{1}{1.08}\right)^{55}}{1-\left(\frac{1}{1.08}\right)}\right) = 22{,}397.48. \quad \square \end{aligned}$$

In some cases a company will sell *term insurance*, which means that the face value is paid only if death occurs within a prescribed period. If the period is n years and the insured is aged x, then the price is denoted $A^{1}_{x:\overline{n|}}$ (for a payment of 1), and the formula is

$$A^{1}_{x:\overline{n|}} = \sum_{t=0}^{n-1} {}_tp_x q_{x+t}\, v^{t+1}. \tag{9.2}$$

Example 9.3

Calculate the price of Rose's insurance in Example 9.1 and Michael's insurance in Example 9.2 if both policies are in force for a term of only 30 years. For Rose, assume $p_x = .94$ for all x.

Solution

In Rose's case, the price is

$$\begin{aligned} 50{,}000 \sum_{t=0}^{29} {}_tp_{38}\, q_{38+t}\, (1.12)^{-t-1} &= 50{,}000 \sum_{t=0}^{29} (.94)^t(.06)(1.12)^{-t-1} \\ &= 50{,}000\left(\frac{.06}{1.12}\right)\left(\frac{1-\left(\frac{.94}{1.12}\right)^{30}}{1-\frac{.94}{1.12}}\right) \\ &= 16{,}579.74. \end{aligned}$$

In Michael's case, we obtain

$$\begin{aligned} &100{,}000\sum_{t=0}^{29} {}_tp_{50}\, q_{50+t}\, (1.08)^{-t-1} \\ &\quad= 100{,}000\sum_{t=0}^{29}\left(\frac{55-t}{55}\right)\left(\frac{1}{55-t}\right)(1.08)^{-t-1} \\ &\quad= \left(\frac{100{,}000}{55}\right)\left(\frac{1}{1.08}\right)\left(\frac{1-\left(\frac{1}{1.08}\right)^{30}}{1-\left(\frac{1}{1.08}\right)}\right) \\ &\quad= 20{,}468.70. \end{aligned}$$

□

We could also talk about ${}_{n|}A_x$, the price for *deferred insurance*, where the policy of face amount 1 is purchased at age x but does not come into force until age $x+n$. This is not as important as the other two cases, so we will not stress it here, but it should be noted that

$$A_x = A^{1}_{x:\overline{n|}} + {}_{n|}A_x . \tag{9.3}$$

Finally, there is n-year *endowment insurance.* In this case the face value is paid if death occurs within a prescribed n-year period or, if the policyholder is still alive at the end of n years, he receives the face value at that time. The price for this benefit, with face value 1, is denoted $A_{x:\overline{n|}}$, and is the sum of n-year term insurance and a pure endowment (see Section 8.1) at age $x+n$. Hence we have

$$A_{x:\overline{n|}} = A^{1}_{x:\overline{n|}} + {}_nE_x. \tag{9.4}$$

In this context the symbol $A_{x:\overline{n|}}^{\ \ \ 1}$ is sometimes used in place of ${}_nE_x$.

Example 9.4

Calculate the price of Rose's insurance in Example 9.1 and Michael's insurance in Example 9.2 if both policies are to be 30-year endowment insurance. For Rose, assume $p_x = .94$ for all x.

Solution

We will use the results of Example 9.3 and simply add a 30-year pure endowment in each case. For Rose, $16{,}579.74 + 50{,}000({}_{30}p_{38})(1.12)^{-30}$, which is $16{,}579.74 + 50{,}000(.94)^{30}(1.12)^{-30} = 16{,}840.52$. For Michael, we obtain $20{,}468.70 + (100{,}000)({}_{30}p_{50})(1.08)^{-30}$, which is $20{,}468.70 + (100{,}000)(\frac{25}{55})(1.08)^{-30} = 24{,}985.85$. □

The reader will show in the exercises that

$$A^{1}_{x:\overline{n|}} < A_x < A_{x:\overline{n|}}. \tag{9.5}$$

Note that the examples given in this section support these inequalities.

