HUA OPERATORS ON BOUNDED HOMOGENEOUS DOMAINS IN $\mathbb{C}^n$ AND ALTERNATIVE REPRODUCING KERNELS FOR HOLOMORPHIC FUNCTIONS.

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0. Introduction

A major theme in the study of function theory on bounded domains in $\mathbb{C}^n$ is the study of the “boundary values” of holomorphic functions on the Bergman-Shilov boundary. Often the study of boundary values involves defining a suitable class of real valued “harmonic” functions. Ideally such a class should:

1. Contain all real and imaginary parts of bounded holomorphic functions.
2. Be describable as “Poisson integrals” over the Bergman-Shilov boundary against a real kernel (the “Poisson” kernel).
3. Be invariant under all bi-holomorphisms of the domain.
4. Be describable as the nullspace $H_L$ of a degenerate-elliptic system $L$ of second order differential operators. (We refer to $H_L$ as the space of $L$-harmonic functions.)

In the literature at least two classes of harmonic functions and their boundary behavior have been investigated: the $\Delta$-harmonic functions where $\Delta$ is the Laplace-Beltrami operator and the Poisson-Szegö integrals of functions on the Bergman-Shilov boundary as defined in [H] and [K]. Neither of these classes is entirely satisfactory in that in general the $\Delta$-harmonic functions fail the second condition and the Poisson-Szegö integrals fail the fourth. ([BV]).

In this work we study several different classes of functions which satisfy some (possibly weakened) form of the above conditions in the context of bounded homogeneous domains in $\mathbb{C}^n$. The study of this class of domains is already both interesting and challenging in that in general for such domains the Bergman-Shilov boundary is much smaller than the topological boundary and the topological boundary is not smooth. (c.f. [S1] and [S2]).

We make heavy use of the fact that any such domain is realizable as a Siegel domain of type I or II. Explicitly let $\mathcal{V} \subset \mathbb{R}^n$ be an open convex cone which does not contain straight lines. We assume that the cone $\mathcal{V}$ is homogeneous i.e. there is an algebraic subgroup $S$ of $\text{Gl}(n, \mathbb{R})$ which acts transitively on $\mathcal{V}$ via the usual representation of $\text{Gl}(n)$ on $\mathbb{R}^n$. (We denote this representation by $\pi$.) $S$ may be taken to be a triangular subgroup which acts simply transitively on $\mathcal{V}$. Suppose

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further that we are given a complex vector space $\mathcal{Z}$ and a Hermitian symmetric bi-linear mapping $K : \mathcal{Z} \times \mathcal{Z} \to \mathbb{C}^n$. We shall assume that

(a) $K(z, z) \in \mathcal{V}$ for all $z \in \mathcal{Z}$
(b) $K(z, z) = 0$ implies $z = 0$

The Siegel domain $\mathcal{D}$ associated with this data is defined as

$$\mathcal{D} = \{(z_1, z_2) \in \mathcal{Z} \times \mathbb{C}^n : \exists z_2 - K(z_1, z_1) \in \mathcal{V}\}.$$

The domain is said to be type I or III depending upon whether or not $\mathcal{Z}$ is non-trivial.

The Bergman-Shilov boundary $\mathcal{B}$ of $\mathcal{D}$ is defined as

$$\mathcal{B} = \{(z_1, z_2) \in \mathcal{Z} \times \mathbb{C}^n : \exists z_2 = K(z_1, z_1)\}.$$

Suppose further that we are given a complex linear algebraic representation $\sigma$ of $S$ in $\mathcal{Z}$ such that

$$K(\sigma(s)z, \sigma(s)w) = \rho(s)K(z, w) \text{ for all } z, w \in \mathcal{Z}.$$

The group $S$ acts on $\mathcal{D}$ by

$$s(z, w) = (\sigma(s)z, \rho(s)w).$$

We let $\mathbb{R}^n$ act on $\mathcal{D}$ by translation:

$$x(z, w) = (z, w + x), \quad x \in \mathbb{R}^n.$$

Finally we let $\mathcal{Z}$ act by

$$z_0(z, w) = (z + z_0, w + 2iK(z, z_0) + iK(z_0, z_0)).$$

These actions generate a completely solvable group $G$ which acts simply transitively on $\mathcal{D}$. The action of the group $G$ extends to $\mathcal{B}$ and the nilpotent group $N$ generated by transformations (0.2) and (0.3) acts simply transitively on $\mathcal{B}$.

Every bounded homogeneous domain in $\mathbb{C}^n$ is biholomorphic to a homogeneous Siegel domain on which the group $G$ described above acts simply transitively. This group plays a fundamental role in our theory.\footnote{The importance of the group $G$ was noticed by Korányi and Stein almost thirty years ago in their study of the Hardy spaces $H^\vartheta(\mathcal{D})$ cf. e.g., [KS1] and [KS2]. Later the group $G$ and its representations played a fundamental role in the work of Rossi and Vergne, [RV1], [RV2].}

In fact in [D] and [DH] a general class of solvable lie groups which includes $G$ were studied. These results apply to the space of bounded $L$-harmonic functions for a single second order degenerate-elliptic $G$-invariant operator $L$ which also satisfies the Hörmander condition. Following Furstenberg, Guivarc'h and Raugi it was shown how to associate with every such operator $G$ a class of boundaries and on each boundary a Poisson kernel $P_L$. It is also proved that every bounded
$L$-harmonic is the integral over the maximal boundary of a function against the corresponding Poisson kernel. Using these results we prove the following which is one of the main results of the current work:

Let $L$ be a $G$-invariant, real, second order operator which satisfies the Hörmander condition and annihilates holomorphic functions on a homogeneous Siegel domain $\mathcal{D}$. The Shilov-Bergman boundary $\mathcal{B}$ is one of the boundaries associated with $L$. Let $P_{L}$ be the corresponding Poisson kernel on $\mathcal{B}$. Then every bounded holomorphic function $F$ on $\mathcal{D}$ is the Poisson integral $F = P_{L}(f)$ of the boundary values $f$ of $F$ on $\mathcal{D}$.

Moreover,

For every homogeneous Siegel domain there exists an operator $L$ as above for which the maximal boundary is $\mathcal{B}$.

In fact for a given homogeneous domain there are many such operators.

Taken together the above results imply that the space of $L$-harmonic functions satisfy conditions (1)-(4) stated above except that in condition (3) invariance under the full automorphism group of the domain is replaced by the weaker condition of invariance under the transitive group $G$. On the other hand condition (4) is strengthened—harmonicity is defined in terms of the nullspace of a single differential operator. This may be viewed as a characterization of the Bergman-Shilov boundary by means of a differential operator suggested by E. M. Stein many years ago.

If the operator $L$ were invariant under all of Aut ($\mathcal{D}$) then of course the stronger form of condition (3) would follow. It general however it seems that the algebra of Aut ($\mathcal{D}$)-invariant differential operators may be just the algebra generated by $\Delta$ the Laplace-Beltrami operator for the Bergman metric on $\mathcal{D}$. In general the $\Delta$-harmonic bounded functions are not reproducible from their boundary values on the Bergman-Shilov boundary except if $\mathcal{D}$ is a product of balls. Thus it seems that in order to retain condition (3) we are forced to consider invariant systems of differential operators. In this work following an idea suggested to us by Nolan Wallach we define a canonical system $HJK$ (the Hua system) in terms of a contraction of $\partial\bar{\partial}$ against the curvature tensor. Our main result concerning this system is:

For every homogeneous Siegel domain, there exists a canonical system which we call the Hua system and denote $HJK$. The space of $HJK$ harmonic functions $H_{HJK}$ is Aut ($\mathcal{D}$)-invariant and every bounded function $F$ in $H_{HJK}$ is a Poisson integral $P_{HJK}(f)$ of a bounded function $f$ on $\mathcal{B}$.

For the Poisson kernel $P_{HJK}$ one can take the Poisson kernel $P_{L}$ on $\mathcal{B}$ corresponding to any $G$-invariant operator $L$ which is subelliptic and is a linear combination of the elements of the system $HJK$. In particular, we may use the Laplace-Beltrami operator $\Delta$ as $L$.

Therefore

Every bounded, $HJK$-harmonic function is the integral over the Bergman-Shilov boundary of a uniquely determined bounded function against the Poisson kernel for $\Delta$. 

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In the case that $\mathcal{D}$ is a Hermitian symmetric tube domain, our $HJK$ system is the one which Johnson and Korányi [JK] generalize earlier work by Hua [HT]. However, the kernel $P_{HJK}$ is equal to the standard Poisson-Szegő kernel $P_{\mathcal{S}}$ on $\mathcal{B}$ (cf. [H] and [K]) iff $\mathcal{D}$ is a Hermitian-symmetric tube domain.

It might appear that the Laplace-Beltrami operator plays a special role in this theory. In fact, this is not the case. It is possible to define a whole class of elliptic second order differential operators for which our results hold. In fact, there are cases where $\Delta$ does not provide the sharpest results. Explicitly, we show that

For the tube domain over the cone of real, positive definite $n \times n$ matrices there exists a single $G$-invariant elliptic second order differential operator $\Delta'$ such that the functions in $H_{\Delta}$, are precisely the Poisson-Szegő integrals of bounded functions on the Bergman-Shilov boundary. The operator $\Delta'$ is a linear combination of the "diagonal" elements of $HJK$.

For $n = 2$ Malliavin and Korányi [KM] (cf. also [J1] and [J2]) exhibited a system $\mathcal{L}$ of two $G$-invariant operators for which $H_{\mathcal{L}}$ consists of Poisson-Szegő integrals of $L^{\infty}$ functions on the Bergman-Shilov boundary.

For an arbitrary symmetric domain Berline and Vergne [BV] exhibited a third order $\text{Aut} (\mathcal{D})$-invariant system $\mathcal{L}$ for which $H_{\mathcal{L}}$ consists of Poisson-Szegő integrals of $L^{\infty}$ functions on the Bergman-Shilov boundary. In [D] and [DH] some probabilistic tools are used so restriction to the second order degenerate elliptic operators is necessary. This also explains why we are unable to go beyond second order systems in the present paper.

Our proofs are inductive, relying both on the characterization of bounded homogeneous domains as Siegel domains of type I and II due to [PS] as well as the structure theory of homogeneous cones due to [V]. We also use the results of [DH].

Section 1. The Hua Operators

In this section, we define the Hua operators in general and compute them in the context of a bounded homogeneous domain.

Let $\mathcal{D}$ be a Kählerian manifold and let $T$ be the (real) tangent bundle for $\mathcal{D}$. (We shall not need to indicate its dependence on $\mathcal{D}$ in our notation.) We assume that the reader is familiar with the basic properties of Kählerian manifolds and their Riemannian connection. (See e.g., [He]). Let

$$T_c = T^{10} \oplus T^{01}$$

be the decomposition of $T_c$ into holomorphic and anti-holomorphic vector fields. ($T_c$ is the complex tangent bundle.) We have a similar decomposition

$$T_c^* = (T^*)^{10} \oplus (T^*)^{01}$$

where $(T^*)^{ij}$ is the annihilator of $T_{ji}$ in $T^*_c$. Hence, $(T^*)^{ij}$ is the dual space of $T^{ij}$.

