MA 301 Test 1, Spring 2006

YOU MAY NOT USE THEOREM 2 IN CHAPTER 1 IN ANY OF THE FOLLOWING PROBLEMS.

TA: Grade 1, 5, 6, 7

(1) Define 8 pts.

$$\lim_{n\to\infty} a_n = L$$

0, 4 or 8 pts. It need not be verbatim, but the meaning should be the same.

Solution

DEFINITION 1. *We say that* $\lim_{n\to\infty} a_n = L$ *provided that for every number* $\epsilon > 0$*, there is a number* N *such that*

$$|a_n - L| < \epsilon \tag{1}$$

for all $n > N$*.*

(2) Use the axioms, together with the definition of $x + y + z$, to prove the following equality, putting in reasons for each step. **DO NOT USE ANY OF THE PROPERTIES (C1)-(C8).** Whenever you use an axiom, state what is being substituted for a, b, and c. 12 pts.

$$(x + y + z) + w = (x + y + w) + z$$

Solution:

$$\begin{aligned}
(x+y+z)+w &= ((x+y)+z)+w && \text{Def. of } x+y+z \\
&= (x+y)+(z+w) && \text{A1, } a = x+y,\ b = z,\ c = w \\
&= (x+y)+(w+z) && \text{A4, } a = z,\ b = w \\
&= ((x+y)+w)+z && \text{A1, } a = x+y,\ b = w,\ c = z \\
&= (x+y+w)+z && \text{Def. of } x+y+w
\end{aligned}$$

(3) Solve for x, stating all axioms or properties used (but when you use an axiom or property, you do *not* have to state what is being substituted for a, b and c). **To save time, you may take as given that if $5x = -5$, then $x = -1$.** 12 pts.

$$7x + 5 = 2x$$

Solution: Assume x satisfies the above equality. Then

$$\begin{aligned}
7x+5 &= 2x \\
(-(2x))+(7x+5) &= (-(2x))+2x \qquad \text{A0,A3} \\
(-(2x)+7x)+5 &= 0 \qquad \text{A1,A4,A3} \\
((-2)x+7x)+5 &= 0 \qquad \text{C5} \\
5x+5 &= 0 \qquad \text{C1,Number fact} \\
(5x+5)+(-5) &= 0+(-5) \qquad \text{A0, A3} \\
5x+(5+(-5)) &= -5 \qquad \text{A1,A4} \\
5x+0 &= -5 \qquad \text{A3} \\
5x &= -5 \qquad \text{A2} \\
x &= -1 \qquad \text{Given}
\end{aligned}$$

Conversely, if $x=-1$, then $7x+5=2x$, showing that -1 is a solution.

12 pts.

(4) Prove the following equality, stating all axioms or properties used (but when you use an axiom or property, you do *not* have to state what is being substituted for a, b and c). **The definition of a^2 is $a^2 = a \cdot a$.**

$$(a+1)(a+2) = a^2 + (3a+2)$$

Solution:

$$\begin{aligned}
(a+1)(a+2) &= a(a+2)+1(a+2) \quad \text{C1} \\
&= (a^2 + a\cdot 2)+(a+2) \quad \text{D, Def. of } a^2\text{, M4, M2} \\
&= (a^2+2a)+(a+2) \quad \text{M4} \\
&= a^2+(2a+(a+2)) \quad \text{A1} \\
&= a^2+((2a+1a)+2) \quad \text{A1, M4,M2} \\
&= a^2+((2+1)a+2) \quad \text{C1} \\
&= a^2+(3a+2) \quad \text{Num. fact}
\end{aligned}$$

16 pts.

(5) Solve for x.

$$\frac{x^2-3}{x} > x-2$$

State all inequality axioms or properties used. You don't need to state other axioms or properties. Be sure to justify any application of a function to both sides of an inequality.

