Society of Actuaries' Textbook

ON

LIFE CONTINGENCIES

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the parameters are usually confined to the ranges shown below:

$$.001 < A < .003,$$
$$10^{-6} < B < 10^{-3},$$
$$1.08 < c < 1.12.$$

8. Select mortality tables

The mortality of a group of insured lives exhibits certain distinctive characteristics which derive from the special nature of such a group. Before an insurance policy is issued, the insurer must be satisfied that the applicant meets certain underwriting standards. Some applicants, because of health conditions or other factors, will not be offered insurance on a standard basis, and some others may be considered uninsurable. As a result of this selection process, a group of lives insured on a standard basis does not constitute a random group, but rather a select group all the members of which have initially satisfied certain criteria of insurability.

It follows that the mortality in such a group will vary not only by age, but also by the duration of the insurance. Thus, a group of lives just insured on a standard basis at age 30 will be subject to a lower rate of mortality during the following year than another group of lives aged 30 who were similarly insured a year ago and are now in the second year of insurance, and the mortality of both of these groups will be lower than that among a third group aged 30 who were insured two years ago at age 28.

If we write $q_{[x]+n}$ for the rate of mortality at attained age $x + n$ among a group of lives insured at age x, the tendency described above may be expressed mathematically by the series of inequalities

$$q_{[x]} < q_{[x-1]+1} < q_{[x-2]+2} < \cdots .$$

Normally the difference $q_{[x-n]+n} - q_{[x-n+1]+n-1}$ diminishes quite rapidly and becomes negligible for practical purposes after some years. Thus, there will probably be no appreciable variation in the mortality experienced by insured groups of the same age which have been insured for durations greater than 10 or 15 years. The period of time during which the effects of selection are still significant is called the *select period*.

This feature of the mortality among insured lives is one that

often be recognized in the construction of mortality tables. *mortality tables*, which show the mortality variation both by and duration, are therefore prepared.

completely select table would consist of a set of mortality one for each age at issue. It is not usually necessary to pre- tables in this extensive form, however, since it is generally ible to assume a uniform select period for all ages at issue, condense the table into a *select and ultimate* form. If it can be assumed, for example, that the effects of selection disappear in 3 years, the table can be shown in the select and ultimate form illustrated in Table 3. The first column, headed $l_{[x]}$, represents the number of lives insured at age x, the symbol $[x]$ denoting a life newly insured at age x. The second column, $l_{[x]+1}$, represents the number of survivors at age $x + 1$ of the $l_{[x]}$ lives insured at age x. The columns headed $l_{[x]+2}$ and l_{x+3} represent the corresponding numbers of survivors of the original $l_{[x]}$ lives at the two succeeding years of age. The select symbol $[x]$ is not used in the l_{x+3} column, since the effects of selection do not carry over into the fourth year, and l_{x+3} is therefore equally representative of the number of survivors of the $l_{[x]}$ lives insured 3 years previously, the $l_{[x-1]}$ lives insured 4 years previously, and so on, and this column constitutes an *ultimate* mortality table. The value of l_{x+4} is found directly below l_{x+3} in the ultimate column. The complete set of values for age at issue x is thus obtained by reading the select values horizontally starting with $l_{[x]}$ and the ultimate values vertically from l_{x+3}. At age 21, for example, there are 944,710 lives insured. The number of survivors at age 23 is found by reading $l_{[21]+2} = 941{,}916$ from the third column of the select portion of the table; and the number of survivors at age 25 is found by reading $l_{25} = 938{,}359$ from the ultimate column. Note that this same value for l_{25} also represents the number of survivors at age 25 of the $l_{[20]}$ lives insured at age 20 and of the $l_{[22]}$ lives insured at age 22.

In order to construct a select table in this form, it is first necessary to have values of $q_{[x]}$, $q_{[x]+1}$, and $q_{[x]+2}$ for each age at issue for the select portion, and a complete set of values of q_{x+3} for the ultimate section (assuming a 3-year select period). Suppose that the earliest age at issue is 20. A convenient radix is then assumed for $l_{[20]}$, and the complete mortality table for this age at issue is constructed using the values of $q_{[20]}$, $q_{[20]+1}$, $q_{[20]+2}$, q_{23}, q_{24}, etc.