9.2 COMMUTATION FUNCTIONS AND BASIC IDENTITIES

We begin this section by introducing two new commutation functions, which will be helpful in solving problems involving insurance. We will also develop some nice relationships between the insurance symbols A_x, Example 9.8

Phyllis, aged 40, purchases a whole life policy of 50,000. If $N_{40} = 5000$, $N_{41} = 4500$ and $i = .08$, find the price of this policy.

Solution

We have

$$A_{40} = 1 - d\ddot{a}_{40} = 1 - \left(\frac{.08}{1.08}\right)\left(\frac{N_{40}}{D_{40}}\right)$$
$$= 1 - \left(\frac{.08}{1.08}\right)\left(\frac{5000}{500}\right).$$

Thus the net single premium is $50{,}000A_{40} = 12{,}962.96$. □

There are many other identities which can be similarly derived, and we will leave these for the exercises.

9.3 INSURANCE PAYABLE AT THE MOMENT OF DEATH

Until now, we have always assumed that insurance is payable at the end of the year following death. In practice, however, this is often not the case and it is more common for insurance to be payable at the moment of death. We will now see how to deal with problems of this type.

In a similar manner to Section 8.3, we first consider the case where a payment of 1 is due at the end of the $\frac{1}{m}^{th}$ part of a year in which death occurs. The net single premium for this insurance to a life aged x is given by

$$A_x^{(m)} = v^{\frac{1}{m}}\left(\frac{\ell_x - \ell_{x+\frac{1}{m}}}{\ell_x}\right) + v^{\frac{2}{m}}\left(\frac{\ell_{x+\frac{1}{m}} - \ell_{x+\frac{2}{m}}}{\ell_x}\right) + \cdots. \tag{9.13}$$

Taking the limit of this expression as m approaches infinity, we obtain the net single premium for insurance payable at the moment of death. Denoting this by $\overline{A}_x$, we have

$$\overline{A}_x = \lim_{m\to\infty} A_x^{(m)} = -\int_0^\infty v^t \cdot \frac{d(\ell_{x+t})}{\ell_x} = -\int_0^\infty v^t \cdot \frac{\ell_{x+t}}{\ell_x} \cdot \frac{d(\ell_{x+t})}{\ell_{x+t}}.$$

This gives us the important formula

$$\bar{A}_x = \int_0^\infty v^t {}_tp_x \mu_{x+t}\, dt. \tag{9.14}$$

Expressions like $\bar{A}^{\,1}_{x:\overline{n}|}, \bar{A}_{x:\overline{n}|}$, and so on, all exist and have the expected meanings. We immediately obtain

$$\bar{A}^{\,1}_{x:\overline{n}|} = \int_0^n v^t {}_tp_x \mu_{x+t}\, dt \tag{9.15}$$

and

$$\bar{A}_{x:\overline{n}|} = \int_0^n v^t {}_tp_x \mu_{x+t}\, dt + v^n {}_np_x. \tag{9.16}$$

Example 9.9

Find the net single premium for a 100,000 life insurance policy, payable at the moment of death, purchased by a life aged 30 if it is assumed that $i = .06$ and ${}_tp_{30} = (.98)^t$ for all t.

Solution

We have $\bar{A}_{30} = \displaystyle\int_0^\infty v^t {}_tp_{30} \mu_{30+t}\, dt.$

Since $\mu_{30+t} = -\dfrac{D({}_tp_{30})}{{}_tp_{30}} = -ln(.98)$, the answer is

$$-100{,}000 \int_0^\infty \left(\frac{.98}{1.06}\right)^t ln(.98)\, dt = -100{,}000 ln(.98) \left[\frac{\left(\frac{.98}{1.06}\right)^t}{ln\left(\frac{.98}{1.06}\right)}\right]_0^\infty$$

$$= \frac{100{,}000 ln(.98)}{ln\left(\frac{.98}{1.06}\right)} = 25{,}745.24. \qquad \square$$

Example 9.10

Repeat Example 9.9 if the 30-year-old is purchasing 20-year endowment insurance instead of whole life insurance.

