MA 301 Test 4, Fall 2005

TA Grades 1-4 and 6

(1) State the "official" definition of "$\lim_{x\to a} f(x) = L$." 4 pts

0 or 4 pts.

DEFINITION 1. *We say that*

$$\lim_{x\to a} f(x) = L$$

provided that for all numbers $\epsilon > 0$ there is a number $\delta > 0$ such that

$$|f(x) - L| < \epsilon$$

for all x satisfying $0 < |x - a| < \delta$.

(2) Use a δ-ϵ argument to prove that 12 pts

$$\lim_{x\to 2} \frac{3x}{x+1} = 2.$$

Scratch Work: *Let $\epsilon > 0$ be given. We want*

$$\left|\frac{3x}{x+1} - 2\right| < \epsilon \quad \textit{2 pts.}$$

$$\left|\frac{x-2}{x+1}\right| = |x-2|\,\frac{1}{|x+1|} < \epsilon$$

2 pts. for simplification

Assume that $x = 2 \pm 1$ (1 pt. for $2 \pm \delta$ (any δ)) so that $1 < x < 3$ and $2 < x + 1 < 3$. Then

$$|x-2|\left|\frac{1}{x+1}\right| < \frac{1}{2}|x-2| \quad \textit{3 pts.}$$

This will be $< \epsilon$ if $|x - 2| < 2\epsilon$.

Proof: Let $\epsilon > 0$ be given *1 pt.* and let $\delta = \min\{1, 2\epsilon\}$ *1 pt.*. Assume that $0 < |x-2| < \delta$.*1 pt.* Then from the scratch work

$$\left|\frac{3x}{x+1} - 2\right| < \epsilon, \quad \text{1 pt.}$$

proving the limit statement.

(3) Use a δ-ϵ argument to prove that 12 pts

$$\lim_{x\to 3}\sqrt{x+1}=2.$$

Scratch Work: *Let $\epsilon>0$ be given. We want*

$$\left|\sqrt{x+1}-2\right|<\epsilon \quad \textit{2 pt.}$$

$$\frac{|\sqrt{x+1}-2|\,|\sqrt{x+1}+2|}{\sqrt{x+1}+2}<\epsilon \quad \textit{1 pt.}$$

$$\frac{|(x+1)-4|}{\sqrt{x+1}+2}<\epsilon$$

$$|x-3|\,\frac{1}{\sqrt{x+1}+2}<\epsilon \quad \textit{1 pt. for simplification}$$

Assume that $x=3\pm 1$ 1 pt. for $3\pm\delta$, any δ so that

$$2<x<4$$

$$3<x+1<5$$

$$\sqrt{3}<\sqrt{x+1}<\sqrt{5}$$

$$2+\sqrt{3}<\sqrt{x+1}+2<2+\sqrt{5} \quad \textit{2 pt.}$$

Then

$$|x-3|\,\frac{1}{\sqrt{x+1}+2}<|x-3|\,\frac{1}{2+\sqrt{3}} \quad \textit{1 pt.}$$

This will be $<\epsilon$ if $|x-1|<(2+\sqrt{3})\epsilon$.

Proof: *Grade same as Problem 2* Let $\epsilon>0$ be given and let $\delta=\min\{1,(2+\sqrt{3})\epsilon\}$. Assume that $0<|x-3|<\delta$. Then from the scratch work

$$\left|\sqrt{x+1}-2\right|<\epsilon,$$

proving the limit statement.

12 pts (4) Use a δ-ϵ argument to prove that

$$\lim_{x\to 1}\frac{1}{2x-1}=1.$$

Solution:

Scratch Work: *Let* $\epsilon > 0$ *be given. We want*

$$\left|\frac{1}{2x-1} - 1\right| < \epsilon \quad \textit{2 pt.}$$

$$\left|\frac{2-2x}{2x-1}\right| = |x-1|\,\frac{2}{|2x-1|} < \epsilon$$

2 pt. for simplification

Assume that $x = 1 \pm .25$ *2 pt. for* $3 \pm \delta$*, any* δ*. Then*

$$.75 < x < 1.25$$
$$1.5 < 2x < 2.5$$
$$.5 < 2x - 1 < 1.5$$

2 pt.: but interval cannot contain 0

Hence

$$|x-1|\,\frac{2}{|2x-1|} < \frac{2}{.5}|x-1|$$

This will be $< \epsilon$ *if* $|x-1| < 4\epsilon$*.*

Proof: *Grade as in Exercise 2* Let $\epsilon > 0$ be given and let $\delta = \min\{.25, 4\epsilon\}$. Assume that $0 < |x-1| < \delta$. Then from the scratch work

$$\left|\frac{1}{2x-1} - 1\right| < \epsilon,$$

proving the limit statement.

(5) Assume that $\lim_{x\to a} f(x) = 2$. Use a δ-ϵ argument to prove that 12 pts

$$\lim_{x\to a}(f(x)+1)^2 = 9.$$

Scratch work: *Let* $\epsilon > 0$ *be given. We want*

$$\left|(f(x)+1)^2 - 9\right| < \epsilon \quad \textit{2 pts.}$$

$$\left|f(x)^2 + 2f(x) - 8\right| < \epsilon$$

$$|f(x)-2|\,|f(x)+4| < \epsilon \quad \textit{2 pts.}$$

The term on the left is our "gold" since it becomes small as x approaches a. The other term is our "trash" which we *will bound. Specifically, we reason that for all x sufficiently close to a, $f(x) = 2 \pm 1$. Thus, for such x,*

$$1 < f(x) < 3 \quad \textit{2 pts.}$$
$$5 < f(x) + 4 < 7 \quad \textit{1 pts.}$$

Hence

$$|f(x) - 2|\,|f(x) + 4| < 7\,|f(x) - 2| \quad \textit{1 pts.}$$

This is $< \epsilon$ if $|f(x) - 2| < \epsilon/7$, which is true for all x sufficiently close to a.

