MA 301 Test 2, Spring 2006

(1) Define
$$\lim_{n\to\infty} a_n = L.$$
6 pts

Solution: No partial credit. It need not be verbatim, but the meaning must be the same.

Definition 1. *Let a_n be some sequence of numbers and let L be a number. We say that $\lim_{n\to\infty} a_n = L$ provided that for every number $\epsilon > 0$, there is a number N such that*
$$|a_n - L| < \epsilon$$
for all $n > N$.

(2) State (carefully) the Sum Theorem from the text. Then prove it.

Theorem (Sum Theorem). *Suppose that a_n and b_n are convergent sequences. Then*
$$\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$$
12 pts

1pt for statement

Proof Let $\epsilon > 0$ be given and set $L = \lim_{n\to\infty} a_n$ and $M = \lim_{n\to\infty} b_n$. *1pt*

Our theorem is equivalent with
$$\lim_{n\to\infty}(a_n + b_n) = L + M$$
From our hypothesis, there are numbers N_1 and N_2 such that
$$|a_n - L| < \frac{\epsilon}{2} \quad \text{for all } n > N_1$$
$$|b_n - M| < \frac{\epsilon}{2} \quad \text{for all } n > N_2$$
4pts

Let N be the maximum of N_1 and N_2. *2pts* Then $n > N$ implies
$$|a_n + b_n - (L + M)| = |(a_n - L) + (b_n - M)|$$
$$\leq |a_n - L| + |b_n - M| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$
5 pts

This fulfills the requirements for the definition of limit, proving our theorem.

General rules for grading explicit limit proofs:
1pts for $|a_n - L| < \epsilon$
2pts for simplifying $|a_n - L| < \epsilon$
4pts for bounding $|a_n - L|$
2pts for solving for N
3pts for "Formal Proof" which must state "Let $\epsilon > 0$ *be given, let* $N = \ldots$ *then, from the scratch work, for* $n > N$, $|a_n - L| < \epsilon$*", or something equivalent to this.*

Similar grading scale for $L = \infty$.

Prof grades 2, 4, 5, 6, 8
TA grades 1, 3, 7, 9

(3) Prove using ϵ:

$$\lim_{n\to\infty} \frac{\sqrt{n+1}}{\sqrt{n}} = 1.$$

12 pts

Scratch Work:

$$|\frac{\sqrt{n+1}}{\sqrt{n}} - 1| < \epsilon$$

$$|\frac{\sqrt{n+1} - \sqrt{n}}{\sqrt{n}}| < \epsilon$$

$$|\frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{n(\sqrt{n+1} + \sqrt{n})}| < \epsilon$$

$$\frac{1}{n(\sqrt{n+1} + \sqrt{n})} < \frac{1}{n\sqrt{n}} < \epsilon$$

$$\frac{1}{n^{\frac{3}{2}}} < \epsilon$$

$$n > \left(\frac{1}{\epsilon}\right)^{\frac{2}{3}}$$

Formal Proof: Let $\epsilon > 0$ be given and let $N = \left(\frac{1}{\epsilon}\right)^{\frac{2}{3}}$. From the scratch work, for $n > N$,

$$|\frac{\sqrt{n+1}}{\sqrt{n}} - 1| < \epsilon$$

proving the result. □

(4) Prove using ϵ:

$$\lim_{n\to\infty} \frac{n \ln n}{\sqrt{n^3+5}} = 0.$$

12 pts

Scratch Work: There is an N_1 such that $\ln n < n^{1/4}$ for $n > N_1$. Hence, for $n > N_1$

$$\begin{aligned} |\frac{n \ln n}{\sqrt{n^3+5}}| &< \frac{n^{\frac{5}{4}}}{\sqrt{n^3+5}} \\ &< \frac{n^{5/4}}{n^{3/2}} \\ &= \frac{1}{n^{1/4}} < \epsilon \end{aligned}$$

provided $n > \frac{1}{\epsilon^4}$.

Formal Proof Let $\epsilon > 0$ be given and let $N = \max\{N_1, \frac{1}{\epsilon^4}\}$. Then, from the scratch work, for $n > N$,

$$|\frac{n \ln n}{\sqrt{n^3+5}}| < \epsilon$$

proving the result.

(5) Suppose that $\lim_{n\to\infty} a_n = 1$. Prove, using ϵ, that 12 pts

$$\lim_{n\to\infty} \frac{3a_n - 1}{a_n} = 2.$$

Scratch Work:

$$|\frac{3a_n - 1}{a_n} - 2| < \epsilon \text{ 2pts}$$

$$|\frac{a_n - 1}{a_n}| < \epsilon \text{ 2pts}$$

Assume $a_n = 1 \pm \frac{1}{2}$–i.e.

$$|a_n - 1| < \frac{1}{2}$$
$$\frac{1}{2} < a_n < \frac{3}{2}$$

2 pts

Hence

$$|\frac{a_n - 1}{a_n}| < 2|a_n - 1| < \epsilon$$

if

$$|a_n - 1| < \frac{1}{2}\epsilon.$$

2 pts

Formal Proof: *4 pts*

Let $\epsilon > 0$ be given. Choose N_1 so that $|a_n - 1| < \frac{1}{2}$ for $n > N_1$ and choose N_2 so that $|a_n - 1| < \frac{1}{2}\epsilon$ for $n > N_2$. Let $N = \max\{N_1, N_2\}$. Let $n > N$. Then, $n > N_1$ and $n > N_2$. From the scratch work, it follows that

$$|\frac{3a_n - 1}{a_n} - 2| < \epsilon.$$

12 pts □

12 pts (6) Suppose that $\lim_{n\to\infty} a_n = 0$. Suppose also that $0 < b_n < 2$ for all n. Prove, using ϵ, that

$$\lim_{n\to\infty} a_n^2 b_n = 0.$$

Solution: Let $\epsilon > 0$ be given.*1pt* Choose N so that $|a_n| < \sqrt{\frac{\epsilon}{2}}$ for $n > N$.*4pt* Then for $n > N$,*1pt*

$$\left|a_n^2 b_n - 0\right| \text{ 2pts} < |2a_n^2| \text{4pts}$$
$$< \epsilon$$

as desired.

