MA 301 Test 1, Spring 2006

YOU MAY NOT USE THEOREM 2 IN CHAPTER 1
IN ANY OF THE FOLLOWING PROBLEMS.

TA: Grade 1, 5, 6, 7

(1) Define \( \lim_{n \to \infty} a_n = L \)

0, 4 or 8 pts. It need not be verbatim, but the meaning should be the same.

Solution

**Definition 1.** We say that \( \lim_{n \to \infty} a_n = L \) provided that for every number \( \epsilon > 0 \), there is a number \( N \) such that

\[
|a_n - L| < \epsilon
\]

for all \( n > N \).

(2) Use the axioms, together with the definition of \( x + y + z \), to prove the following equality, putting in reasons for each step. **DO NOT USE ANY OF THE PROPERTIES (C1)-(C8).** Whenever you use an axiom, state what is being substituted for \( a, b, \) and \( c \).

\[
(x + y + z) + w = (x + y + w) + z
\]

Solution:

\[
(x + y + z) + w = ((x + y) + z) + w \quad \text{Def. of } x + y + z
\]
\[
= (x + y) + (z + w) \quad \text{A1, } a = x + y, b = z, c = w
\]
\[
= (x + y) + (w + z) \quad \text{A4, } a = z, b = w
\]
\[
= ((x + y) + w) + z \quad \text{A1, } a = x + y, b = w, c = z
\]
\[
= (x + y + w) + z \quad \text{Def. of } x + y + w
\]

(3) Solve for \( x \), stating all axioms or properties used (but when you use an axiom or property, you do not have to state what is being substituted for \( a, b \) and \( c \)). **To save time, you may take as given that if** \( 5x = -5 \), then \( x = -1 \).

\[
7x + 5 = 2x
\]
Solution: Assume $x$ satisfies the above equality. Then
\[ 7x + 5 = 2x \]
\[ (-2x) + (7x + 5) = (-2x) + 2x \quad \text{A0, A3} \]
\[ (-2x) + 7x + 5 = 0 \quad \text{A1, A4, A3} \]
\[ ((-2)x + 7x) + 5 = 0 \quad \text{C5} \]
\[ 5x + 5 = 0 \quad \text{C1, Number fact} \]
\[ (5x + 5) + (-5) = 0 + (-5) \quad \text{A0, A3} \]
\[ 5x + (5 + (-5)) = -5 \quad \text{A1, A4, A3} \]
\[ 5x + 0 = -5 \quad \text{A3} \]
\[ 5x = -5 \quad \text{A2} \]
\[ x = -1 \quad \text{Given} \]

Conversely, if $x = -1$, then $7x + 5 = 2x$, showing that $-1$ is a solution.

(4) Prove the following equality, stating all axioms or properties used (but when you use an axiom or property, you do not have to state what is being substituted for $a$, $b$ and $c$). The definition of $a^2$ is $a^2 = a \cdot a$.

\[ (a + 1)(a + 2) = a^2 + (3a + 2) \]

Solution:
\[ (a + 1)(a + 2) = a(a + 2) + 1(a + 2) \quad \text{C1} \]
\[ = (a^2 + a \cdot 2) + (a + 2) \quad \text{D, Def. of } a^2, \text{ M4, M2} \]
\[ = (a^2 + 2a) + (a + 2) \quad \text{M4} \]
\[ = a^2 + (2a + (a + 2)) \quad \text{A1} \]
\[ = a^2 + ((2a + 1a) + 2) \quad \text{A1, M4, M2} \]
\[ = a^2 + ((2 + 1)a + 2) \quad \text{Cl} \]
\[ = a^2 + (3a + 2) \quad \text{Num. fact} \]

16 pts.