Solution:

8 pts./case

From (I1), we have three cases: $x < 0$, $x = 0$, and $x > 0$. The fraction is not defined if $x = 0$. Hence we have the following cases:

Case 1: $x > 0$**.** Assume that x satisfies the given inequality. Then

$$\begin{aligned} \frac{x^2-3}{x} &> x-2 \\ x^2-3 &> x(x-2) \quad \text{I4} \\ x^2-3 &> x^2-2x \\ -3 &> -2x \quad \text{I3} \\ \frac{3}{2} &< x \quad \text{E3} \end{aligned}$$

5 pt.

Since $\frac{3}{2} > 0$, our conclusion is $x > \frac{3}{2}$*2 pt.*. Conversely*1 pt.*, if $x > \frac{3}{2}$, the above reasoning may be reversed, showing that $(\frac{3}{2}, \infty)$ is part of the solution.

Case 2 is scored as in Case 1.

Case 2: $x < 0$**.** Assume that x satisfies the given inequality. Then

$$\begin{aligned} \frac{x^2-3}{x} &> x-2 \\ x^2-3 &< x(x-2) \quad \text{E3} \\ x^2-3 &< x^2-2x \\ -3 &< -2x \quad \text{I3} \\ \frac{3}{2} &> x \quad \text{E3} \end{aligned}$$

Since $\frac{3}{2} > 0$, our conclusion is $x < 0$. Conversely, if $x < 0$, the above reasoning may be reversed, showing that $(-\infty, 0)$ is part of the solution. The full solution is $(-\infty, 0) \cup (\frac{3}{2}, \infty)$.

(6) Prove that if $b > 1$, then 12 pts.

$$\frac{\sqrt{b-1}}{b} \leq \frac{1}{2}$$

State all inequality axioms or properties used. You don't need to state other axioms or properties. Be sure to justify any application of a function to both sides of an inequality.

Scratch Work: *6 pt.*

$$\begin{aligned}\frac{\sqrt{b-1}}{b} &\leq \frac{1}{2}\\ 2\sqrt{b-1} &\leq b\\ 4(b-1) &\leq b^2\\ 0 \leq & b^2-4b+4\\ 0 \leq & (b-2)^2\end{aligned}$$

-2 pt. if they continue this as $0 < b-2$. O.K. if they write $0 < |b-2|$

Proof: *6 pt.*

$$\begin{aligned} 0 \leq & (b-2)^2 \quad \text{E7}\\ 0 \leq & b^2-4b+4\\ 4b-4 &\leq b^2 \quad \text{I3}\\ 4(b-1) &\leq b^2\\ 2\sqrt{b-1} &\leq b \quad y=\sqrt{x} \text{ is increasing, } b>1.\end{aligned}$$

-2 pt. if increasing is not mentioned

$$\frac{\sqrt{b-1}}{b} \leq \frac{1}{2} \quad \text{I4}$$

12 pts. (7) Find numbers $C > 0$, and N such that for all $x > N$,

$$Cx^4 < x^4 - 3x^3 - 4x^2 + 4$$

You must explain the reasoning—just stating a value of C and N is not sufficient.

Solution:

2 pt.

$$x^4 - 3x^3 - 4x^2 + 4 > x^4 - 3x^3 - 4x^2.$$

Also

$$\begin{aligned}4x^2 &< \frac{1}{3}x^4\\ 12 &< x^2\\ \sqrt{12} &< x\end{aligned}$$

3 pt.

and

$$\begin{aligned} 3x^3 &< \frac{1}{3}x^4 \\ 3 &< \frac{1}{3}x \\ 9 &< x \end{aligned}$$

3 pt.