12 pts (7) Prove, using M, that

$$\lim_{n\to\infty} \frac{n^4 + \ln n}{n + 2\sqrt{n} + 1} = \infty$$

Scratch Work:

$$\begin{aligned}\frac{n^4+\ln n}{n+2\sqrt{n}+1} &> \frac{n^4}{n+2\sqrt{n}+1}\\ &> \frac{n^4}{n+2n+n}\\ &= \frac{1}{4}n^3 > M\end{aligned}$$

provided

$$n > (4M)^{\frac{1}{3}}.$$

Proof Let $M > 0$ be given. Let $N = (4M)^{\frac{1}{3}}$. Then for $n > M$, from the scratch work,

$$\frac{n^4+\ln n}{n+2\sqrt{n}+1} > M$$

proving the result.

(8) Find the max, min, sup, and inf of the following set and prove your answer. 12 pts

$$S = \{\frac{2n+1}{n+1} \mid n \in N\}.$$

Solution: Max does not exist, $\sup S = 2$, min = inf = $\frac{3}{2}$.
2pts
Proof

Sup

$$\begin{aligned}\frac{2n+1}{n+1} &< 2\\ 2n+1 &< 2n+2\\ 1 &< 2\end{aligned}$$

3pts
Reversing the above argument shows that 2 is an upper bound.

If 2 is not the LUB, there is an $\epsilon > 0$ such $2-\epsilon$ is an upper bound.

However

$$2-\epsilon < \frac{2n+1}{n+1}$$

2pts
holds if

$$\begin{aligned}-\epsilon &< \frac{2n+1}{n+1} - 2 \\ \epsilon &> \frac{1}{n+1} \\ n &> \frac{1}{\epsilon} - 1\end{aligned}$$

2pts
showing that $2 - \epsilon$ is not an upper bound. Hence 2 is the LUB.

min $= \frac{3}{2}$

$$\frac{3}{2} = \frac{2 \cdot 1 + 1}{1+1} \in S.$$

1 pt Hence, it suffices to show that $\frac{3}{2}$ is a lower bound which we do as follows:

$$\begin{aligned}\frac{2n+1}{n+1} &\geq \frac{3}{2} \\ 2(2n+1) &\geq 3(n+1) \\ n &\geq 1\end{aligned}$$

2pts
which is true for all $n \in \mathbb{N}$. Reversing the above argument shows that $\frac{3}{2}$ is a lower bound. □

(9) You are given the following information. Use this information to answer the question below. This question is intended to test your ability to read the proof of Lemma 1 which is given on the second page of this test. *Your solution must follow the proof of this result. Other methods will not give credit.*

10 pts

$$\begin{aligned}\frac{n^4}{n^5+7} &= 0 \pm \epsilon \quad \text{for } n > \frac{1}{\epsilon} \\ (\sqrt{n} - \sqrt{n+1}) &= 0 \pm \epsilon \quad \text{for } n > \frac{1}{\epsilon^2}\end{aligned}$$

Question: *According to the proof of Lemma 1*, for which value of N does

$$\left(\frac{n^4}{n^5+7}\right)\left(\sqrt{n}-\sqrt{n+1}\right) = 0 \pm 10^{-8}$$

for all $n > N$? Explain.

Solution: From the given

$$|\frac{n^4}{n^5+7}| < 10^{-4} \quad 2pts \qquad \text{for } n > \frac{1}{10^{-4}} \quad 2pts$$

$$\left|\sqrt{n}-\sqrt{n+1}\right| < 10^{-4} \quad 2pts \qquad \text{for } n > \frac{1}{10^{-8}} \quad 2pts$$

Hence,

$$|\left(\frac{n^4}{n^5+7}\right)\left(\sqrt{n}-\sqrt{n+1}\right)| < 10^{-8}$$

for $n > \max\{\frac{1}{10^{-4}}, \frac{1}{10^{-8}}\}$ *2pts.*

Lemma 1, Lemma 2, and Proposition 1

LEMMA 1. *Suppose that* $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0$. *Then* $\lim_{n\to\infty} a_n b_n = 0$.

Proof Let $\epsilon > 0$ be given. We need to show that there is an N such that for all $n > N$,

$$|a_n b_n - 0| = |a_n b_n| \quad = |a_n|\,|b_n| < \epsilon \tag{1}$$

for all $n > N$.

This will be true if both of the following inequalities hold for all $n > N$:

$$\begin{aligned} |a_n| &< \sqrt{\epsilon} \\ |b_n| &< \sqrt{\epsilon} \end{aligned} \tag{2}$$

However, since $\lim_{n\to\infty} a_n = 0$, we know that there is a number N_1 such that the first inequality in (2) holds for all $n \geq N_1$. Similarly, there is a number N_2 such that the second inequality in (2) holds for all $n \geq N_2$. Both inequalities hold for $n > N = \max\{N_1, N_2\}$. Hence, inequality (2) holds for all $n > N$, proving our lemma. □

LEMMA 2. *If C is any constant, then* $\lim_{n\to\infty} Ca_n = C \lim_{n\to\infty} a_n$.

PROPOSITION 1. *For all* $k > 0$

$$\lim_{n\to\infty} \frac{1}{n^k} = 0.$$