We define an operator $\overline{\partial} \partial : C^\infty (\mathcal{D}) \rightarrow \Gamma((T^*)^{10} \otimes (T^*)^{01})$ in local holomorphic coordinates by

$$\overline{\partial} \partial f = \frac{\partial^2 f}{\partial \overline{z}_j} dz_i \otimes \partial \overline{z}_j.$$
Now let \( g \) be the Riemannian structure for \( \mathcal{D} \) and let \( \nabla \) denote the corresponding Riemannian connection. For a \( C^\infty \) function \( f \) we define a 2-tensor by
\[
\nabla^2 f(X,Y) = (XY - \nabla_X Y)f
\]

Then on a Kählerian manifold we have the following:

(1.1) Lemma. For all \( f \in C^\infty(\mathcal{D}) \Gamma \partial \bar{\partial} f = \nabla^2 f| T^{10} \times T^{01} \)

Proof This follows immediately from the above formula for \( \nabla^2 f \) and the observation that for all \( i \) and \( j \Gamma \)
\[
\nabla Z_j Z_i = 0 \quad \text{and} \quad \nabla Z_j \bar{Z}_i = 0
\]

where \( Z_j = \frac{\partial}{\partial z_j} \). (See the material below formula (12) in [He]).

A usual we define the curvature operator by
\[
R(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}
\]

where \( X \) and \( Y \) are complex vector fields. We interpret \( R \) as a \( \text{End}(T_c, T_c) \) valued two form on \( \mathcal{D} \). (We extend \( \nabla \) to \( T_c \) by complex linearity.) We also extend \( g \) to the complex tangent bundle by complex linearity. We shall let \( H \) be the Hermitian form on \( T_c \) defined by
\[
H(Z, W) = \frac{1}{2} g(Z, \overline{W}).
\]

Let \( \{ E_1, \ldots, E_n \} \subset T^{10}_c \) be a local orthonormal frame for \( T^{10}_c \) (orthonormal with respect to \( H \)). For \( f \in C^\infty(\mathcal{D}) \Gamma \) we define
\[
(1.2) \quad HJK(f) = - \sum \overline{\partial} f(E_i, E_j) R(E_i, E_j)| T^{01}
\]

It is easily seen that this is independent of the orthonormal frame.

It is clear that \( HJK \) annihilates holomorphic functions. The next lemma will establish that \( HJK \) is real in the sense defined in the introduction.

(1.3) Lemma. For all \( Z \) and \( W \) in \( \Gamma(T^{01}) \) and all \( f \in C^\infty(\mathcal{D}) \Gamma \)
\[
H(HJK(f)Z, W) = H(Z, HJK(\overline{f})W).
\]

Proof This follows from formula (1.2) along with the observations that for all \( UTVTZ \) and \( W \) in \( \Gamma(T_c) \Gamma \)
\[
\]
and for all $Z$ and $W \in \Gamma(T^{01})$,
\[
\overline{\partial \partial f(W, Z)} = \overline{\partial \partial f(Z, W)}.
\]

The next proposition establishes that every Hua-harmonic function is in the kernel of $\Delta$ where $\Delta$ is the Laplace-Beltrami operator.

(1.4) Proposition. For all $f \in C^\infty(D)\Gamma$
\[
\text{Tr } HJK(f) = \Delta(f).
\]

Proof. We note first that for all $Z$ and $W$ in $T^{01}_x\Gamma$
\[
\text{Tr } R_x(Z, W)|T^{10}_x = -r_x(Z, W).
\]

where $r$ is the Ricci curvature. (This formula follows easily from formula (5) in [He] and with the identity $R^k_{ij} = R^k_{ij} = -R^k_{ij'1}$ where the notation is as in [He] loc. cit.)

From Proposition 3.6 in [He] along with formula (1.2) above $\Gamma$ we see that
\[
(1.5) \quad \text{Tr } HJK(f) = 2 \sum \overline{\partial \partial f(E_i, \overline{E_j})} H(E_i, E_j) = 2 \sum \overline{\partial \partial f(\overline{E_i}, E_i)}.
\]

On the other hand, it is known that $\Delta f$ is the contraction of $\nabla^2 f$ ([O] p. 86.) It is easily seen from Lemma (1.1) that this is exactly the quantity on the right. □

Our next goal is to compute a formula for $HJK$ in the case that $D$ is a bounded homogeneous domain. Thus in view of [PS] and [V] we may assume that there is a connected, simply connected Lie group $G$ which acts simply transitively on $D$ and that this action is real analytic in the $G$-variable and is holomorphic in the $D$ variable. We let $x_0$ be a fixed base point in $D$.

Let $\mathcal{G}$ denote the Lie algebra of $G$. In general we shall adopt the convention that upper case Roman letters will be used to denote Lie groups and that the corresponding upper case script letter will automatically denote the corresponding Lie algebra.

The complex tangent space $(T_x)_{x_0}$ may be identified with $\mathcal{G}_c$ and $G$-invariant vector fields on $D$ with left-invariant vector fields on $G$. The set of elements $X$ in $\mathcal{G}_c$ which annihilate holomorphic functions at $x_0$ is denoted by $\mathcal{P}$. Clearly $\mathcal{P}$ is a complex subalgebra of $\mathcal{G}_c$. Since left translation preserves holomorphic functions $\Gamma$ vector field $X \in \mathcal{P}$ is a section of the bundle $T^{01}$. We let $Q = \overline{\mathcal{P}}$. The vector fields valued in $Q$ define the sections of the bundle $T^{10}$.

Note that since $T_c = T^{10} \oplus T^{01}\Gamma$ we have
\[
\mathcal{G}_c = \mathcal{P} \oplus Q
\]
Let $\pi_Q$ be the projection to $Q$ along $P$. For each $Z \in \mathcal{P}_G\Gamma$ we define an operator $M(Z) : Q \to Q$ by

$$M(Z)(X) = \pi_Q([Z, X]).$$

(1.6)

To compute the $HJK$ operator we must compute the connection. Notice that for $X \in \Gamma(T^{10})\Gamma Z \in \Gamma(T^{01})\Gamma$ and $f$ any (local) holomorphic function

$$\nabla_Z X(f) = \nabla_X Z(f) + [X, Z]f = [X, Z]f$$

since the torsion is zero and the connection preserves holomorphic type. It follows that

$$\nabla_Z X(f) = \pi([X, Z])(f)$$

(1.7)

where $\pi$ is the projection to $T^{10}$ along $T^{01}$ in $T_c$. Hence $\nabla_Z X(f) = \pi([X, Z])$. In particular for $Z \in \mathcal{P}_G X \in Q$ we have

$$\nabla_Z X = M(Z)(X).$$

Since the connection is real we may also state that

$$\nabla_X Z = M(X)(Z),$$

where

$$M(X)Z = \overline{M(X)\overline{Z}}.$$

We will also need to know the connection on other types of forms. Since the Riemannian structure is invariant the form $g$ is defined by a scalar product $g$ on $G$ and $H$ is defined as above by a Hermitian scalar product (still called $H$) on $G_c$. For $Z \in \mathcal{Q}_G\Gamma\Gamma X \in Q$ we define an operator $M^*(Z) : Q \to Q$ by the identity

$$H(M^*(Z)X, Y) = H(X, M(Z)Y)$$

where $X$ and $Y$ range over $Q$. Thus $M^*(Z)$ is the adjoint in $H$ of $M(Z)$.

(1.8) Proposition. Let $Z$ and $X$ be elements of $Q$. Then

$$\nabla_Z X = -M^*(Z)(X).$$

Proof On a Kähler manifold the connection preserves holomorphic types. Therefore $\nabla_Z X$ is of type $(1, 0)$. Furthermore

$$\overline{\nabla_Z X} = \nabla_{\overline{Z}} X$$

We compute:

$$ZH(X, Y) = H(\nabla_Z X, Y) + H(X, \nabla_{\overline{Z}} Y).$$
for $Y \in \Gamma(T^{10})$. But for $XTY$ $G$-invariant $\Gamma Z H(X, Y) = 0$ and so

$$H(\nabla_Z X, Y) = -H(X, \nabla_Z Y) = -H(X, M(Z)Y) = -H(M^*(Z)X, Y).$$

Again, since the connection is real, we may write

$$\nabla_Z X = \nabla_Z Y = -M^*(Z)X$$

where $M^*(Z)X = M^*(Z)X$. 

Next, we compute the curvature. Our result is:

**(1.9) Theorem.** For $X$ and $Z$ in $Q$ and $W \in \mathcal{P}\Gamma$ the form $R$ defined below is the curvature tensor at the identity $e$ of $G$.


**Proof** It follows easily that at $e$

$$(\nabla_Z \nabla_W - \nabla_W \nabla_Z)X = (-M^*(Z)M(W) + M(W)M^*(Z))X$$

Also,

$$[Z, W] = \nabla_Z W - \nabla_W Z = M(Z)W - M(W)Z$$

It follows that at $e$

$$R(Z, W)X = (-M^*(Z)M(W) + M(W)M^*(Z) - M(M(Z)W) - M^*(M(W)Z)X.$$ 

This proves our formula.

**Section 2. Hua operators on type I Domains**

The Siegel domain of type I associated with a homogeneous regular cone $\mathcal{V}$ (as described in the introduction) is the domain in $\mathbb{C}^n$ defined by

$$\mathcal{E} = \mathbb{R}^n + i\mathcal{V}.$$ 

i.e. for such domains the space $\mathcal{Z}$ is trivial. Let $S$ an algebraic subgroup $S$ of $\text{Gl}(n, \mathbb{R})$ which acts transitively on $\mathcal{V}$ via the usual action of $\text{Gl}(n)$ on $\mathbb{R}^n$. It is a result of [V] that $S$ may be taken to be a triangular subgroup which acts simply transitively on $\mathcal{V}$. We may also assume that $S$ contains $tI$ for all $t \in \mathbb{R}^+$. We shall let $e \in \mathcal{V}$ be a fixed base point.

The group $S$ acts on $\mathcal{E}$ by matrix multiplication. We let $\mathcal{M} = \mathbb{R}^n$ thought of as a commutative Lie algebra. The corresponding Lie group $M$ is $\mathbb{R}^n$ under addition. This group acts on $\mathcal{E}$ by translation. These two actions generate a simply transitive subgroup $G$ of the automorphism group of $\mathcal{E}$. The group $G$ is the semi-direct product $G = M \times_s S$ where the $S$ action on $M$ is matrix multiplication. We
shall identify $\mathcal{M}$ and $\mathcal{S}$ with the corresponding subalgebras of $\mathcal{G}$ and hence $\mathcal{M}$ and $\mathcal{S}$ with subgroups of $\mathcal{G}$.

Let $\rho$ be the representation of $\mathcal{S}$ on $\mathbb{R}^n$ defined by letting $\mathcal{S}$ act on $\mathbb{R}^n$ by matrix multiplication. We shall also let $\rho$ denote the action of the Lie algebra $\mathcal{S}$ on $\mathbb{R}^n$ obtained by differentiating $\rho$. Since $\mathcal{S}$ acts simply transitively the mapping $\sigma$ of $\mathcal{S}$ into $\mathbb{R}^n$ defined by

$$\kappa(X) = \rho(X)c$$

is a vector space isomorphism. We extend $\rho$ and $\kappa$ to $\mathcal{S}_c$ by complex linearity. Then we have the following:

**2.1 Lemma.** $\mathcal{P} = \{(\kappa(Y), iY)|Y \in \mathcal{S}_c\}$.

*Proof.* We consider $\mathcal{E} \subset \mathbb{R}^n \times \mathbb{R}^n$. Then the tangent space at $ic$ is $\mathbb{R}^n \times \mathbb{R}^n$. The tangent space is also identified with $\mathcal{G} = \mathcal{M} \times_s \mathcal{S}$. The identification is defined by mapping $(X, Y) \in \mathcal{G}$ into $(X, \kappa(Y)) \in \mathbb{R}^n \times \mathbb{R}^n$. Under this identification the space defined in the statement of the lemma maps onto the Cauchy-Riemann operators proving the lemma. $\square$

It follows that the complex structure on the tangent space is defined by the mapping $J : (\kappa(X), Y) \to (-\kappa(Y), X)$.