TABLE 3

SECTION OF SELECT AND ULTIMATE TABLE

$[x]$	$l_{[x]}$	$l_{[x]+1}$	$l_{[x]+2}$	l_{x+3}	$x+3$
20	946,394	945,145	943,671	942,001	23
21	944,710	943,435	941,916	940,202	24
22	942,944	941,652	940,108	938,359	25
23	941,143	939,835	938,265	936,482	26
24	939,279	937,964	936,379	934,572	27
25	937,373	936,061	934,460	932,628	28
26	935,433	934,123	932,507	930,651	29
27	933,467	932,151	930,520	928,631	30
28	931,488	930,156	928,491	926,560	31
29	929,476	928,119	926,421	924,429	32
30	927,422	926,040	924,290	922,220	33

$[x]$	$d_{[x]}$	$d_{[x]+1}$	$d_{[x]+2}$	d_{x+3}	$x+3$
20	1,249	1,474	1,670	1,799	23
21	1,275	1,519	1,714	1,843	24
22	1,292	1,544	1,749	1,877	25
23	1,308	1,570	1,783	1,910	26
24	1,315	1,585	1,807	1,944	27
25	1,312	1,601	1,832	1,977	28
26	1,310	1,616	1,856	2,020	29
27	1,316	1,631	1,889	2,071	30
28	1,332	1,665	1,931	2,131	31
29	1,357	1,698	1,992	2,209	32
30	1,382	1,750	2,070	2,306	33

$[x]$	$q_{[x]}$	$q_{[x]+1}$	$q_{[x]+2}$	q_{x+3}	$x+3$
20	.00132	.00156	.00177	.00191	23
21	.00135	.00161	.00182	.00196	24
22	.00137	.00164	.00186	.00200	25
23	.00139	.00167	.00190	.00204	26
24	.00140	.00169	.00193	.00208	27
25	.00140	.00171	.00196	.00212	28
26	.00140	.00173	.00199	.00217	29
27	.00141	.00175	.00203	.00223	30
28	.00143	.00179	.00208	.00230	31
29	.00146	.00183	.00215	.00239	32
30	.00149	.00189	.00224	.00250	33

[illegible] produces the entire ultimate column. The select values for [illegible] at issue over 20 must now be filled in so that the ultimate [illegible] already obtained represent the correct number of survivors [illegible] each age at issue. This is done by working backwards from the ultimate values. To fill in the select values for age at issue 21, we compute $p_{[21]+2}$ and then obtain $l_{[21]+2} = \dfrac{l_{24}}{p_{[21]+2}}$. Similarly,

$$l_{[21]+1} = \frac{l_{[21]+2}}{p_{[21]+1}} \text{ and } l_{[21]} = \frac{l_{[21]+1}}{p_{[21]}}.$$

When the complete l_x table has been constructed in this way, the d_x table is formed from the relations,

$$d_{[x]} = l_{[x]} - l_{[x]+1},$$
$$d_{[x]+1} = l_{[x]+1} - l_{[x]+2},$$
$$d_{[x]+2} = l_{[x]+2} - l_{x+3},$$

and in the ultimate column

$$d_{x+n} = l_{x+n} - l_{x+n+1}, \qquad n > 2.$$

Table 3 is based on the actual experience of a group of insured lives. The degree of selection inherent in this table can be seen by inspection. At age 25, for example, the select $q_{[25]}$ is seen to be just 70% of the ultimate rate q_{25}. Similarly, $q_{[25]+1}$ is 84% of q_{26}, and $q_{[25]+2}$ is 94% of q_{27}, and these ratios illustrate the way in which the influence of initial selection diminishes.

