MA 301 Test 3, Spring 2005

TA grades: 1, 3, 4, 5, 7
Prof. Grades 2, 6, 8, 9, 10

(1) Question 1: Study for Test 4.

State the “official” definition of “\( \lim_{x \to a} f(x) = L \).”

0, 5, 9 or 10 pts. The 9 pts. is if they omit 0 < |x - a|.

**Definition 1.** We say that \( \lim_{x \to a} f(x) = L \)

provided that for all numbers \( \epsilon > 0 \) there is a number \( \delta > 0 \)

such that

\[ |f(x) - L| < \epsilon \]

for all \( x \) satisfying \( 0 < |x - a| < \delta \).

(2) Assume that it is given that \( y = f(x) \) is increasing on \([0, 5]\)

and decreasing on \( [5, \infty) \). (See the figure below for a possible

graph of \( f \).) Let \( a_n = f(n) \) and \( s = \sum_1^\infty a_n \).

10 pts

(a) Find a specific value of \( n, a \) and \( b \) such that the following

inequality is guaranteed to hold. Choose both \( n \) and \( a \)
as large as possible and \( b \) as small as possible, consistent

with the information provided. Justify your answer with

a diagram. You may either use the figure below or draw

your own.

\[ a_1 + a_2 + \cdots + a_n < \int_a^b f(x) \, dx \]

**Solution:**

\[ a_1 + a_2 + a_3 + a_4 \leq \int_1^5 f(x) \, dx \]

For the figure draw 5 rectangles of width 1 with their

left edges beginning at \( x = 1, 2, 3, 4 \) respectively and

extending up to the curve. The above inequality is true

because the rectangles have their top edges below the

curve since \( f \) is increasing over \([0, 5]\).
Figure 1

(b) Find a specific value of $n$ and $a$ such that the following inequality is guaranteed to hold. Choose $a$ and $n$ as small as possible, consistent with the information provided. Justify your answer with a diagram. You may either use the figure below or draw your own.

$$s - s_n < \int_a^\infty f(x) \, dx$$

Solution:

$$s - s_5 \leq \int_5^\infty f(x) \, dx$$

For the figure draw rectangles of width 1 with their right edges beginning at $x = 6, 7, 8, 9, 10, \ldots$ and extending up to the curve. The sum of the areas of these rectangles is $s - s_5$. The above inequality is true because the rectangles have their top edges below the curve since $f$ is decreasing over $[5, \infty)$. 
(3) Question 3: Study for Test 4.
Prove that \( Z = \frac{1}{\sqrt{1 + \sqrt{2}}} \) is irrational. You may assume that \( \sqrt{2} \) is irrational. You MAY NOT use Proposition 1 from Chapter 9. 10 pts

Solution: Assume that \( Z \) is rational. 2 pt
Then \( Z = \frac{p}{q} \) where \( p \) and \( q \) are integers 2 pt. It must be stated somewhere in the solution that \( p \) and \( q \) are integers. with \( q \neq 0 \). Then

\[ \frac{p}{q} = \frac{1}{\sqrt{1 + \sqrt{2}}} \quad 2 \text{ pts.} \]
\[ \frac{q}{p} = \sqrt{1 + \sqrt{2}} \]
\[ \frac{q^2}{p^2} = 1 + \sqrt{2} \]
\[ \frac{q^2 - p^2}{p^2} = \sqrt{2} \quad 2 \text{ pts.} \]
Since $q^2 - p^2$ and $p^2$ are both integers 2 pts., we conclude that $\sqrt{2}$ is rational, which is nonsense. Hence $Z$ must be irrational.

10 pts

(4) Write a sum that expresses $s$ to within $\pm 10^{-3}$ where

$$ s = \sum_{1}^{\infty} \frac{2}{(2n + 1)^3}. $$

**Solution:**

According to Theorem 1,

$$ s - s_n < \int_{n}^{\infty} \frac{2}{(2x + 1)^3} \, dx $$$$ = \frac{1}{2(2n + 1)^2} $$

4 pts., -2 if they miss evaluate integral

This will be less than $10^{-3}$ provided

$$ \frac{1}{2(2n + 1)^2} < 10^{-3} $$

$$ 2(2n + 1)^2 > 10^3 $$

$$ 2n + 1 > \left( \frac{10^3}{2} \right)^{1/2} $$

$$ n > \frac{1}{2} \left( \frac{10^3}{2} \right)^{1/2} - \frac{1}{2} = 10.68 $$

4 pts. They may have the wrong answer for the integral. As long as they attempt to solve for $n$ they get these points.

Hence

$$ s = \sum_{1}^{11} \frac{2}{(2n + 1)^3} \pm 10^{-3}. $$

2 pts.