Solution

Now we want

$$
\begin{aligned}
100{,}000\bar{A}_{30:\overline{20}|} &= 100{,}000(\bar{A}^{\;1}_{30:\overline{20}|} + v^{20}\,{}_{20}p_{30}) \\
&= 100{,}000\int_0^{20} v^t\,{}_tp_{30}\mu_{30+t}\,dt + 100{,}000v^{20}\,{}_{20}p_{30} \\
&= -100{,}000 ln(.98)\left[\frac{\left(\frac{.98}{1.06}\right)^t}{ln\left(\frac{.98}{1.06}\right)}\right]_0^{20} + 100{,}000\left(\frac{.98}{1.06}\right)^{20} \\
&= \frac{100{,}000 ln(.98)}{ln\left(\frac{.98}{1.06}\right)}\left[1-\left(\frac{.98}{1.06}\right)^{20}\right] + 100{,}000\left(\frac{.98}{1.06}\right)^{20} \\
&= 41{,}202.36.
\end{aligned}
$$

□

Example 9.11

Repeat Examples 9.9 and 9.10 if we now assume $\delta = .06$ and $\ell_x = 105 - x$, $0 \le x \le 105$.

Solution

We now have ${}_tp_{30} = \frac{75-t}{75}$ and $\mu_{30+t} = \frac{1}{75-t}$ for $0 \le t \le 75$. Then

$$
\begin{aligned}
100{,}000\bar{A}_{30} &= 100{,}000\int_0^{75} e^{-.06t}\left(\frac{1}{75}\right)dt \\
&= \frac{100{,}000}{75}\left[\frac{e^{-.06t}}{-.06}\right]_0^{75} \\
&= 21{,}975.36.
\end{aligned}
$$

For the endowment insurance,

$$
\begin{aligned}
100{,}000\bar{A}_{30:\overline{20}|} &= 100{,}000\int_0^{20} e^{-.06t}\left(\frac{1}{75}\right)dt + 100{,}000e^{-.06(20)}\left(\frac{55}{75}\right) \\
&= \frac{100{,}000}{75}\left[\frac{1-e^{-1.2}}{.06}\right] + 100{,}000e^{-1.2}\left(\frac{55}{75}\right) \\
&= 37{,}616.59.
\end{aligned}
$$

□

In Section 8.3, the approximate formula $\bar{a}_x = a_x + \frac{1}{2}$ was derived, along with the corresponding formula for commutation functions $\overline{N}_x = N_x - \frac{1}{2}D_x$. In the case of insurance payable at the moment of death, the relationships are the very different looking $\bar{A}_x = \frac{i}{\delta}A_x$ and $\overline{M}_x = \frac{i}{\delta}M_x$. (See Exercise 9-39.)

Using integration by parts on the expression for $\bar{A}_x$ given by (9.14), we obtain the important identity

$$\bar{A}_x = 1 - \delta\bar{a}_x. \tag{9.17}$$

The reader should note the similarity of this to the formula $A_x = 1 - d\ddot{a}_x$ given by Formula (9.12b).

9.4 VARYING INSURANCE

In Section 3.6 we studied interest-only varying annuities such as $(Ia)_{\overline{n}|}$ and $(Da)_{\overline{n}|}$, and in Section 8.4 the corresponding life annuities, such as $(Ia)_x$, were introduced. Now we consider the analogous situation for life insurance.

We will denote *increasing whole life insurance* by $(IA)_x$. This is a policy which provides a death benefit of 1 in the first year, 2 in the second year, and so on, increasing by 1 per year, payable at the end of the year of death. We note that

$$\begin{aligned}
(IA)_x &= \sum_{t=0}^{\infty}(t+1)\,{}_tp_x\, q_{x+t}\, v^{t+1} \\
&= \sum_{t=0}^{\infty}(t+1)\frac{C_{x+t}}{D_x} \\
&= \frac{1}{D_x}(C_x + 2C_{x+1} + 3C_{x+2} + \cdots) \\
&= \frac{1}{D_x}\Big((C_x + C_{x+1} + \cdots) + (C_{x+1} + C_{x+2} + \cdots) + \cdots\Big) \\
&= \frac{1}{D_x}(M_x + M_{x+1} + M_{x+2} + \cdots), \qquad (9.18)
\end{aligned}$$

which leads to

$$ {}_tV_x = \frac{P_{x+t} - P_x}{P_{x+t} + d}, \tag{9.36} $$

since $\frac{1}{\ddot{a}_{x+t}} = P_{x+t} + d$.