There is an algebraic description of the general homogeneous cone which is due to Vindberg which we shall require. We define a product $\Delta$ on $\mathcal{S}$ by the equality

$$X\Delta Y = \kappa^{-1}(\rho(X)\rho(Y)c)$$

Since $\rho$ is a Lie algebra representation it is easily seen that for all $X$ and $Y$ in $\mathcal{S}\Gamma$

$$X\Delta Y - Y\Delta X = [X, Y].$$

The operation just introduced is useful in describing the operator $M$ introduced above. Let $X = (i\kappa(B), B) \in \mathcal{Q}$ and $Z = (-i\kappa(A), A) \in \mathcal{P}$. Then

$$[Z, X] = (i\rho(A)\kappa(B) + i\rho(B)\kappa(A), [A, B])$$

$$= (i\rho(A)\kappa(B), A\Delta B) - (-i\rho(B)\kappa(A), B\Delta A)$$

In view of (1.7) it follows that

$$M(Z)X = (i\kappa(A\Delta B), A\Delta B).$$

Our next goal is to explicitly compute the operator $M^*(Z)X$ for $Z \in \mathcal{Q}_0$. For this we shall also require an algebraic description of the Riemannian structure of the domain. Assume that the Riemannian structure in question is that derived from the Bergman metric. Since this structure is $G$-invariant it is defined by a scalar product $g$ on the Lie algebra $\mathcal{G}$. Koszul ([1], Formula 4.5) proved the existence of a functional $\beta \in \mathcal{G}^*$ such that this scalar product is given by

$$g(X, Y) = \beta([X, Y]).$$
This functional has a very simple description in terms of the “normal decomposition” of \(S\) which will be explained after Proposition (2.6). Since \(g\) is \(J\)-invariant \(\Gamma\)

\[
(2.4) \quad \beta([JX, JY]) = -\beta([J^2X, Y]) = \beta([X, Y])
\]

We shall not explicitly use any other information concerning \(\beta\) other than the fact that formula (2.3) defines a \(J\)-invariant positive-definite scalar product. Proving our results in this generality seems necessary in order to carry out the inductive portion of the proof (see Section 6).

Notice that for all \(A\) and \(B\) in \(S\)

\[
g((0, B), (0, A)) = \beta([(-\kappa(B), 0), (0, A)]) = \beta((\kappa(A\Delta B), 0)) = \xi(A\Delta B),
\]

where \(\xi \in S^*\) is defined by

\[
\xi(A) = \beta((\kappa(A), 0)).
\]

Note that one consequence of the above is that the expression

\[
(A, B) = \xi(A\Delta B)
\]

defines a scalar product on \(S\).

Using formula (2.4) and the fact that \(\mathcal{M}\) is abelian \(\Gamma\)it is easily seen that \(\beta\) is zero on \([S, S]\). Moreover \(\mathcal{M}\) and \(S\) are orthogonal. This easily implies:

\[
(2.5) \text{Lemma. For } X = (i\kappa(A), A) \text{ and } Z = (i\kappa(B), B)\Gamma \text{ we have}
\]

\[
H(Z, X) = \xi(A\Delta B).
\]

To describe \(M^*\Gamma\) we shall require the ‘dual’ product on \(S\). We define a product ‘\(\square\)’ on \(S\) by the equality

\[
(A\Delta B, C) = (B, A\square C).
\]

This product is in fact the ‘\(\Delta\)’ product on \(S\) induced from the dual cone although we shall not require this fact.

Now let \(X = (i\kappa(A), A) \in \mathcal{Q}\) and \(Z = (i\kappa(B), B) \in \mathcal{Q}\). Then we have the following proposition which follows easily from formula (2.2).

\[
(2.6) \text{Proposition. } M^*(Z)X = (i\kappa(B\square A), B\square A).
\]

To obtain more precise results we shall need to use the structure theory of clans due to Vindberg. Let \(r\) be the rank of \(S\). (The dimension of the maximal torus in \(S\).) Vindberg proves (Proposition 8\(\Gamma\)p.374) that \(S\) has a ‘normal decomposition’. This means that there is a direct sum decomposition

\[
S = \bigoplus_{1 \leq i \leq j \leq r} S_{ij}
\]
where

$$ (2.7) \quad \text{For each } 1 \leq i \leq r \Gamma \text{ } S_{ii} \text{ is spanned by a single element } e_{ii} \text{ such that } e_{ii} \Delta e_{ii} = e_{ii}. $$

$$ (2.8) \quad \text{For } 1 \leq i \leq j \leq k \Gamma $$

$$ S_{ij} \Delta S_{jk} \subset S_{ik} \text{ and } $$

$$ S_{jk} \Delta S_{ik} + S_{ik} \Delta S_{jk} \subset S_{ij}. $$

$$ (2.9) \quad S_{ij} \Delta S_{kl} = \{0\} \text{ if } j \neq k \text{ and } j \neq l. $$

$$ (2.10) \quad \text{Let } i < j \text{ and let } s_{ij} \in S_{ij}. \text{ Then } $$

$$ e_{ii} \Delta s_{ij} = \frac{1}{2} s_{ij} = e_{jj} \Delta s_{ij} $$

$$ s_{ij} \Delta e_{jj} = s_{ij}. $$

$$ (2.11) \quad \text{The functional } \xi \text{ is zero on } S_{ij} \text{ for } i < j \Gamma \text{ and by definition } \Gamma \xi(e_{ii}) = g(e_{ii}, e_{ii}). \text{ Therefore } \beta \text{ is zero on } \sum_{i<j} \sigma(S_{ij}) \times S. $$

We refer to the above properties as the ‘properties of the normal decomposition’. In fact (2.11) can be derived easily from (2.7)-(2.10) the orthogonality of the decomposition \( \mathcal{G} = \mathcal{M} \times \mathcal{S} \) and the invariance of \( g \) under \( J \). To understand the meaning of (2.7)-(2.11) it helps to keep the following example in mind.

**Example: 1** Let \( \mathcal{X} \) be the set of \( n \times n \text{ real symmetric matrices} \) and let \( \mathcal{V} \subset \mathcal{X} \) be the cone of positive definite matrices. The group \( S \) of all invertible upper-triangular matrices with positive diagonal acts simply transitively on \( \mathcal{V} \) by means of the representation \( \rho \) defined by

$$ \rho(S)X = XS^t. $$

The differentiated representation of \( \mathcal{S} \) then is given by

$$ \rho(A)X = AX + XA^t. $$

We choose \( c = I \) as our base point. Then

$$ \kappa(A) = \rho(A)I = A + A^t. $$

If \( B \in \mathcal{S} \Gamma \text{ then } B \Delta A \) is defined by

$$ B \Delta A = \kappa^{-1}(\rho(B)(A + A^t)) = \kappa^{-1}(B(A + A^t) + (A + A^t)B^t). $$

The space \( S_{ij} \) are just the space of matrices which are non-zero only in the \((i, j)\) position. The elements \( e_{ii} \) are the diagonal matrices which have \( 1/2 \) in the \((i, i)\) entry and all other entries zero. The functional \( \xi \) may be taken to be the trace. The properties for the normal decomposition are easily verified in this case.

We shall also need information on how \( \Box \) interacts with the normal decomposition. This is most easily stated in terms of the spaces
\[ T_{ij} = S_{r-j} r-i. \]

The following is a simple consequence of the observation that the normal decomposition is an orthogonal decomposition.

**Proposition.** The operation \( \Box \) satisfies the properties of the normal decomposition with respect to the spaces \( T_{ij} \).

One requirement for the boundary theory which we utilize is a detailed knowledge of the root structure of \( G \). This too is readily obtained from the normal decomposition. Let \( \mathcal{A} \subseteq \mathcal{S} \) be the span of the \( e_{ii} \) and let

\[ \mathcal{N} = \sum_{i < j} S_{ij}. \]

Then \( \mathcal{N} \) is the unipotent radical for \( \mathcal{S} \) and \( \mathcal{A} \) is the maximal torus for both \( \mathcal{S} \) and \( \mathcal{G} \). Let \( \lambda \in \mathcal{A}^* \). Then \( \lambda \) is said to be a root if there are non-zero \( X \) such that for all \( D \in \mathcal{A} \Gamma \)

\[ [D, X] = \lambda(D)X. \]

Such \( X \) are called root vectors. We shall let \( \mathcal{M}_\lambda \) and \( \mathcal{N}_\lambda \) denote respectively the spaces of the root vectors for \( \lambda \) in \( \mathcal{M} \) and in \( \mathcal{N} \). The set of all roots will be denoted \( \mathcal{R} \).

Let \( \{\lambda_1, \lambda_2, \ldots, \lambda_r\} \subset \mathcal{A}^* \) be the dual basis to the \( e_{ii} \) basis. We shall leave the following to the reader:

**Proposition.** Let \( i \leq j \). If \( S_{ij} \neq 0 \), then both \( (\lambda_i + \lambda_j)/2 \) and \( (\lambda_i - \lambda_j)/2 \) are roots. The corresponding root spaces are, respectively, \( \sigma(S_{ij}) \subseteq \mathcal{M} \) and \( S_{ij} \subseteq \mathcal{S} \).

Now we shall introduce some notation. Let \( e_i = \xi(e_{ii}) \). Also \( \Gamma \) for \( i < j \Gamma \) we let \( d_{ij} \) be the dimension of \( S_{ij} \). We choose a basis \( e^\alpha \) for \( S_{ij} \) such that

\[ (e^\alpha, e^\gamma_j) = \delta_{\alpha, \gamma} e_i. \]

This basis turns out to be more convenient than an orthonormal basis due to the following:

**Lemma.** For \( i < j \Gamma \)

\[ e^\alpha_i \Delta e^\gamma_j = \delta_{\alpha, \gamma} e_{ii}. \]

**Proof** Since \( S_{ii} \) is one dimensional \( \Box e^\alpha_i \Delta e^\gamma_j = ce_{ii} \) for some scalar \( c \). Furthermore

\[ ce_i = \xi(e^\alpha_i \Delta e^\gamma_j) = (e^\alpha_i, e^\gamma_j) = c_i \delta_{\alpha, \gamma}. \]

This proves the lemma. \( \Box \)
Next we shall require an orthonormal basis for $Q$. For this we define for all $i \leq j$

\begin{equation}
E_{ij}^\alpha = (Y_{ij}^\alpha + \sqrt{-1}X_{ij}^\alpha)/\sqrt{e_i}
\end{equation}

where

\begin{align*}
Y_{ij}^\alpha &= (0, e_{ij}^\alpha) \\
X_{ij}^\alpha &= (\kappa(e_{ij}^\alpha), 0).
\end{align*}

(If $i = j$ we interpret $\alpha = 1$ and $e_{ii}^\alpha = e_{ii}$) It follows easily from Lemmas (2.5) and (2.15) that the $E_{ij}^\alpha$ define an orthonormal basis for $Q$.