Proof: Let $\epsilon > 0$ 1 pts. be given and choose $\delta_1 > 0$ 1 pts. so that

$$|f(x) - 2| < 1$$

for $0 < |x - a| < \delta_1$.

Choose $\delta_2 > 0$ such that

$$|f(x) - 2| < \epsilon/7 \quad \text{1 pts.}$$

for $0 < |x - a| < \delta_2$. Let $\delta = \min\{\delta_1, \delta_2\}$ 1 pts.. From the scratch work, $0 < |x - a| < \delta$ implies that

$$\left|(f(x) + 1)^2 - 9\right| < \epsilon$$

proving the limit statement.

12 pts (6) Let $f(x) = (x + 1)^4$.

(a) Find constants m and M such that

$$m \le f''(x) \le M$$

for all $x \in [0, 3]$. (You need not prove that your values really work.)

Solution *3 pts.*

$$f(x) = (x + 1)^4$$
$$f'(x) = 4(x + 1)^3$$
$$f''(x) = 6(x + 1)^2$$

For $0 \le x \le 3$, $1 < x + 1 < 4$. *This statement is optional.* Hence

2 pts.

$$6 < 6(x+1)^2 < 6 \cdot 4^2 = 96$$

(b) Use the information from part 6a and integration to obtain constants a_0, a_1, C and D such that

$$Cx^2 \le (x+1)^4 - (a_0 + a_1 x) < Dx^2$$

holds for all $x \in [0,3]$. *Do not use Maclaurin's theorem on p. 187. Your method, however, should parallel the proof of this theorem.*

Solution:

We integrate the last inequality from 0 to x twice:

$$x < 4(x+1)^3\Big|_0^x < 96x \quad \text{2 pts.}$$

$$x < 4(x+1)^3 - 4 < 96x \quad \text{1 pts.}$$

$$\frac{x^2}{2} < (x+1)^4\Big|_0^x - 4x < 48x^2 \quad \text{2 pts.}$$

$$\frac{x^2}{2} < (x+1)^4 - (1+4x) < 48x^2 \quad \text{1 pts.}$$

Hence $C = \frac{1}{2}$, $D = 48$, $a_0 = 1$, $a_1 = 2$. *2 pt.*

(7) Suppose that $f(x)$ is continuous at $x = a$. Prove that $h(x) = \frac{f(x)}{1+f(x)^2}$ is also continuous at $x = a$. You may use the product and ratio theorems for limits of functions from Chapter 10. 12 pts

Solution:

$$\begin{aligned}
\lim_{x\to a} h(x) &= \lim_{x\to a} \frac{f(x)}{1+f(x)^2} \quad \text{3 pts.} \\
&= \frac{\lim_{x\to a} f(x)}{\lim_{x\to a}(1+f(x)^2)} \quad \text{Quotient Theorem} \quad \text{3 pts.} \\
&= \frac{\lim_{x\to a} f(x)}{(1+(\lim_{x\to a} f(x))^2)} \quad \text{Product and Sum Theorems} \quad \text{3 pts.} \\
&= \frac{f(a)}{1+f(a)^2} = h(a) \quad \text{Continuity of } f \quad \text{3 pts.}
\end{aligned}$$

Since $\lim_{x\to a} h(x) = h(a)$, h is continuous at $x = a$.

(8) Suppose that f is continuous at every x in $[0, \pi]$ and that for all x in this interval, $0 \le f(x) \le 1$. Prove that there is

an $x \in [0, \pi]$ such that $f(x) = \cos x$. *Be explicit about the theorem(s) you are using and how you are using it (them).* FYI: $\cos \pi = -1$.

Solution: Let

$$g(x) = f(x) - \cos x. \quad \text{3 pts.}$$

Then

$$g(0) = f(0) - 1 \leq 0. \quad \text{2 pts.}$$

If $f(0) = 1$ then $f(0) = \cos 0$ so there is an x such that $f(x) = \cos x$ 2 pts.. Hence we may assume that $g(0) < 0$.

Also

$$g(\pi) = f(1) - \cos \pi = f(1) + 1 > 0. \quad \text{2 pts.}$$

Since $g(0) < 0$ and $g(\pi) > 0$, it follows *from the Intermediate Value Theorem* 3 pts. that there is an x between 0 and 1 such that $g(x) = 0$; hence there is an x such that

12 pts $f(x) = \cos x$.

(9) Use a δ-ϵ argument to prove Theorem 3 on p. 164 of the

12 pts notes:

THEOREM 3 (Sequence). *Let $f(x)$ be continuous at a and let x_n be a sequence such that $\lim_{n\to\infty} x_n = a$. Then*

$$\lim_{n\to\infty} f(x_n) = f(a).$$

Proof Let $\epsilon > 0$ 1 pt. be given. Since $\lim_{x\to a} f(x) = f(a)$, there is a $\delta > 0$ such that

(1) $$|f(x) - f(a)| < \epsilon. \quad \text{4 pt.}$$

for $|x - a| < \delta$, $x \neq a$. This inequality holds even if $x = a$ since in this case the left hand quantity is zero. 1 pt.

But, since $\lim_{n\to\infty} x_n = a$, there is an N such that

$$|x_n - a| < \delta \quad \text{4 pt.}$$

for all $n > N$. Replacing x with x_n in (1) shows that

$$|f(x_n) - f(a)| < \epsilon \quad \text{3 pt.}$$

for $n > N$, which proves our theorem.