Sometimes it is important to know the connection between the terminal reserves of two successive policy years. Problems of this type can either be worked out from first principles or by using the formula we will now derive. Again recall that ${}_tV_x = A_{x+t} - P_x \cdot \ddot{a}_{x+t}$. We note that A_{x+t} can be thought of as the value of the first year's benefit, vq_{x+t}, plus the present value of the remaining benefits, which is given by $vp_{x+t} \cdot A_{x+t+1}$. Similarly, $\ddot{a}_{x+t} = 1 + vp_{x+t} \cdot \ddot{a}_{x+t+1}$. Thus we have

$$ \begin{aligned} {}_tV_x &= vq_{x+t} + vp_{x+t} \cdot A_{x+t+1} - P_x(1+vp_{x+t} \cdot \ddot{a}_{x+t+1}) \\ &= vp_{x+t}(A_{x+t+1} - P_x \cdot \ddot{a}_{x+t+1}) + vq_{x+t} - P_x \\ &= (vp_{x+t}) \cdot {}_{t+1}V_x + vq_{x+t} - P_x \,. \end{aligned} \tag{9.37} $$

This formula can be used either to obtain ${}_tV_x$ given ${}_{t+1}V_x$, or to go in the other direction. A nicer symmetrical form is obtained by transposing P_x and multiplying by $1 + i$, obtaining

$$ (1 + i)({}_tV_x + P_x) = q_{x+t} + p_{x+t}({}_{t+1}V_x). \tag{9.38} $$

EXERCISES

9.1 Basic Concepts

9-1. Find the price of whole life insurance with a face value of 100,000 sold to a person aged 40 in each of the following cases.

(a) $p_x = .96$ for all x and $i = .09$.

(b) $\ell_x = 1000\left(1 - \frac{x}{115}\right)$ and $i = .13$.

9-2. Repeat Question 1 if the payment of 100,000 is to be made at the end of the 5-year period in which death occurs.

9-3. Repeat Question 1 if the insurance is a term policy for 30 years.

9-4. Repeat Question 1 for 30-year endowment insurance.

9-5. Repeat Question 1 if the policy has a face value of 100,000 for the first 30 years only. If the insured survives to age 70, he is paid 70,000 and the remaining 30,000 is retained as a whole life benefit.

9-6. Repeat Question 1, if in each case, $i^{(12)} = .12$, but the benefits are still paid yearly.

9-7. Prove that the identity $A^{\,1}_{x:\overline{n}|} < A_x < A_{x:\overline{n}|}$ is true for all x and for all n.

9-8. Prove that $_{n}\,|\,A_x < A_x$ if $n \geq 1$. Give a verbal explanation for this inequality.

9-9. Julio's mortality for $1 \leq t \leq 4$ is assumed to be governed by the law $_tp_x = .3(4-t)$. Harold's mortality for $1 \leq t \leq 5$ is governed by $_tp_x = .25(5-t)$. If $i = .07$, find the price at time 0 of an insurance policy which will pay 100,000 at the end of the year in which the *second* of Julio or Harold dies.

9-10. Repeat Question 9 if the insurance is paid at the end of the year in which the *first* of the two men dies.

9-11. Repeat Question 9 if the policy is for 2-year endowment insurance.

9-12. Prove the identity $A_x = v(q_x + p_x A_{x+1})$.

9-13. Give a verbal argument for the identity in Question 12.

9-14. Prove that $A_x = (1-A_{x+n})A^{\,1}_{x:\overline{n}|} + A_{x:\overline{n}|} \cdot A_{x+n}$.