Considered as vector fields on $\mathcal{E}T$ the elements $E_{ij}^\alpha$ form an orthonormal frame field for $T^{10}$. We may therefore compute the HJK operators from formula (1.2). Actually it turns out that we only require the ‘strongly diagonal’ HJK operators. These are the operators defined by

\begin{equation}
HJK_m f = H(HJK(f)E_{mm}, E_{mm}).
\end{equation}

Our main result of this section is the following:

\begin{equation}
\textbf{Theorem.}
\end{equation}

\begin{equation}
HJK_m = c_m^{-1}(\Delta_m - \frac{d_m + 2}{c_m}Y_{mm} - \sum_{i<j} d_m Y_{ii})
\end{equation}

where $d_m = \sum_{m<j} d_{mj}$ and

\begin{equation}
\Delta_m = \sum_{i \leq m, \alpha} c_i^{-1}(Y_{im}^\alpha)^2 + (X_{im}^\alpha)^2 + \sum_{m \leq j, \alpha} c_m^{-1}(Y_{mj}^\alpha)^2 + (X_{mj}^\alpha)^2).
\end{equation}

\textbf{Proof} From Lemma (1.1) and Theorem (1.9) we have

\begin{equation}
HJK(f)_m = \sum f_{ij, kl}^\alpha C_{ij, kl}^\alpha
\end{equation}

where

\begin{equation}
C_{ij, kl}^\alpha = -H(R(E_{ij}^\alpha, E_{kl}^\beta)E_{mm}, E_{mm})
\end{equation}

\begin{equation}
= H(M(E_{kl}^\beta)E_{mm}, M(E_{ij}^\alpha)E_{mm}) - H(M^*(E_{ij}^\alpha)E_{mm}, M^*(E_{kl}^\beta)E_{mm})
\end{equation}

\begin{equation}
+ H(M(M(E_{ij}^\alpha)E_{kl}^\beta)E_{mm}, E_{mm}) + H(M^*(M(E_{kl}^\beta)E_{ij}^\alpha)E_{mm}, E_{mm}).
\end{equation}

and

\begin{equation}
f_{ij, kl}^\alpha = [\bar{E}_{ij}^\alpha E_{kl}^\beta - M(\bar{E}_{ij}^\alpha)E_{kl}^\beta] f
\end{equation}
The sum is over all indices with \(1 \leq i \leq j \leq r, \Gamma i \leq k \leq l \leq r\) and \(1 \leq \beta \leq d_{kl}\).

Our first observation is that if \((i, j, \alpha) \neq (k, l, \beta)\) then \(C_{i,j,k,l}^{\alpha,\beta} = 0\). In fact each term in formula (2.19) is zero. (This follows from the normal decomposition properties \((2.6)\Gamma (2.14)\) and the observation that the \(S_{ij}\) spaces are mutually orthogonal.)

Next we shall record a series of formulae which the reader may readily verify. We set
\[
Z_i = E_{ii}/\sqrt{c_i} = c_i^{-1}(Y_{ii} + \sqrt{-1}X_{ii}).
\]
For \(i < j\)
\[
M(\overline{E}_{ij}^{\alpha})E_{mm} = \frac{\delta_{j,m}}{\sqrt{c_m}}E_{im}^{\alpha}
\]
\[
M^*(E_{ij}^{\alpha})E_{mm} = \frac{\delta_{i,m}}{\sqrt{c_m}}E_{mj}^{\alpha}
\]
\[
M(\overline{E}_{ij}^{\alpha})E_{ij} = \frac{1}{\sqrt{c_i}}E_{ii} = Z_i
\]
\[
M(M(E_{ij}^{\alpha})\overline{E}_{ij}^{\alpha})E_{mm} = \delta_{i,m}E_{mm}
\]
\[
M^*(M(\overline{E}_{ij}^{\alpha})E_{ij}^{\alpha})E_{mm} = \delta_{i,m}E_{mm}
\]
It follows that for \(i \leq j\)
\[
C_{i,j,i,j}^{\alpha,\alpha} = \frac{1}{c_m}(\delta_{j,m} - \delta_{i,m} + 2\delta_{i,m}) = \frac{1}{c_m}(\delta_{j,m} + \delta_{i,m})
\]
and
\[
(2.20)\quad f_{i,j,i,j}^{\alpha,\alpha} = (\overline{E}_{ij}^{\alpha}E_{ij}^{\alpha} - Z_i)f.
\]
We sum the terms with indices \((m, j)\Gamma j \geq m\) and \((i, m)\Gamma i \leq m\) separately. Note also that for each pair \((i, j)\) there are \(d_{ij}\) possible values of \(\alpha\). Note also that \(d_{ii} = 1\). We get
\[
c_mHJK_m = \tilde{\Delta}_m - \frac{d_{m} + 2}{c_m}(Y_{mm} + \sqrt{-1}X_{mm}) - \sum_{i \leq m} \frac{d_{im}}{c_i}(Y_{ii} + \sqrt{-1}X_{ii})
\]
where \(d_m = \sum_{m < j} d_{mj}\) and
\[
\tilde{\Delta}_m = \sum_{i \leq m, \alpha} \overline{E}_{im}^{\alpha}E_{im}^{\alpha} + \sum_{m \leq j, \alpha} \overline{E}_{mj}^{\alpha}E_{mj}^{\alpha}.
\]
From Lemma (1.3) \(HJK_m(f) = \overline{HJK_m}(f)\). Thus \(HJK_m\) is a real operator. Taking real parts proves the desired formula.\(\square\)
Our proof of our main theorem will be an inductive argument based upon the fact that every bounded homogeneous domain may be built up from a lower dimensional domain. To explain this we introduce two subalgebras of $S$. We define

$$S_{1*} = \sum_{1 \leq m \leq d} S_{1m}$$

$$S_{>1} = \sum_{2 \leq i \leq j \leq d} S_{ij}$$

Clearly $S_{1*}$ is a Lie ideal in $S$ and $S_{>1}$ is a complimentary Lie subalgebra. We define subspaces of $\mathbb{R}^n$ by

$$M_{1*} = \kappa(S_{1*})$$

$$M_{>1} = \kappa(S_{>1})$$

Then $M_{1*}$ is $S$ invariant under $\rho$. We identify $M_{>1}$ with the quotient $\mathbb{R}^n/M_{1*}$. The image $V_{>1}$ in $M_{>1}$ of the cone $V$ is a cone which is homogeneous under $S/S_{1*} = S_{>1}$. (See [V]1.) It follows that

$$G_{>1} = M_{>1}S_{>1} \subset G$$

acts simply transitively on the tube domain over $V_{>1}$. We use the functional $\beta_{>1} = \beta|G_{>1}$ to define the Riemannian structure on $G_{>1}$. Let $HJK_{>1}$ be the corresponding Hua system for $G_{>1}$.

We shall identify $G_{>1}$ with the quotient $G/G_{1*}$ where

$$G_{1*} = M_{1*}S_{1*}.$$ 

Note that $G_{1*}$ is normal in $G$. This identification allows us to consider functions on $G_{>1}$ as functions on $G$ which are constant on cosets of $G_{1*}$. Under these identifications the strongly diagonal Hua operators on $G$ reduce to those on $G_{>1}$ in the sense of the lemma below. This lemma is a direct consequence of Theorem (2.18).

**Lemma.** Let $HJK_{>1}$ be the Hua system for $G_{>1}$ under the Riemannian structure defined above. Then, for all $f \in C^\infty(G)$ which are constant on $G_{1*}$ cosets,

$$(HJK_{>1})_{m} f = HJK_{m+1}(f)$$

for all $r \geq m \geq 2$. $\square$

### Section 3. Hua Operators on Type II Domains

Let $\mathcal{D}$ be a Siegel domain of type II as described in the introduction (nontrivial $Z$). The group $G$ generated by the actions $(0.1) \Gamma(0.2) \Gamma(0.3)$ may be algebraically described as follows. Let $\phi = 3K$. We let $\mathcal{M} = Z \times \mathbb{R}^n$ with the Lie structure

$$[(z_1, t_1), (z_2, t_2)] = (0, 4\phi(z_1, z_2)).$$
The corresponding group is $\mathcal{M}$ with the product

$$(z_1, t_1) \cdot (z_2, t_2) = (z_1 + z_2, t_1 + t_2 + 2\phi(z_1, z_2)).$$

However, following our convention of denoting Lie groups by upper case Roman letters, we shall denote this space by $M$ when it is considered as a group. This matches with the notation for tube domains which correspond to the case $\mathcal{Z} = 0$. From now on, $M$ will be understood in this larger sense.

Let $G = M \times_s S$ where $s(z, t)s^{-1} = (\sigma(s)z, \rho(s)t)$. Then $\Gamma G$ is a completely solvable group which acts simply transitively on $D$. The corresponding identification of $G$ with $D$ is defined by

$$(z, t), s \rightarrow (z, t + i\rho(s)c + iK(z, z)).$$

We shall let $T = \mathbb{R}^n \times_s S \subset G$. Note that $T$ is the group of the Type I domain $\mathcal{E} = \mathbb{R}^n + i\mathcal{Y}$. The Lie algebra of $T$ will be denoted by $\mathcal{T}$.

We identify the tangent space of $D$ at $ic$ with $\mathcal{G}$. Let $J : \mathcal{G} \rightarrow \mathcal{G}$ define the complex structure. From formula (3.1) it is easily seen that $J : \mathcal{T} \rightarrow \mathcal{T}$ and on this set acts as described below Lemma (2.1). It also follows from formula (3.1) that on $\mathcal{Z}\mathcal{T}J$ is just multiplication by $i$. Next, we assume that the Riemannian structure may be defined by a formula such as formula (2.3) above where $\beta \in \mathcal{G}^*$. Notice that then $\beta|\mathcal{T}$ defines a Riemannian structure for $\mathcal{T}$.

As before, we shall also let $\sigma$ denote the representation of $\mathcal{S}$ in $\mathcal{Z}$ obtained by differentiating $\sigma$. Since (by assumption) $\sigma$ is algebraic, we know that $\sigma(\mathcal{A})$ is diagonalizable over $\mathbb{R}$. Thus, we may decompose $\mathcal{Z}$ into a direct sum of root spaces for $\mathcal{A}$ under $\sigma$. Let $\{\tau_1, \tau_2, \ldots, \tau_k\}$ be the set of root functionals in $\mathcal{A}^*$. The following is well known. We include the proof for sake of completeness.

(3.2) Lemma. Let $\lambda_i$ be as above Proposition (2.14) Then

$$\{\tau_1, \tau_2, \ldots, \tau_k\} \subset \{\lambda_1/2, \lambda_2/2, \ldots, \lambda_r/2\}.$$ 

Proof. Let $Z \in \mathcal{Z}$ be a root vector for $\mathcal{A}$ under $\sigma$ corresponding to the root functional $\tau \in \mathcal{A}^*$. Then $\Gamma U = K(Z, Z)$ is a non-zero root vector for $\rho$ corresponding to $2\tau$. It follows from Proposition (2.14) that $\tau = (\lambda_i + \lambda_j)/4$ for some choice of $i \leq j$. We need to show that necessarily $i = j$. Suppose that $i < j$. Let $X = \kappa^{-1}(U) \in S_{ij}$. Then for all $A \in \mathcal{A}$

$$\sigma(A)\sigma(X)Z = \sigma([A, X])Z + \sigma(X)\sigma(A)Z = \gamma(A)\sigma(X)Z$$

where

$$\gamma = (\lambda_i - \lambda_j)/2 + \tau = \frac{3}{4}\lambda_i - \frac{1}{4}\lambda_j.$$ 

We know from the previous paragraph that such a functional cannot be a root for $\sigma$. Hence $\sigma(X)Z = 0$. But then $\rho(X)U = 0$. We obtain our contradiction by noting that then $X\Delta X = \kappa^{-1}(\rho(X)\rho(X)c) = \kappa^{-1}(\rho(X)U) = 0$ proving that $X = U = 0$ and hence that $Z = 0$. □
From now on $R\Gamma, M, \Gamma, \Lambda$ defined before Proposition (2.14) will be understood in this more general situation i.e. when $M = Z \oplus \mathbb{R}^n$. We assume that our Riemannian structure is defined via a functional $\beta \in G^*$ as in formula (2.3) above.