In evaluating probabilities of death or survival involving select lives, care should be taken to relate all select functions to the correct age at entry. Note the following examples based on the data of Table 3:

$${}_2p_{[22]} = \frac{l_{[22]+2}}{l_{[22]}} = \frac{940{,}108}{942{,}944}$$

$${}_5p_{[20]} = \frac{l_{25}}{l_{[20]}} = \frac{938{,}359}{946{,}394}$$

$$p_{[24]+1} = \frac{l_{[24]+2}}{l_{[24]+1}} = \frac{936{,}379}{937{,}964}$$

$${}_{2|}q_{[23]+1} = \frac{d_{26}}{l_{[23]+1}} = \frac{1{,}910}{939{,}835}$$

Although the table we have used here for illustration involves a select period of three years, it should not be inferred that this is the normal period during which the effects of selection are significant. The length of the select period depends on the nature of the underlying data, and the effects of selection may persist noticeably for many years. A number of standard American tables have 5-year select periods; one of the classical British tables was constructed with a 10-year select period; and the continuous mortality investigation carried on by the Society of Actuaries maintains a 15-year select period.

Although the above discussion has been concerned with the mortality of insured lives, the analysis is perfectly general and applies to any group in which mortality varies by duration as well as by age. Select mortality is found in the same form among annuitants as among insured lives, although in this case the effect is produced by self-selection on the part of the annuitants themselves, since usually only those persons who think themselves to be in good health will consider the purchase of an annuity.

Values of the force of mortality at select ages may be estimated from mortality table data by using the methods of Section 5. Using the analogue of formula (1.21), for example,

$$\mu_{[x]+2} \doteq \frac{8(l_{[x]+1} - l_{[x]+3}) - (l_{[x]} - l_{[x]+4})}{12 l_{[x]+2}}$$

(assuming the select period to be greater than 4 years). Note that this formula could not be used to approximate $\mu_{[x]+1}$, since it would then involve the meaningless symbol $l_{[x]-1}$. The analogue of formula (1.22) could be used in this case.

A mortality table which is constructed from records of insured lives without regard to age at issue, all durations from zero on being included, and all lives of the same attained age being grouped together, is called an *aggregate* table. Another type of mortality table is the *ultimate* table, which is constructed from records of insured lives with the experience of the first years after issue omitted in order to eliminate the effects of selection. The ultimate column of any select and ultimate table constitutes in itself an ultimate mortality table. The American Experience table, one of the earliest tables to be used extensively for insurance purposes in the United States, is a well-known example of an ultimate table.

The International Actuarial Notation

The International Congress of Actuaries adopted in 1898 a system of notation for actuarial literature. In addition to prescribing definite symbols for the common actuarial functions, this code indicates the notational principles to be followed in adopting symbols for new functions that may be needed. In this way, it provides for the development of a consistent actuarial notation that will always be intelligible to anyone familiar with the code.

At the end of each chapter the notational ideas first introduced in that chapter will be summarized. These summaries will consist of three parts: (1) the fundamental principles, (2) the application of these principles to the specific types of function discussed in the current chapter, (3) consideration of any exceptional points.

Fundamental notational principles used in Chapter 1

A. Actuarial functions are represented by a *principal symbol* with associated *prefixes and suffixes* which may be either *subscripts* or *superscripts*. The principal symbol expresses the general nature of the function. The prefixes and suffixes each limit some phase of the generalization, so that the completed symbol constitutes a precise definition of the function.

B. A *suffixed subscript* (appearing at the lower right-hand corner of the principal symbol) indicates the conditions relative to ages and the order of succession of the events.

C. A *prefixed subscript* (appearing at the lower left-hand corner) indicates the conditions relative to the duration of the operations and to their position with regard to time.

D. A vertical bar with a prefixed subscript indicates a period of deferment.

Application to mortality table functions

1. The following principal symbols are used in this chapter:

l, a number living;
d, a number dying;
p, a probability of living;
q, a probability of dying;
μ, a force of mortality.

2. With the above principal symbols, a suffixed subscript indicates the age of the life involved (Principle B); e.g., l_x. If the sub-