9-15. Obtain a formula for the net single premium at age x for an insurance policy which pays 1 at the end of 10 years if death occurs within that period, or at the end of the year of death if death occurs after 10 years.

9-16. Assume that a single rate of mortality, q_{x+n}, is increased to $q_{x+n} + k$ for some constant k. All other values of q_y remain unchanged. Show that A_x will be increased by the amount $kv^{n+1}\,{}_np_x(1 - A_{x+n+1})$.

9-17. The net single premium for a pure endowment of 10,000 issued at age x for n years is 8000 if the premium is to be returned in the event of death before age $x+n$. If the premium is not returned, the net single premium is 7000. Find the net single premium for a pure endowment of 10,000 issued at the same age and for the same period if half of the net premium is to be returned in the event of death.

9.2 Commutation Functions and Basic Identities

9-18. Herman, aged 45, purchases a whole life policy of 100,000. Find the net single premium if (a) $M_{45} = 250$ and $D_{45} = 520$; (b) $N_{45} = 8000$, $D_{45} = 520$ and $i = .04$.

9-19. Repeat Question 18(a) for a 20-year term insurance policy given $M_{65} = 35$.

9-20. Express the net single premium for the following policy, issued to a person aged 30, in terms of commutation functions: 50,000 if death occurs in the next 20 years, 100,000 if death occurs in the 20-year period after that, and 10-year endowment insurance of 50,000 after that.

9-21. Prove each of the following identities:

(a) $A_x = v - da_x$

(b) $A^{1}_{x:\overline{n}|} = v\ddot{a}_{x:\overline{n}|} - a_{x:\overline{n}|}$

(c) $A_{x:\overline{n}|} = v\ddot{a}_{x:\overline{n}|} - a_{x:\overline{n-1}|}$

(d) $A_{x:\overline{n}|} = 1 - d\ddot{a}_{x:\overline{n}|}$

(e) $a_{x:\overline{n}|} = \dfrac{v - A_{x:\overline{n+1}|}}{d}$

(f) $M_x = D_x - dN_x$

(g) $\dfrac{1 - ia_{x:\overline{t-1}|}}{1+i} = \dfrac{M_x - M_{x+t} + D_{x+t}}{D_x}$

9-22. (a) Find the rate of interest if $a_x = 15.5$ and $A_x = .25$.

(b) Given $M_x = 3000$, $M_{x+1} = 2800$ and $q_x = .01$, find D_{x+1}.

9-23. Ronald, aged 40, purchases a whole life policy paying 10,000 during the next 10 years and 20,000 during the ten years after that. If Ronald is still alive at age 60, he will receive 200 at the end of each month for the rest of his life. Given $M_{40} = 750$, $M_{50} = 600$, $M_{60} = 420$, $D_{40} = 4500$, $D_{60} = 1100$ and $N_{60} = 10{,}000$, find the net single premium for this policy.

9-24. Angela, aged 55, purchases a deferred life annuity of 4000 per year commencing at age 65. Before age 65 there is a 10,000 death benefit payable at the end of the year of death. Find the net single premium for this package given $N_{55} = 250$, $N_{56} = 230$, $N_{65} = 110$, $N_{66} = 95$ and $i = .03$.

9-25. Repeat Question 24 if the life annuity is guaranteed, payable to either Angela or her estate, for 50 years.

9.3 Insurance Payable at the Moment of Death

9-26. (a) Find the net single premium for a 50,000 life insurance policy, payable at the moment of death, to a life aged 40 if it is assumed that $i = .06$ and ${}_tp_{40} = (.97)^t$ for all t.

(b) Do part (a) if the 40-year-old is purchasing 30-year term insurance instead of whole life insurance.

(c) Do part (a) if the 40-year-old is purchasing 30-year endowment insurance instead of whole life insurance.

9-27. (a) Find the price of a 100,000 life insurance policy, payable at the moment of death, bought by a life aged 40 if it is assumed that $\delta = .05$ and $\mu_x = .04$ for all x.

(b) Do part (a) if the 40-year-old is purchasing 30-year term insurance instead of whole life insurance.