**Corollary.** The functional $\beta$ is zero on $Z$.

*Proof.* Let $A \in A$ and $Z \in Z$. Then $JA \in \sigma(S)$. Hence

$$\beta([A, Z]) = \beta([JA, JZ]) = 0.$$ 

Our corollary follows since from Lemma (3.2) $\Gamma \text{Ad } A$ maps $Z$ onto $Z$.\square

We let $Z_i$ denote the root space corresponding to $\lambda_i/2$ in $Z$. Then

$$Z = \sum Z_i.$$ 

Furthermore in $L\Gamma$,

$$[Z_i, Z_j] \subset M_{ij}.\tag{3.4}$$

(Recall that $M_{ij} = \kappa(S_{ij})$).

The subalgebra $Q$ is the set of all elements

$$X - \sqrt{-1}JX$$

where $X \in G_c$. For each $i$ we define $Q_i$ to be the set of all elements $X - \sqrt{-1}JX$ as above where $X \in (Z_i)_c$. For any pair of indecies $(i, j)$ we define $Q_{ij}$ to be the set of such elements where $X \in (M_{ij})_c$. Clearly the spaces $Q_{ij}$ and $Q_i$ together span $Q$. Furthermore the space

$$Q_T = \sum Q_{ij}$$

is the algebra which we called $Q$ in the last section relative to the domain $E$. Let $f_i$ be the complex dimension of $Q_i$.

For any subscript $\alpha$ we define $P_\alpha = \overline{Q_\alpha}$. Then from formula (3.4) $\Gamma$

$$[Q_i, P_j] \subset (M_{ij})_c = P_{ij} + Q_{ij}.\tag{3.5}$$

Indeed let $X \in Q_i, Y \in Q_j$. Then

$$H(X, Y) = \frac{1}{2} g(X, Y) = \frac{1}{2} \beta([JX, JY]) = 0.$$ 

One immediate conclusion is that the spaces $Q_i$ are mutually $H$-orthogonal because $\beta$ is zero on $M_{ij}$ for $i < j$.

We choose a basis for $Q$ consisting of

(a) The basis $E_{ij}^\alpha$ for $Q_T$ defined in formula (2.16).

(b) An $H$-orthonormal basis $Z_j^\alpha = X_j^\alpha - \sqrt{-1}Y_j^\alpha$ for each $Z_i$ where $1 \leq \alpha \leq f_i$ and $X_j^\alpha$ and $Y_j^\alpha$ are real.
It is clear that this defines an orthonormal basis for $Q$. We shall use this basis to compute the Hua operators. Again though we are only interested in the strongly diagonal Hua operators. These are still defined by formula (2.17). The analogue of Theorem (2.18) for a Siegel II domain is the following

(3.6) Theorem. Let $HJK_m^T$ be the operator defined as $HJK_m$ in Theorem (2.18). Then, for the case at hand,

$$HJK_m = HJK_m^T + c_m^{-1}(\sum_{\alpha}(X_{m}^\alpha)^2 + (Y_{m}^\alpha)^2 - \frac{f_{m}}{c_m}Y_{mm}).$$

Proof It is clear that from formula (2.19) $HJK_m = HJK_m^T + \sum f_{i,j}^{\alpha,\gamma} C_{i,j}^{\alpha,\gamma}$ where

$$C_{i,j}^{\alpha,\gamma} = H(M(Z_i^\gamma)E_{mm}, M(Z_i^\alpha)E_{mm}) - H(M^*(Z_i^\alpha)E_{mm}, M^*(Z_j^\gamma)E_{mm}) + H(M(M(Z_i^\gamma)Z_j^\gamma)E_{mm}, E_{mm}) + H(M^*(M(Z_j^\gamma)Z_i^\alpha)E_{mm}, E_{mm}).$$

(3.7) and

$$f_{i,j}^{\alpha,\gamma} = [Z_i^\alpha Z_j^\gamma - M(Z_i^\alpha Z_j^\gamma)]f$$

To get (3.7) one has to prove that $H(R(E_{ij}^\alpha, Z_i^\gamma)E_{mm}, E_{mm}) = 0$ which is an easy but a tedious calculation based on two facts:

(3.8) $M(Z_i^\beta)E_{i,j}^\alpha = 0$

and

(3.9) $M(E_{ij}^\alpha)Z_k^\beta \subset (Z_k)_c$.

Indeed since $P$ is a subalgebra we have

$$M(Z_i^\beta)E_{ij}^\alpha = \pi_\Phi [Z_i^\beta, Y_{ij}^\alpha + iX_{ij}^\alpha] = \pi_\Phi [Z_i^\beta, Y_{ij}^\alpha - iX_{ij}^\alpha] = 0.$$ 

Analogously

$$\pi_\Phi [E_{ij}^\alpha, Z_k^\beta] = \pi_\Phi [E_{ij}^\alpha, Z_k^\beta] = [E_{ij}^\alpha, Z_k^\beta],$$

which for $i = j$ is included in $(Z_k)_c$ and for $i < j$ belongs to the root space $\frac{\lambda_i - \lambda_j}{2} + \frac{\lambda_k}{2}$ which is zero.

We claim that $C_{i,j}^{\alpha,\gamma} = 0$ unless $i = j = m$ and $\alpha = \gamma$. Furthermore in this case we get $c_m^{-1}$.

To prove this let $X = A - \sqrt{-1}JA$ be an element of $Q_T$ where $A \in A$. Then $JA \in M$ and hence

$$[X, Z_i^\alpha] = [A - iJA, Z_i^\alpha] = [A + iJA, Z_i^\alpha] \in P.$$
It follows that
\[ M(\overline{X})Z_i^\alpha \text{ and } M(\overline{Z}_i^\alpha)X = 0, \]

In particular, the first term to the right of the equality in formula (3.7) is zero.

Next, from formula (3.5), we note that \( M(\overline{Z}_i^\alpha)Z_j^\alpha \) belongs to \( Q_{ij} \). Hence, the third term on the right in formula (3.7) will be zero unless \( i = j = m \). The same is true for the fourth term since this term is just the conjugate of the third. The following lemma clearly finishes the proof of our claim. In fact, this will also finish the proof of Theorem (3.6). □

**Lemma.**

\[
M(\overline{Z}_m^\alpha)Z_m^\alpha = \frac{\delta_{\alpha,\gamma}}{\sqrt{c_m}} E_{mm}.
\]
\[
M^*(Z_i^\alpha)E_{mm} = \frac{\delta_{i,m}}{\sqrt{c_m}} Z_m^\gamma.
\]

**Proof** For the first equality, we note that \( \kappa(S_{mm}) \) is one dimensional. Hence, from formula (3.4) there is a complex constant \( C^{\alpha,\gamma} \) such that

\[
[Z_m^\alpha, Z_m^\gamma] = C^{\alpha,\gamma} X_{mm}.
\]

Computing this constant is simple. If we apply \( \beta \) to both sides of the above, we find that
\[
2H(Z_m^\alpha, Z_m^\gamma) = \beta([JZ_m^\gamma, \overline{Z}_m^\alpha]) = -i C^{\alpha,\gamma} c_m.
\]

Thus,
\[
C^{\alpha,\gamma} = \frac{2i}{c_m} \delta_{\alpha,\gamma}.
\]

On the other hand, from formula (2.16),
\[
X_{mm} = \frac{-i \sqrt{c_m}}{2} (E_{mm} - \overline{E}_{mm}).
\]

The first equality follows by applying \( \pi_Q \) to formula (3.11).

For the second equality, recall that \( M(\overline{Z}_i^\alpha) \) is zero on \( Q_T \). It follows that \( M^*(Z_i^\alpha)E_{mm} \) is \( H \)-orthogonal to \( Q_T \) and hence belongs to \( Z_c \). Also,
\[
H(M^*(Z_i^\alpha)E_{mm}, Z_j^\gamma) = H(E_{mm}, M(\overline{Z}_i^\alpha)Z_j^\gamma).
\]

This is zero unless \( i = j = m \). In which case, the first part of the lemma proves our result. □

**Section 4. Alternative reproducing kernels for holomorphic functions**

In this section, we consider \( G \)-invariant real second order elliptic degenerate operators \( L \) on \( \Omega \) which annihilate holomorphic functions. We are going to apply the
boundary theory of [DH] in order to show that there are many real kernels on \( M \Gamma \) which reproduce holomorphic functions.

Let \( L \) be a real (i.e. \( Lf = \overline{Lf} \)) second order operator which annihilates holomorphic functions and \( x_0 \in \mathcal{D} \) a fixed base point. In local coordinates around \( x_0 \) we have

\[
(4.1) \quad L = \sum c_{kj} \frac{\partial}{\partial \bar{z}_k} \frac{\partial}{\partial z_j},
\]

with \( c_{jk} = \overline{c_{k,j}} \). Therefore writing \( L \) in terms of partial derivatives \( \frac{\partial}{\partial x_k} \Gamma \frac{\partial}{\partial y_k} \) we obtain an operator with real coefficients and being elliptic degenerate means that the second order symbol of \( L \) is positive semi-definite.

Since \( L \) is \( G \)-invariant we may write it in terms of left-invariant vector fields on \( G \). The identification (3.1) of \( G \) with \( D \) defines global coordinates for \( G \). Let \( \mathcal{X}_k \) be the left-invariant vector field on \( G \) which equals \( \frac{\partial}{\partial \bar{z}_k} \) at \( e \) in these coordinates. In view of Lemma (1.1) we have

\[
L = \sum c_{kj}(\bar{\mathcal{X}}_k \mathcal{X}_j - \nabla_{\bar{\mathcal{X}}_k} \mathcal{X}_j) = \sum c_{kj}(\bar{\mathcal{X}}_k \mathcal{X}_j - M(\bar{\mathcal{X}}_k) \mathcal{X}_j),
\]

where We choose a basis of \( Q \) as in the previous section. Let \( \{ E_{ij}^\alpha \} \Gamma 1 \leq i \leq j \leq r \Gamma 1 \leq a \leq d \) be the basis for \( Q_T \) and \( \{ Z_j^\alpha \} \Gamma 1 \leq j \leq k \Gamma 1 \leq a \leq f_j \) the basis for \( \mathcal{Z} \). Therefore

\[
L = \sum C_{ij,kl}^{\alpha,\beta}(\bar{E}_{ij}^\alpha E_{kl}^\beta - M(\bar{E}_{ij}^\alpha)E_{kl}^\beta) + \sum C_{i,j}^{\alpha,\beta}(\bar{Z}_i^\alpha Z_j^\beta - M(\bar{Z}_i^\alpha)Z_j^\beta)
\]

\[
+ \sum C_{ij,k}^{\alpha,\beta}(\bar{E}_{ij}^\alpha Z_k^\beta - M(\bar{E}_{ij}^\alpha)Z_k^\beta) + \sum C_{i,k,li}^{\alpha,\beta}(\bar{Z}_i^\alpha E_{kl}^\beta - M(\bar{Z}_i^\alpha)E_{kl}^\beta).
\]

The condition \( Lf = \overline{Lf} \) implies that \( L \) belongs to the enveloping algebra of \( G \) i.e. can be written as

\[
L = \mathcal{Y}_1^2 + ... + \mathcal{Y}_m^2 + \mathcal{Y}_0
\]

for some \( \mathcal{Y}_0, ..., \mathcal{Y}_m \in \mathcal{G} \).