(5) Solve for $x$.
\[ \frac{x^2 - 3}{x} > x - 2 \]

State all inequality axioms or properties used. You don’t need to state other axioms or properties. Be sure to justify any application of a function to both sides of an inequality.
Solution:
8 pts./case
From (I1), we have three cases: $x < 0$, $x = 0$, and $x > 0$. The fraction is not defined if $x = 0$. Hence we have the following cases:

Case 1: $x > 0$. Assume that $x$ satisfies the given inequality. Then

$$\frac{x^2 - 3}{x} > x - 2$$
$$x^2 - 3 > x(x - 2) \quad \text{I4}$$
$$x^2 - 3 > x^2 - 2x$$
$$-3 > -2x \quad \text{I3}$$

$$\frac{3}{2} < x \quad \text{E3}$$

5 pt.
Since $\frac{3}{2} > 0$, our conclusion is $x > \frac{3}{2}$. Conversely, if $x > \frac{3}{2}$, the above reasoning may be reversed, showing that $(\frac{3}{2}, \infty)$ is part of the solution.

Case 2 is scored as in Case 1.

Case 2: $x < 0$. Assume that $x$ satisfies the given inequality. Then

$$\frac{x^2 - 3}{x} > x - 2$$
$$x^2 - 3 < x(x - 2) \quad \text{E3}$$
$$x^2 - 3 < x^2 - 2x$$
$$-3 < -2x \quad \text{I3}$$

$$\frac{3}{2} > x \quad \text{E3}$$

Since $\frac{3}{2} > 0$, our conclusion is $x < 0$. Conversely, if $x < 0$, the above reasoning may be reversed, showing that $(-\infty, 0)$ is part of the solution. The full solution is $(-\infty, 0) \cup (\frac{3}{2}, \infty)$.

(6) Prove that if $b > 1$, then

$$\frac{\sqrt{b} - 1}{b} \leq \frac{1}{2}$$
State all inequality axioms or properties used. You don’t need to state other axioms or properties. Be sure to justify any application of a function to both sides of an inequality.

**Scratch Work:** 6 pt.

\[
\frac{\sqrt{b-1}}{b} \leq \frac{1}{2} \\
2\sqrt{b-1} \leq b \\
4(b-1) \leq b^2 \\
0 \leq b^2 - 4b + 4 \\
0 \leq (b - 2)^2
\]

-2 pt. if they continue this as \( 0 < b - 2 \). O.K. if they write \( 0 < |b - 2| \)

**Proof:** 6 pt.

\[
0 \leq (b - 2)^2 \quad E7 \\
0 \leq b^2 - 4b + 4 \\
4b - 4 \leq b^2 \quad I3 \\
4(b-1) \leq b^2 \\
2\sqrt{b-1} \leq b \quad y = \sqrt{x} \text{ is increasing, } b > 1. \\
-2 pt. if increasing is not mentioned \\
\frac{\sqrt{b-1}}{b} \leq \frac{1}{2} \quad I4
\]

12 pts. (7) Find numbers \( C > 0 \), and \( N \) such that for all \( x > N \),

\[
Cx^4 < x^4 - 3x^3 - 4x^2 + 4
\]

You must explain the reasoning—just stating a value of \( C \) and \( N \) is not sufficient.

**Solution:**

2 pt.

\[
x^4 - 3x^3 - 4x^2 + 4 > x^4 - 3x^3 - 4x^2.
\]
Also
\[ 4x^2 < \frac{1}{3}x^4 \]
\[ 12 < x^2 \]
\[ \sqrt{12} < x \]

3 pt.
and
\[ 3x^3 < \frac{1}{3}x^4 \]
\[ 3 < \frac{1}{3}x \]
\[ 9 < x \]

3 pt.
Hence, if \( x > \max\{\sqrt{12}, 9\} \) (or \( x > 9 \)) 2 pt.

\[ x^4 - 3x^3 - 4x^2 + 4 > \frac{1}{3}x^4. \]

Thus \( C = \frac{1}{3} \) and \( N = \max\{\sqrt{12}, 9\} \) work. 2 pt.

(8) For each pair of functions below (i) determine which is dominate, (ii) prove your answer by finding a number \( N \) fulfilling the requirements of Definition 1 from Chapter 3. You must prove that the stated value of \( N \) works. 16 pts.