Hence, if $x > \max\{\sqrt{12}, 9\}$ (or $x > 9$) *2 pt.*

$$x^4 - 3x^3 - 4x^2 + 4 > \frac{1}{3}x^4.$$

Thus $C = \frac{1}{3}$ and $N = \max\{\sqrt{12}, 9\}$ work. *2 pt.*

(8) For each pair of functions below (i) determine which is dominate, (ii) prove your answer by finding a number N fulfilling the requirements of Definition 1 from Chapter 3. *You must prove that the stated value of N works.* 16 pts.

$$\begin{array}{lll} \text{(a)} & (1.1)^x, & 5x^2 \\ \text{(b)} & x^{1/5}, & (\ln x)^3 \end{array}$$

Solution to (a): $(1.1)^x$ is the dominate function. For $x > 5$, $5x^3 < x^4$. Also

$$\begin{aligned} x^4 &< (1.1)^x \\ 4\ln x &< x\ln(1.1) \\ \ln x &< \frac{\ln(1.1)}{4}x \end{aligned}$$

From Proposition 2, this is true if $x > \frac{4}{a^2}$ where $a = \frac{\ln(1.1)}{4}$.

Solution to (b): $x^{1/5}$ is the dominate function. To prove this we must find N so that for $x > N$,

$$\begin{aligned} (\ln x)^3 &< x^{1/5} \\ \ln x &< x^{1/15} \end{aligned}$$

Replace x by $x^{1/15}$ in Proposition 2. Then, for $x^{1/15} > 4/a^2$,

$$\begin{aligned} \ln x^{1/15} &< ax^{1/15} \\ \frac{1}{15}\ln x &< ax^{1/15} \\ \ln x &< 15ax^{1/15} \end{aligned}$$

Let $a = \frac{1}{15}$. Our inequality holds for $x > \left(\frac{4}{a^2}\right)^{15}$.

The Field Axioms

Let a, b, and c be real numbers.

A0: Addition is a well defined process which takes pairs of real numbers a and b and produces from then one single real number $a + b$.

A1: $a + (b + c) = (a + b) + c$.

A2: There is a real number 0 such that for all real numbers a $a + 0 = a$.

A3: For every real number a there is a real number $-a$ such that $a + (-a) = 0$.

A4: $a + b = b + a$.

M0: Multiplication is a well defined process which takes pairs of real numbers a and b and produces from then one single real number ab.

M1: $a(bc) = (ab)c$.

M2: There is a real number 1 such that for all real numbers a $a1 = a$.

M3: For every real number $a \neq 0$, there is a real number a^{-1} such that $aa^{-1} = 1$.

M4: $ab = ba$.

D: $a(b + c) = ab + ac$.

Z: $0 \neq 1$

Theorem (Theorem 1, Chapter 1). *Let a, b, and c be real numbers. Then*

C1: $(a + b)c = ac + bc$.

C2: $0a = 0$

C3: $-a = (-1)a$.

C4: $-(ab) = (-a)b = a(-b)$.

C5: $-(-a) = a$.

C6: *If $a \neq 0 \neq b$, then $(ab)^{-1} = a^{-1}b^{-1}$.*

C7: $(a^{-1})^{-1} = a$.

C8: $-(a + b) = (-a) + (-b)$.

Order Axioms

I1: For real numbers a and b, one and only one, of the following statements must hold: $a < b$, $b < a$. $a = b$.
I2: If $a < b$ and $b < c$, then $a < c$.
I3: If $a < b$ and c is any real number, then $a + c < b + c$.
I4: If $a < b$ and $c > 0$, then $ac < bc$.

THEOREM (Theorem 1, Chapter 2). *Let a, b, c, and d be real numbers. Then*

E1: If $a < b$ and $c < d$, then $a + c < b + d$.
E2: If $0 < a < b$ and $0 < c < d$, then $0 < ac < bd$.
E3: If $a < b$ and $c < 0$, then $ac > bc$.
E4: If $0 < a < b$, then $\frac{1}{a} > \frac{1}{b} > 0$.
E5: If $a < 0$ and $b < 0$ then $ab > 0$.
E6: If $a \neq 0$, $a^2 > 0$.
E7: If $a \in \mathbb{N}$, $a > 0$.

PROPOSITION (2, Chapter 3). *Let $0 < a < 1$. Then $\ln x < ax$ for $x > 4/a^2$.*