For the rest of the paper we assume that \( L \) satisfies the Hörmander condition i.e

\[
(4.3) \quad \mathcal{Y}_1, ..., \mathcal{Y}_m \text{ generate } \mathcal{G} \text{ as a Lie algebra}
\]

The same condition is satisfied by \( \pi_A(L) \Gamma \) where \( \pi_A(L) \) is the image of \( L \) under the canonical homomorphism \( \pi : S \to A = G/M \Gamma \text{ i.e } \pi_A(L) \) is elliptic. Since the second order part of \( \pi_A(L) \) is equal to

\[
\sum_{i,j=1}^r C_{ij,ij}^{\alpha,\alpha} \mathcal{Y}_i^\alpha \mathcal{Y}_j^\alpha = \sum_{i,j=1}^r C_{ii,jj} \mathcal{Y}_i \mathcal{Y}_j
\]

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we have

\[ C_{ii,ii}^{\alpha,\alpha} = C_{ii,ii} > 0. \]

In order to determine the Poisson boundary for \( L \) we need to know the \( \mathcal{A} \)-component \( Y_0 \) of the first order part of \( L \). For this we compute the contribution of each term in (4.2). In view of (3.8) and (3.9) \( \Gamma M(\mathbf{Z}_i^\alpha)E_{kl} = 0 \) and \( M(\mathbf{E}_{ij})Z_k^\beta \in (\mathcal{Z}_k)_c \). Hence

\[ H(M(\mathbf{E}_{ij})Z_k^\beta, E_{mm}) = 0 \]

and

\[ H(M(\mathbf{Z}_i^\alpha)Z_j^\beta, E_{mm}) = 0. \]

Moreover \( M(\mathbf{Z}_i^\alpha)Z_j^\beta \in Q_{ij} \) and by Lemma (3.10)

\[ M(\mathbf{Z}_i^\alpha)Z_j^\beta = \frac{\delta_{\alpha,\beta}}{\sqrt{c_i}} E_{ii}. \]

By orthogonality of \( \mathcal{P} \) and \( \mathcal{Q} \) Lemma (3.10) and (2.2)

\[ H(M(\mathbf{E}_{ij})E_{ij}^\beta, E_{mm}) = \xi((Y_{ij}^\alpha \Delta Y_{ij}^\beta) \Delta Y_{mm}), \]

which in view of the properties of \( \Delta \) is nonzero if and only if \( \alpha = \beta \Gamma i = k \Gamma j = l \) and \( m = i \). Then we have

\[ M(\mathbf{E}_{ij})E_{ij}^\alpha = \frac{1}{\sqrt{c_i}} E_{ii}. \]

Therefore

\[ L = L_0 - Y_0, \]

where

\[ Y_0 = \sum_{i=1}^r \frac{1}{c_i} \left( \sum_{j \geq i, \alpha} C_{ij,ij}^{\alpha,\alpha} + \sum_{\alpha} C_{ii,i}^{\alpha,\alpha} \right) Y_{ii}. \]

We claim that for every \( 1 \leq i \leq r \)

\[ \sum_{j \geq i, \alpha} C_{ij,ij}^{\alpha,\alpha} + \sum_{\alpha} C_{ii,i}^{\alpha,\alpha} > 0. \]

Since \( L \) has a nonnegative second order symbol \( \Gamma L f(x_0) \geq 0 \) for \( f \) having minimum at \( x_0 \). Let \( \Gamma \) in local coordinates around \( x_0 \) \( \Gamma f = |z_j|^2 \). Then in the notation of (4.1)

\[ L f(x_0) = c_{jj} \frac{\partial}{\partial z_j} \frac{\partial}{\partial z_j} |z_j|^2 = c_{jj}. \]

Hence \( C_{ij,ij}^{\alpha,\alpha} C_{ii,i}^{\alpha,\alpha} \geq 0 \) and so (4.4) implies (4.6).
The following proposition sums up our considerations

(4.7) Proposition. A $G$-invariant, real, second order operator $L$ on $\mathcal{D}$, which annihilates holomorphic functions and satisfies the Hörmander condition, can be written in the form

$$(4.8) \quad L = L_0 - \sum_{m=1}^{r} b_m Y_{mm},$$

where $b_m > 0$ and $L_0$ is a left-invariant second order operator with the first order part contained in $\mathcal{M} \oplus \mathcal{N}$. $\square$

Notice that both the Laplace-Beltrami operator $\Delta$ and the ‘diagonal Laplacian’

$$\Delta_{\text{diag}} = \sum_{m} HJK_m$$

belong to the class of the operators described in Proposition (4.7). The first statement follows from formula (1.5) and for the second from (2.17) and Lemma (1.3).

The vector $Y_0$ will play a special role in our discussion. We wish to apply the boundary theory of [DH] to operators described in Proposition (4.7). We let $R^+$ denote the set of roots $\lambda$ such that

$$\lambda(Y_0) > 0$$

and $R^- = R \setminus R^+$. (This set corresponds to $\Delta_1(L)$ on p.8 of [DH]). Note that our $Y_0$ is $-Z_0$ in the notation of [DH].)

It is important to notice that $R^+$ is non-empty. In fact it is clear from Proposition (2.14) that for all $i \leq j$ such that $S_{ij} \neq 0 \Gamma$

$$(\lambda_i + \lambda_j)/2 \in R^+.$$ 

Also

$$\lambda_i/2 \in R^+.$$ 

The root spaces corresponding to these functionals span $\mathcal{M}$. We let

$$\mathcal{N}^+ = \sum_{\lambda \in R^+} \mathcal{N}_\lambda$$

and

$$\mathcal{N}^- = \sum_{\lambda \in R^-} \mathcal{N}_\lambda.$$ 

In the notation of [DH] $\Gamma loc. \text{cit.}\Gamma \mathcal{N}_0(L) = \mathcal{N}^-$ and $\mathcal{N}_1(L) = \mathcal{M} \oplus \mathcal{N}^+$. Note that both $\mathcal{N}^\pm$ are subalgebras of $\mathcal{S}$. Since

$$S = N^+ N^- A$$
the homogeneous space \( B = G/N^-A \) is identifiable with the nilpotent Lie group \( MN^+ \). We refer to \( B \) as the maximal boundary for \( L \). With this identification \( M \) is contained in it. Let \( dx \) be Haar measure on \( MN^+ \). According to the main result of [DH] there exists a bounded positive \( C^\infty \) function \( P \) on \( B \) such that

\[
\int P(x)dx = 1.
\]

(4.9)

(4.10) For all \( L^\infty \) functions \( f \) on \( B \) the function

\[
F(g) = \int_B f(gx)P(x)\,dx
\]

satisfies \( LF = 0 \). (Here \( \Gamma x \rightarrow gx \) denotes the action of \( G \) on the coset space \( G/N^-A \).)

(4.11) If \( F \) is a bounded solution to \( LF = 0 \) then there is a unique \( L^\infty \) function \( f \) on \( B \) which expresses \( F \) as above. This function is called the boundary value of \( F \).

Finally we have

(4.12) Proposition. There is a choice of constants in (4.2) such that the maximal boundary for \( L \) is \( M \). In particular, the Poisson kernel for such \( L \) reproduces holomorphic functions in the sense of (4.9)-(4.11).

Proof We consider \( L \) of the form

\[
L = \sum C_{ij,ij}^\alpha \alpha (E_{ij}^\alpha E_{ij}^\alpha - M(E_{ij}^\alpha E_{ij}^\alpha)) + \sum C_{ii,ii}^\alpha (Z_i^\alpha Z_i^\alpha - M(Z_i^\alpha Z_i^\alpha)),
\]

which is clearly elliptic if all \( C_{ij,ij}^\alpha \alpha \) and \( C_{ii,ii}^\alpha \alpha \) are greater than 0. Hence we have to find positive \( C_{ij,ij}^\alpha \alpha \) and \( C_{ii,ii}^\alpha \alpha \) such that

\[
(\lambda_i - \lambda_j)(Y_0) = \frac{1}{c_i} \left( \sum_{k \geq i,\alpha} C_{ik,ik}^\alpha \alpha + \sum_{\alpha} C_{i,i}^\alpha \alpha \right) + \frac{1}{c_j} \left( \sum_{k \geq j,\alpha} C_{jk,jk}^\alpha \alpha + \sum_{\alpha} C_{j,j}^\alpha \alpha \right) \geq 0,
\]

for \( i < j \) which is very easy. Assume we can satisfy (4.13) for \( i < i_0 \) and all \( j > i \). To get (4.13) for \( i = i_0 \) and \( j > i_0 \) we increase \( C_{jk,jk}^\alpha \alpha \) and \( C_{j,j}^\alpha \alpha \) sufficiently which does not change positivity of \( (\lambda_i - \lambda_j)(Y_0) \) for \( i < i_0 \). □

It turns out that all \( L \) from Proposition (4.7) give rise to reproducing kernels on \( M \) although their maximal boundaries may be larger. For that we have to explain the idea of a boundary for \( L \) and consider not only the maximal boundary but also the smaller ones. We shall say that a subalgebra \( N^0 \) of \( N \) is homogeneous if it is normalized by \( A \). For any such algebra \( \Gamma \) there is an \( A \)-invariant subspace \( N^1 \) of \( N \) such that

\[
N = N^1 \oplus N^0.
\]
Let $\mathcal{N}^0 \supset \mathcal{N}^-$ be some homogeneous subalgebra of $\mathcal{N}$. Then according to [DH] Theorem (4.7) the homogeneous space $\tilde{B} = G/N^0A$ is a boundary for $L$. This means that there is a probability measure $\tilde{P}$ on $G/N^0A$ such that the functions

$$F(g) = \int_{G/N^0A} f(gx)\tilde{P}(x)dx, \ f \in L^\infty(G/N^0A)$$

are $L$-harmonic. $\tilde{P}$ is closely related to $P$.

Clearly $M = G/S$ is a boundary for $L$. Let $P_M$ be the corresponding Poisson kernel. We are going to prove that $P_M$ reproduces bounded holomorphic functions. This is not totally obvious unless $B = G/S$ and $P_M = P$.

The boundary $\tilde{B}$ can be realised as $MN^1 \Gamma N^1 = \exp N^1 \Gamma$ with an appropriate action of $G$. Indeed the mapping

$$MN^1 \times N^0 \ni (x_1, x_0) \rightarrow x_1x_0 \in MN$$

is a diffeomorphism between $MN^1 \times N^0$ and $MN$ and so every $x \in MN$ can be written in a unique way as

$$x = x_1x_0, \ x_1 \in MN^1, x_0 \in N^0.$$ 

We have a well defined projection $\pi_{N^1} : MN \rightarrow MN^1 \Gamma$ given by

$$\pi_{MN^1}(x) = x_1.$$ 

In this terms (4.14) becomes

$$F(g) = \int_{MN^1} f(\pi_{MN^1}(gx))\tilde{P}(x)dx, \ f \in L^\infty(MN^1)$$

and $x \rightarrow \pi_{MN^1}(gx)$ is the action of $G$ on $MN^1$ corresponding to the action $x \rightarrow gx$ in $B = G/N^0A$ realization.

(4.15) follows from a more general fact. One can obtain this kind of decomposition of a connected and simply connected nilpotent Lie group as far as the assumptions of the following lemma are satisfied.