(c) Do part (a) if the 40-year-old is purchasing 30-year endowment insurance instead of whole life insurance.

9-28. Repeat Question 27 if we continue to use $\delta = .05$ but now assume that the life is subject to a de Moivre's survival function with terminal age 110.

9-29. Do Question 28(a) if we assume the insurance is to be deferred for 20 years.

9-30. George is informed that the net single premium for a 200,000 whole life insurance policy, payable at the moment of death, is 70,000. If George is subject to a constant force of mortality $\mu_x = .03$, and if δ is also constant, find the net single premium for a 5-year deferred life policy of 200,000.

9-31. Helen is informed that the net single premium for 100,000 of whole life insurance, payable at the moment of death, is 65,000. If Helen is subject to a constant force of mortality $\mu_x = .0275$, and if δ is also constant, find the actuarial present value of a whole life annuity of 5000 per year payable continuously to Helen.

9-32. Brenda purchases life insurance which will pay 100,000 if she dies during the next 5 years and 200,000 if she dies after that. The benefits are payable at the moment of death. It is known that $\delta = .08$, and $\mu_x = .04$ for the next 8 years and $\mu_x = .05$ thereafter. Find the net single premium for this insurance.

9-33. (a) If the force of interest is increased, but the force of mortality stays the same, does $\bar{A}_x$ increase or decrease? Explain your answer.

(b) If the force of mortality is increased, but the force of interest stays the same, does $\bar{A}_x$ increase or decrease? Explain your answer.

9-34. Using integration by parts, derive the formula $\bar{A}_x = 1 - \delta\bar{a}_x$.

9-35. (a) Derive the formula $\bar{A}^{\,1}_{x:\overline{n}|} = 1 - \delta\bar{a}_{x:\overline{n}|} - v^n\, {}_np_x$.

(b) Derive the formula $\bar{A}_{x:\overline{n}|} = 1 - \delta\bar{a}_{x:\overline{n}|}$.

9-36. Assuming a constant force of mortality μ_x and a constant force of interest δ, find n such that $\bar{A}_{x:\overline{n}|} = 2\bar{A}^{\,1}_{x:\overline{n}|}$.

9-37. Herb purchase a 100,000 whole life insurance policy, with the benefit payable at the moment of death. The policy pays an additional 50,000 if death occurs during the first 5 years due to specified cause (a). Assuming $\delta = .07$, the force of decrement due to cause (a) is .005, and the force of decrement due to causes other than (a) is .045, find the net single premium for Herb's insurance.

9-38. A 20-year-old purchases a 100,000 whole life policy with benefit payable at the moment of death. Given $\delta = .05$ and $\mu_x = .02$ for all x, find each of the following:
- (a) The net single premium for this policy.
- (b) A number X such that the insurance company is 80% certain that the value at time of purchase of their eventual payout will be less than or equal to X. (Such a number X is sometimes called the 80^{th} percentile of the present value of the benefit.)

9-39
- (a) Assuming uniform distribution of deaths throughout the year of age x, show that $\bar{A}^{\,1}_{x:\overline{1|}} = \frac{i}{\delta} \cdot A^{\,1}_{x:\overline{1|}}$.
- (b) Extend the result of part (a) to show that $\bar{A}_x = \frac{i}{\delta} \cdot A_x$.

9.4 Varying Insurance

(Note: Unless stated otherwise, all exercises in Sections 9.4 and 9.5 assume death benefits payable at the end of the year of death.)

9-40. Tim, aged 50, purchases a whole life policy. In the first year his benefit is 20,000, and benefits increase by 5000 per year thereafter. Find the net single premium given $D_{50} = 500$, $R_{50} = 2300$ and $M_{50} = 220$.

9-41. Repeat Question 40 if Tim's policy is only to last for 20 years. In addition to the above data, assume $M_{70} = 60$ and $R_{70} = 400$.

9-42. Repeat Question 40 if Tim's policy is to increase as stated up to and including age 70, and then remain constant at 120,000 per year thereafter. Use the data given in Questions 40 and 41.