(5) Write a sum that expresses $s$ within $\pm 10^{-3}$ where

$$ s = \sum_{1}^{\infty} \frac{2}{(2n + 1)^3 + 17 \ln(n + 2) + 5}. $$

5 pts
Solution:

\[ \frac{2}{(2n+1)^3 + 17 \ln(n+2) + 5} \leq \frac{2}{(2n+1)^3} \]

3 pts.

From Problem 4, 11 terms suffice. Hence

\[ s = \sum_{1}^{11} \frac{2}{(2n+1)^3 + 17 \ln(n+2) + 5} \pm 10^{-3}. \]

2 pts.

(6) Prove, using \( M \), that the following series diverges. 10 pts

\[ \sum_{1}^{\infty} \frac{1}{\sqrt{n+3}} \]

Scratch work: According to Theorem 4

\[ s_n \geq \int_{1}^{n+1} (x+3)^{-\frac{1}{2}} \, dx \]

\[ = 2 \left( (4+n)^{\frac{1}{2}} - 2 \right) \]

\[ = 2\sqrt{4+n} - 4 \]

2 pts+1 pts

Then \( s_n \) is greater than \( M \) if:

\[ 2\sqrt{4+n} - 4 > M \quad 2 \text{ pts} \]

\[ \sqrt{4+n} > \frac{1}{2} M + 2 \]

\[ n > \left( \frac{1}{2} M + 2 \right)^2 - 4 \quad 2 \text{ pts} \]

Proof: Let \( M > 0 \) be given. 1 pt Let \( N = (\frac{1}{2} M + 2)^2 - 4.1 \) pt From the scratch work, for \( n > N, s_n > M \), 1 pt proving that \( \lim_{n \to \infty} s_n = \infty \).

(7) Is the following series convergent or divergent? Prove your answer. 10 pts

\[ \sum_{1}^{\infty} \frac{\ln n}{n^{1.1} + 1} \]
Solution: Convergent. There is an $N > 0$ such that for all $n > N$,

$$\ln n < n^{0.05} \quad (\text{or } n^a \text{ for any } 0 < a < .1).$$

3 pts

Then (3 pts for dealing with $N$)

$$\sum_{n=N+1}^{\infty} \frac{\ln n}{n^{1.1} + 1} < \sum_{n=N+1}^{\infty} \frac{n^{0.05}}{n^{1.1}} = \sum_{n=N+1}^{\infty} \frac{1}{n^{1.05}} < \infty$$

3 pts

since $\sum \frac{1}{n^p} < \infty$ for $p > 1.1$ pts It follows that

$$\sum_{n=1}^{\infty} \frac{\ln n}{n^{1.1} + 1} < \infty$$

proving convergence.

15 pts

(8) For which values of $p$, $p \geq 0$, is the following series:

(a) Divergent?
(b) Conditionally convergent?
(c) Absolutely convergent?

You must justify all of your answers.

$$\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n^5 + 1}}{n^p + 2}$$

Solution:

$$\frac{\sqrt{n^5 + 1}}{n^p + 2} \sim \frac{n^{2.5}}{n^p} \sim \frac{1}{n^{p-2.5}}.$$  

(a) If $2.5 \geq p > 0$, the series diverges since $\lim_{n \to \infty} \frac{\sqrt{n^5 + 1}}{n^{p-2.5}} \neq 0$ in this case.

(b) If $3.5 \geq p > 2.5$ the series converges conditionally because it is an alternating series and $\lim_{n \to \infty} \frac{\sqrt{n^5 + 1}}{n^{p-2.5}} = 0$ in this case.

(c) If $p > 3.5$ the series will converge absolutely since $\sum \frac{1}{n^q}$ converges for $q > 1$. 

(9) What is the set of \( x \) for which the following series converges? You need not prove your answer. However, you should explain your reasoning. 

\[
\sum_{n=1}^{\infty} \frac{\ln n}{2^n(n+1)} x^n
\]

**Solution:** \(-2 \leq x < 2\). 3 pts

This is the same as

\[
\sum_{n=1}^{\infty} \frac{\ln n}{n+1} \left(\frac{x}{2}\right)^n.
\]

If \( |x| > 1 \), this diverges because \( \left(\frac{x}{2}\right)^n \) grows exponentially while \( \frac{\ln n}{n+1} \) decays slowly. (Or one can say that in this case \( \lim_{n \to \infty} |a_n| = \infty \).) 2 pts

Similarly, if \( |x| < 1 \) it converges because \( \left(\frac{x}{2}\right)^n \) decays exponentially while \( \frac{\ln n}{n+1} \) decays. 3 pts

If \( x = 2 \), it diverges because \( \frac{\ln n}{n+1} > \frac{1}{n+1} \) for \( n > N \). If \( x = -2 \), it converges since it is an alternating series and \( \lim_{n \to \infty} \frac{\ln n}{n+1} = 0 \). 2 pts

(10) **Question 10:** Study for Test 4.