(4.17) Lemma. Let a nilpotent Lie algebra $\mathcal{N} = \mathcal{N}_1 \oplus N_0$ be a sum of two linear subspaces $\mathcal{N}_1$ and $\mathcal{N}_0$. Assume that we can find a basis $E_1, \ldots, E_n$ of $\mathcal{N}$ such that every $E_j$ belongs either to $\mathcal{N}_1$ or to $\mathcal{N}_0$ and, in coordinates $x = \exp(x_1E_1 + \ldots + x_nE_n)$, the multiplication in $N = \exp \mathcal{N}$ is given by

$$(xy)_i = x_i + y_i + T_i(x_1, \ldots, x_{i-1}, y_1, \ldots, y_{i-1})$$

with $T_i \in C^\infty(N)$ independent of $x_i, \ldots, x_n, y_1, \ldots, y_n$. Then

$$\exp N_1 \times \exp N_0 \ni (x_1, x_0) \rightarrow x_1x_0 \in N$$
is a diffeomorphism. □

For the proof of Lemma (4.17) which by all means is standard see e.g. the preliminaries of [DH]. Although Lemma (4.17) is not formulated there in the above form the proof is essentially the same as the proof of Lemmas (1.21)Γ(1.22)Γ(1.25) there. In every such situation we are going to consider the corresponding projections \( \pi^1_N \) and \( \pi^0_N \).

In view of Lemma (4.17) we can decompose \( N^+ \) as

\[
N^+ = N^1 N^2,
\]

where \( N^2 = \exp N_2 \) and \( N_2 = N^+ \cap N^0 \). Notice that \( N^2 \) is a subgroup. Moreover in view of (4.16) we have

\[
\bar{P}(x) = \int_{N^2} P(xy)dy, \quad x \in MN^1.
\]

Indeed identifying \( f \in C_b(MN^1) \) with a continuous bounded function on \( MN \) constant on the right cosets of \( N^0 \Gamma \) we have

\[
F(g) = \int_{MN^1 \times N^2} f(\pi_{MN^1N^2}(gxy))P(xy)dxdy,
\]

(4.19)

\[
= \int_{MN^1 \times N^2} f(\pi_{MN^1}(gxy))P(xy)dxdy = \int_{MN^1} f(\pi_{MN^1}(gx))\left( \int_{N^2} P(xy)dy \right)dx.
\]

On the other hand

\[
F(g) = \int_{MN^1} f(\pi_{MN^1}(gx))\bar{P}(x)dx,
\]

which proves (4.18). In particular

\[
P_M(x) = \int_{N^+} P(xy)dy, \quad x \in M.
\]

Now we are ready to formulate the main result of this section

**(4.21) Theorem.** Let \( L, L_1 \) be \( G \)-invariant (not necessarily distinct), real, second order operators on \( D \), which annihilate holomorphic functions and satisfy the Hörmander condition. Assume that the maximal boundary for \( L_1 \) is \( M \). Let \( P_M \) be the \( L \)-Poisson kernel on \( M \), and \( F \) a bounded function, which is at the same time \( L \) and \( L_1 \) harmonic. Then there is \( f \in L^\infty(M) \) such that

\[
F(g) = \int_M f(\pi_M(gx))P_M(x)dx.
\]

(4.22)
In particular, bounded holomorphic functions are reproducible from their boundary values on \( M \) via the kernel \( P_M \).

**Remark.** The last chapter of the paper will be devoted to the proof of an analogous theorem with the Hua diagonal operators playing the role of \( L_1 \). This involves somewhat more work because except for the case of the tube over the cone of symmetric real \( r \times r \) matrices (see the next section) it is not known whether or not there is a linear combination of Hua diagonal operators which satisfies the hypotheses of \( L_1 \).

For the proof of both theorems we need a technical lemma which will be formulated and proved below. Before that we must introduce some notation. Elements of \( C_b(B) \) are defined by right \( N^- \)-invariant continuous functions on \( MN^+ \) while elements of \( C_b(\bar{B}) \) are defined by right \( N^0 \)-invariant functions. We say that an element of \( C_b(B) \) ‘reduces’ to \( \bar{B} \) if it is defined by a right \( N^0 \)-invariant function on \( MN \). In this case as in (4.19) we may write

\[
F(g) = \int_{MN^+} f(\pi_{MN}(gx))P(x) \, dx
\]

\[
(4.23) \quad = \int_{MN^1 \times N^2} f(\pi_{MN}(gx_{1}x_{2}))P(x_{1}x_{2}) \, dx_{1} \, dx_{2}
\]

\[
= \int_{M N^1} f(\pi_{MN}(gx_{1}))\bar{P}(x_{1}) \, dx_{1}.
\]

Let \( Y \in A \). We say that \( Y \) is contractive on \( G \) if \( \ad Y \) has only non-negative eigenvalues. In this case we let \( N^0_Y \) be the span of the positive eigenspaces in \( \mathcal{M} + \mathcal{N} \) and \( \mathcal{N}^0_Y \) be the centralizer of \( Y \) in \( \mathcal{M} + \mathcal{N} \). Note that \( N^0_Y \) is an ideal in \( \mathcal{M} + \mathcal{N} \).

**Lemma.** Let \( F \) be a bounded, \( L \)-harmonic function. Assume that the \( L \)-boundary value \( f \) is continuous on the maximal boundary \( B = MN^+ \). Let \( Y \in A \) be contractive. Then

\[
\lim_{t \to -\infty} F((\exp tY)g) = F_Y(g)
\]

converges uniformly on compact sets in \( G \) and defines an \( L \)-harmonic function with continuous boundary function \( f_Y \). Both \( F_Y \) and \( f_Y \) are constant on right cosets of \( N^0_Y \) in \( G \) and in \( MN \) respectively. Additionally, \( f_Y \) and \( f \) agree on \( N^1 \cap MN^+ \). If \( f \) reduces to \( G/AN^0 \), then \( f_Y \) will reduce to \( \bar{G}/AN^0 \), where \( N^1 \) is the subgroup generated by \( N^0_Y \) and \( N^0 \).

**Proof.** Given \( g \in G \) we write

\[
g = an_Y^1 n_Y^0
\]

relative to the decomposition \( G = AN_Y^1 N_Y^0 \). (The assumptions of Lemma (4.17) are clearly satisfied because \( \mathcal{N}_Y^1 \cap \mathcal{N}_Y^0 \) together contain all of the eigenspaces of \( \ad Y \).) We define

\[
g(t) = (\exp tY)g(\exp -tY).
\]
Then
\[ g(t) = an_1^1 n_Y^0(t) \]

Let
\[ I^1 = N_Y^1 \cap MN^+ \text{ and } I^0 = N_Y^0 \cap MN^+. \]

Notice that also \( II^1 \Gamma I^0 \) are composed of the whole eigenspaces and so
\[ MN^+ = I^1 I^0. \]

Then
\[ F(\exp tY) = \int_{I^1} \left( \int_{I^0} f(x^1 x^0(t)) P(x^1 x^0) \, dx^0 \right) \, dx^1. \]

When \( t \to -\infty \) then \( x^0(t) \to e \) and (4.25) converges to
\[ \int_{I^1} f(x^1) P(x^1) \, dx^1, \]

where
\[ \bar{P}(x^1) = \int_{I^0} P(x^1 x^0) \, dx^0. \]

More generally for \( g \in GT \)
\[ F((\exp tY)g) = F(an_1^1 n_Y^0(t)(\exp tY)) \]
\[ = \int_{I^1} \left( \int_{I^0} f(\pi_{MN^+}(an_1^1 n_Y^0(t)x^1 x^0(t))) P(x^1 x^0) \, dx^1 \right) \, dx^0. \]

As \( t \to -\infty \Gamma n_Y^0(t) \to e \Gamma x^0(t) \to e \) and we see that
\[ F_Y(g) = \int_{I^1} f(\pi_{MN^+}(an_Y x^1)) \bar{P}(x^1) \, dx^1 = \int_{I^1} f(\pi_{I^1}(an_Y x^1)) \bar{P}(x^1) \, dx^1. \]

The convergence of the limit as well as the fact that \( F_Y \) is constant on right cosets of \( N_Y^0 \) follows.

Each of the functions \( g \to F((\exp tY)g) \) is \( L \)-harmonic since \( L \) is left invariant.

Our limit will converge in the \( C_c^\infty \) topology due to the hypoellipticity of \( L \) [B].

From formula (4.26) and formula (4.27) \( f_Y \) is the function on \( MN \) defined by
\[ f_Y(x^1 x^0 n^-) = f(x^1) \]

for all \( x^1 \in I^1 \Gamma x^0 \in I^0 \) and \( n^- \in N^- \Gamma \) so the agreement of \( f_Y \) and \( f \) is proved. Both functions will be considered as functions on \( MN \) constant on \( N^- \) right cosets.

Writing \( x \in MN \) as
\[ x = x^1 x^0 n^- \quad x^1 \in I^1, \ x^0 \in I^0, \ n^- \in N^-, \]
for \( y \in N_Y^0 \) we have
\[
f_Y(x^1 x^0 n^- y) = f_Y(x^1 x^0 n^- y(n^-)^{-1} n^-) = f_Y(x^1 x^0 n^- y(n^-)^{-1}) = f(x^1) = f_Y(x^1 x^0 n^-),
\]
which proves that \( f_Y \) is right \( N_Y^0 \) invariant as a function on \( MN \). Finally suppose that \( f \) is constant on right cosets of \( N^0 \) where \( N^0 \) is a homogeneous subgroup of \( N \). Since \( N^0 \) is homogeneous we see that
\[
N^0 = (N^0 \cap (MN^+))(N^0 \cap N^-) = (N^0 \cap I^1)(N^0 \cap I^0)(N^0 \cap N^-).
\]
Now since \( N_Y^0 \) is a normal subgroup of \( N \) and \( f_Y \) is right \( N_Y^0 \) and \( N^- \) invariant for \( y \in N_Y^0 \) we have
\[
f_Y(x^1_Y x^0_Y y) = f_Y(x^1_Y y y^{-1} x^0_Y y) = f_Y(x^1_Y y).
\]
Now decomposing \( y \) as
\[
y = y^1 y^0 y^- \quad y^1 \in I^1, \; y^0 \in I^0, \; y^- \in I^-\]
we obtain
\[
f_Y(x^1_Y y) = f_Y(x^1_Y y^1) = f(x^1_Y y^1) = f_Y(x^1_Y x^0_Y).
\]
We see that \( f_Y \) is constant on all \( N^0 \) right cosets. Since \( f_Y \) is also constant on \( N_Y^0 \) cosets we see that it is constant on all right cosets. Since \( f_Y \) is also right \( N^0 \) invariant, \( f_Y \) is constant on right cosets of \( S \). Indeed, \( f_Y \) is right \( N^0 \) invariant, and so
\[
\tilde{f}(g) = f(g_1 g). \quad \text{Then by (4.16) (with \( N^1 = N^+ \) the boundary value} \tilde{f} \text{of} \tilde{F} \text{satisfies}
\]
\[
(4.28) \quad \tilde{f}(x) = f(\pi_{MN^+}(g_1 x)), \quad x \in MN^+.
\]
But when both \( f \Gamma \tilde{f} \) are considered as \( N^- \) right invariant functions on \( N \Gamma \), (4.28) becomes
\[
\tilde{f}(x) = f(g_1 x), \quad x \in MN.
\]
Therefore by (4.23) \( \Gamma f \) reduces to \( M \). This proves Theorem (4.21) in the case where the boundary value is continuous.
Actually the general case of Theorem (4.21) also follows. If \( F \) is an arbitrary \( L \) and \( L_1 \)-harmonic function and \( \phi \in C_c^\infty(MN^+) \) then the convolution \( F_\phi(g) = \phi \ast F(g) = \int_{MN^+} \phi(x)F(\tau^{-1}g) \, dx \) is \( \Gamma L_1 \)-harmonic with continuous boundary value \( \phi \ast f \). (See Lemma (4.9) of [DH].) Hence \( \phi \ast f \) treated as a function on \( MN \) is constant on \( S \)-cosets. Letting \( \phi \) range over an approximate identity proves Theorem (4.21).