(a) \((1.1)^x, \quad 5x^2 \)
(b) \(x^{1/5}, \quad (\ln x)^3 \)

Solution to (a): \((1.1)^x\) is the dominate function. For \( x > 5, \) \( 5x^3 < x^4. \) Also
\[ x^4 < (1.1)^x \]
\[ 4 \ln x < x \ln(1.1) \]
\[ \ln x < \frac{\ln(1.1)}{4} x \]

From Proposition 2, this is true if \( x > \frac{4}{a^2} \) where \( a = \frac{\ln(1.1)}{4}. \)
Solution to (b): \(x^{1/5}\) is the dominate function. To prove this we must find \(N\) so that for \(x > N\),

\[
\begin{align*}
(ln x)^3 &< x^{1/5} \\
\ln x &< x^{1/15}
\end{align*}
\]

Replace \(x\) by \(x^{1/15}\) in Proposition 2. Then, for \(x^{1/15} > 4/a^2\),

\[
\begin{align*}
\ln x^{1/15} &< ax^{1/15} \\
\frac{1}{15} \ln x &< ax^{1/15} \\
\ln x &< 15ax^{1/15}
\end{align*}
\]

Let \(a = \frac{1}{15}\). Our inequality holds for \(x > \left(\frac{4}{a^2}\right)^{15}\).

The Field Axioms

Let \(a, b,\) and \(c\) be real numbers.

A0: Addition is a well defined process which takes pairs of real numbers \(a\) and \(b\) and produces from them one single real number \(a + b\).

A1: \(a + (b + c) = (a + b) + c\).

A2: There is a real number 0 such that for all real numbers \(a\)
\[a + 0 = a\]

A3: For every real number \(a\) there is a real number \(-a\) such that \(a + (-a) = 0\).

A4: \(a + b = b + a\).

M0: Multiplication is a well defined process which takes pairs of real numbers \(a\) and \(b\) and produces from them one single real number \(ab\).

M1: \(a(bc) = (ab)c\).

M2: There is a real number 1 such that for all real numbers \(a\)
\[a1 = a\]

M3: For every real number \(a \neq 0\), there is a real number \(a^{-1}\) such that \(a a^{-1} = 1\).

M4: \(ab = ba\).

D: \(a(b + c) = ab + ac\).

Z: \(0 \neq 1\)

Theorem (Theorem 1, Chapter 1). Let \(a, b,\) and \(c\) be real numbers. Then

C1: \((a + b)c = ac + bc\).

C2: \(0a = 0\)
C3: \(-a = (-1)a.\)
C4: \(-(ab) = (-a)b = a(-b).\)
C5: \(-(-a) = a.\)
C6: If \(a \neq 0 \neq b,\) then \((ab)^{-1} = a^{-1}b^{-1}.\)
C7: \((a^{-1})^{-1} = a.\)
C8: \(-(a + b) = (-a) + (-b).\)
Order Axioms

I1: For real numbers $a$ and $b$, one and only one, of the following statements must hold: $a < b$, $b < a$, $a = b$.
I2: If $a < b$ and $b < c$, then $a < c$.
I3: If $a < b$ and $c$ is any real number, then $a + c < b + c$.
I4: If $a < b$ and $c > 0$, then $ac < bc$.

Theorem (Theorem 1, Chapter 2). Let $a$, $b$, $c$, and $d$ be real numbers. Then

E1: If $a < b$ and $c < d$, then $a + c < b + d$.
E2: If $0 < a < b$ and $0 < c < d$, then $0 < ac < bd$.
E3: If $a < b$ and $c < 0$, then $ac > bc$.
E4: If $0 < a < b$, then $\frac{1}{a} > \frac{1}{b} > 0$.
E5: If $a < 0$ and $b < 0$ then $ab > 0$.
E6: If $a \neq 0$, $a^2 > 0$.
E7: If $a \in \mathbb{N}$, $a > 0$.

Proposition (2, Chapter 3). Let $0 < a < 1$. Then $\ln x < ax$ for $x > 4/a^2$. 