Find an explicit one-to-one correspondence between the set of odd integers and the integers that are multiples of 3. 10 pts

**Solution:**

\[ f(n) = 3 \frac{n + 1}{2} . \]

The following is worth 5 pts. (It is not really “explicit”.)

\[
\begin{array}{ccccccc}
\ldots & -3 & -1 & 1 & 3 & 5 & 7 & \ldots \\
\ldots & -3 & 0 & 3 & 6 & 9 & 12 & \ldots
\end{array}
\]

**Theorem (2').** Suppose \( a_n > 0 \) for all \( n \) and \( f(x) \) is an integrable, decreasing function on \([0, \infty)\) such that \( a_n = f(n) \) for all \( n \in \mathbb{N} \). Then \( s = \sum_{1}^{\infty} a_n \) exists if

\[
\int_{0}^{\infty} f(x) \, dx < \infty
\]
Proof Each \( a_n \) is the length of a line segment drawn from the point \((n,0)\) on the \(x\)-axis to the graph of \( y = f(x) \) as in Figure 3.

The area of a rectangle of width one having this line segment as its right edge is \( a_n \). (See Figure 4). This rectangle also lies entirely below the graph of \( y = f(x) \) since this graph is decreasing.

Since the left side of the first rectangle extends to \( x = 0 \),

\[
(1) \quad s_n = a_1 + a_2 + \cdots + a_n \leq \int_0^n f(x) \, dx \leq \int_0^\infty f(x) \, dx.
\]

Finally, since the \( a_n \) are all positive, \( s_n \) is an increasing sequence. From the Bounded Increasing Theorem, \( \lim s_n \) either exists or equals \( \infty \). Formula (1) proves that the limit is not \( \infty \). Hence the limit exists, proving the convergence of the sum. \( \Box \)

Various Results From The Text

Proposition (1, p.89). If \( \lim_{n \to \infty} a_n \neq 0 \), then \( \sum_{1}^{\infty} a_n \) cannot converge.
Theorem (1, p.89). Suppose $a_n > 0$ for all $n$ and $f(x)$ is an integrable, decreasing function on $[0, \infty)$ such that $a_n = f(n)$ for all $n \in \mathbb{N}$. Then

$$s - s_n \leq \int_n^\infty f(x) \, dx$$

Theorem (2, p.89). Suppose $a_n > 0$ for all $n$ and $f(x)$ is an integrable, decreasing function on $[0, \infty)$ such that $a_n = f(n)$ for all $n \in \mathbb{N}$. Then $s = \sum_1^\infty a_n$ exists if there is a $k$ such that

$$\int_k^\infty f(x) \, dx < \infty$$

Theorem (3, p.91). The following series converges for all $p > 1$.

$$\sum_1^\infty \frac{1}{n^p}$$

Remark 1: The series in Theorem 3 above diverges if $p \leq 1$.

Theorem (4, p.94). Suppose $a_n > 0$ for all $n$ and $f(x)$ is an integrable, decreasing function on $[0, \infty)$ such that $a_n = f(n)$ for all $n \in \mathbb{N}$. Then

$$s_n \geq \int_1^{n+1} f(x) \, dx$$

Theorem (5, p. 95). Suppose that $0 \leq a_n \leq b_n$ for all $n$. Then $\sum_1^\infty a_n$ will converge if $\sum_1^\infty b_n$ converges.

Theorem (6, p. 96). Suppose that in Theorem 5 above, the sum of the first $N$ $b_n$ approximates $\sum_1^\infty b_n$ to within $\pm \epsilon$. Then the same will be true for $a_n$: i.e. the sum of the first $N$ $a_n$ will approximate $\sum_1^\infty a_n$ to within $\pm \epsilon$.

Theorem (7, p. 98). Let $x$ be a real number. Then the series on the right side of the following equality converges if, and only if,
$|x| < 1$. Furthermore, when it converges, it converges to the stated value.

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \ldots$$

\textbf{Remark 2:} $\sum_1^{\infty} a_n$ converges if and only if there is an $N$ such that $\sum_N^{\infty} a_n$ converges.

\textbf{Theorem (1, p. 111).} Let $a_n$ be a sequence of real numbers. Then $\sum_1^{\infty} a_n$ will converge if $\sum_1^{\infty} |a_n|$ converges.

\textbf{Theorem (2, p. 114).} Suppose that $a_n$ is a positive, decreasing sequence where $\lim_{n \to \infty} a_n = 0$. Then

$$s = \sum_1^{\infty} (-1)^n a_n$$

converges. Furthermore

$$|s - s_n| < a_{n+1}$$