Section 5. Tube domain over the cone of symmetric positive definite \( r \times r \) matrices.

In this section let \( D \) be the tube domain over the cone of symmetric positive definite \( r \times r \) matrices. By \( \Delta \) we denote the Laplace-Beltrami operator on \( D \). For a sequence of strictly positive numbers \( a = (a_1, \ldots, a_r) \) let

\[
L^a = \sum_{m=1}^{r} a_m HJK_m
\]

be a linear combination of strongly diagonal Hua operators. We are going to prove that the operators \( L^a \) having \( M \) as the maximal boundary play a special role on \( D \) --they characterize the classical Poisson-Szegö integrals from the Shilov boundary. This means that a bounded function is a Poisson-Szegö integral if and only if it is \( L^a \)-harmonic. Unfortunately this nice characterization is not true for other symmetric tube domains because then there are no \( L^a \) having \( M \) as the maximal boundary.

We begin by proving the existence of \( L^a \) which have the \( M \) as their maximal boundary. In the case of the cone of symmetric positive definite \( r \times r \) matrices \( c_{ij} = \frac{1}{2} \Gamma d_{ij} = 1 \) and \( d_m = r - m \Gamma \) and so the strongly diagonal Hua operators have the form

\[
HJK_m = 4(\frac{1}{2} \Delta_m - (r + 2 - m)Y_{mm} - \sum_{i < m} Y_{ii}).
\]

(5.1) Lemma. There is a choice of \( a_1, \ldots, a_r \) such that the maximal boundary for \( L^a \) is \( M \).

Proof Let \( Y = -\sum_{m=1}^{r} a_m((r + 2 - m)Y_{mm} + \sum_{i < m} Y_{ii}) \). We need \( a_1, \ldots, a_r \) such that

\[
(\lambda_i - \lambda_j)(Y) \geq 0 \quad \text{for } i < j.
\]

We start with \( \lambda_i - \lambda_r \Gamma i < r \). Then

\[
(\lambda_i - \lambda_r)(Y) = -\lambda_i \lambda_r (2a_rY_{rr} + a_rY_i + \sum_{i < m < r} a_mY_i) = a_r - \sum_{i < m < r} a_m.
\]

Whenever \( a_m < \frac{4a_r}{r} \Gamma m = 1, \ldots, r - 1 \Gamma (5.2) \) is true. Assume we can satisfy (5.2) for \( j > m \). We have

\[
(\lambda_i - \lambda_m)(Y) = (-\sum_{j > m} a_j(Y_i + Y_m) - a_m((r + 2 - m)Y_m + Y_i) - \sum_{i < j < m} a_jY_i - a_i(r + 2 - i)Y_i)
\]
Clearly making \( a_j \gamma i \leq j < m \) small enough we can satisfy (5.2) for \( j = m. \Box \)

**Remark.** The above lemma is not true for other symmetric tube domains i.e. when \( d_{ij} = 2, 4, 8. \)

Let \( P^\Delta_M \) be the \( \Delta \)-Poisson kernel on \( M \) while \( P^a_M \) be \( I^a \)-Poisson kernel on \( M. \) The main theorem of this section is

(5.3) **Theorem.** Let \( I^a \) be as above with the maximal boundary being \( M. \) For a bounded function \( F \) the following are equivalent

(5.4) There is \( f \in L^\infty(M) \) such that \( F(g) = \int_M f(\pi_M(gx))P^\Delta_M(x) \, dx \)

(5.5) \( L^a F = 0. \)

Moreover, for all such \( L^a \) the kernel \( P^a_M \) is equal to \( P^\Delta_M. \)

**Remark.** The proof relays heavily on the Johnson-Korányi result [JK] saying that for a bounded \( FT(5.4) \) is equivalent to be Hua harmonic.

**Proof** Implication (5.4)\( \rightarrow \) (5.5) follows directly from the result of Johnson-Korányi mentioned above. For the converse we first prove that \( P^a_M = P^\Delta_M. \) For a function \( f \in C_c(M) \) let

(5.6) \( P^\Delta_M f(g) = \int_M f(\pi_M(gx))P^\Delta_M(x) \, dx. \)

Since \( P^\Delta_M f \) is \( L^a \)-harmonic in view of Theorem (3.8) of [DH] there is \( h \in L^\infty(M) \) such that

(5.7) \( P^\Delta_M f(g) = P^a_M h(g) = \int_M h(\pi_M(gx))P^a_M(x) \, dx. \)

Convolving (5.6) and (5.7) from the left by \( \phi \in C_c(M) \) we have

\[ P^\Delta_M(\phi * f)(g) = P^a_M(\phi * h)(g). \]

Indeed

\[
P^\Delta_M(\phi * f)(g) = \int_M \int_M \phi(y^{-1})f(y\pi_M(gx))P^\Delta_M(x) \, dydx
\]

\[ = \int_M \phi(y^{-1})P^\Delta_M f(yg) \, dy = \int_M \phi(y^{-1})P^a_M h(yg) \, dy = P^a_M(\phi * h)(g). \]
Let now \( g(t) = \exp t \sum Y_{ii} \). Letting \( t \to -\infty \) we see that
\[
\phi \ast f(e) = \phi \ast h(e).
\]
This proves \( f = h \) and
\[
(P_M^a)^2 = P_M^a.
\]
Now if \( L^a F = 0 \) holds for a bounded function \( F \) then by Theorem (3.8) [DH]
\[
F = P_M^a f
\]
for an \( f \in L^\infty(M) \) which in view of (5.8) implies (5.4). \( \square \)

**Section 6. Hua harmonic functions**

The main result of this section is the following

**Theorem**. Let \( F \) be a bounded function on \( G \) annihilated by strongly diagonal Hua operators and harmonic with respect to an operator \( L \) satisfying the assumptions of Proposition (4.7). Then there is \( f \in L^\infty(M) \) such that
\[
F(g) = \int_M f(\pi_M(g x)) P_M(x) \, dx,
\]
where \( P_M \) is the \( L \)-Poisson kernel on \( M \).

**Remark.** Clearly we could use \( \Delta_{\text{diag}} \) as \( L \) in (6.1). In this case any \( F \) which is annihilated by the Hua system is automatically annihilated by \( L \) since \( \Delta_{\text{diag}} \) is the sum of the strongly diagonal Hua operators. Thus we produce a single Poisson kernel which is capable of representing Hua harmonic functions. We find the more general formulation of Proposition (6.3) remarkable in that the maximal boundary of \( L \) would typically be considerably larger than \( M \). The above proposition says that just being Hua harmonic forces the \( L \)-boundary function to reduce to a smaller boundary.

To prove Theorem (6.1) we need the following proposition:

**Proposition**. Suppose that a bounded function \( F \) is annihilated by the strongly diagonal Hua operators and, together with its \( L \)-boundary function \( f \), is constant on cosets of \( M \) in \( G \). Then \( F \) is a constant function.

Once Proposition (6.3) has been proved then Theorem (6.1) will follow as in the proof of Theorem (4.21).

To prove Proposition (6.3) we let \( f \) be the (continuous) \( L \)-boundary function for \( F \). By assumption \( F \) and \( f \) are now constant on cosets of \( M \). Effectively we may ignore \( M \) and consider all the functions and the operators as functions and
operators on $S$. This means that $F$ is $\pi_S(L)$-harmonic and instead of the Hua operators $\Delta_m \Gamma m = 1, \ldots, r$ we have

\[ \pi_S(\Delta_m) = c_m^{-1} \left( \sum_{i \leq m, \alpha} c_i^{-1}(Y_{im}^\alpha)^2 + \sum_{m \leq i, \alpha} c_m^{-1}(Y_{mj}^\alpha)^2 - \frac{d_m + f_m + 2}{c_m} Y_{mm} - \sum_{i < m} \frac{d_{im}}{c_i} Y_{ii} \right). \]

Therefore now our goal is to prove

**(6.4) Proposition.** Suppose that bounded $\pi(L)$-harmonic function $F$ is annihilated by $\pi(\Delta_m)$, $m = 1, \ldots, r$. Then $F$ is constant.

To prove Proposition (6.4) we have to formulate Lemma (4.24) in a more general situation. Let $S = NA$ be a semi-direct product of a connected and simply connected nilpotent Lie group $N$ and the group $A = \mathbb{R}^r$ with a diagonal action of $A$ on $N$. Let $L = Y_1^2 + \ldots + Y_m^2 + Y_0$ be a left-invariant operator on $S$ satisfying the Hörmander condition with the maximal boundary $N/N^+$ identified as in section 4 with $N^+ \Gamma N^+ = \sum_{\lambda \in \mathbb{R}^+} \Lambda_\lambda \Gamma N^+ = \sum_{\lambda \in \mathbb{R}^+} \Lambda_\lambda$.

**Lemma.** Let $Y \in A$ be a bounded, $L$-harmonic function on $S$. Assume that the $L$-boundary value $f$ is continuous on the maximal boundary $N^+$. Let $Y \in A$ be contractive. Then

\[ \lim_{t \to -\infty} F((\exp tY)g) = F_Y(g) \]

converges uniformly on compact sets in $S$ and defines an $L$-harmonic function with continuous boundary function $f_Y$. Both $F_Y$ and $f_Y$ are constant on right cosets of $N_Y^1$ in $S$ and in $N$ respectively. Additionally, $f_Y$ and $f$ agree on $N_Y^1 \cap N^+$. If $f$ reduces to $N/N^0$, then $f_Y$ will reduce to $S/N^1$, where $N^1$ is the subgroup generated by $N_Y^1$ and $N^0$. □

Still in the above general setting we have.

**(6.6) Lemma.** Let $N/N^0$ be a boundary for $L$ identified, as in section 4, with $N^1$ being a complement to $N^0$ in the sense of $N = N^1 N^0$. Assume that

\[ F(s) = \int_{N^1} f(\pi_{N^1}(sx)) \, \tilde{P}(x) \, dx \]

and $Y \in A + N^0$ centralize $N^1$. Then $F$ is constant under right translation by $\exp tY$ for all $t \in \mathbb{R}$.

**Proof** The proof is immediate because

\[ \pi_{N^1}(s \exp tY x) = \pi_{N^1}(sx \exp tY) = \pi_{N^1}(sx). \]
Let $N^+$ be the maximal boundary for $\pi(L)$ and $F$ a bounded
$\pi(L)$-harmonic function. Its boundary function $f$ defined originally on $N^+$ is
extended to $N$ by
\begin{equation}
(f(x^+ x^-) = f(x^+)
\end{equation}
and considered as a function on $N$.

We assume by induction that Proposition (6.4) is known for all domains of rank
less than $r$.

(6.8) Lemma. The function $f$ is constant on cosets of $N_{>1}$ in $S$, where $N_{>1} = S_{>1} \cap N$.

Proof We apply Lemma (6.5) with $Y = Y_{11}$. Then $N_Y^1$ is exactly $N_{>1}$ and $N_Y^0$ is $N_{1s}^1 = S_{1s} \cap N$. The function $F_Y$ is annihilated by $\pi_S(\Delta_m)\Gamma m = 1, \ldots, r$ and
is constant on cosets of $N_{1s}$ in $S$. From Lemma (2.21) $\Gamma F_Y$ is annihilated by the
image under $\pi_S$ of strongly diagonal $HJ\Gamma_{>1}$ operators and hence by induction $\Gamma$ is constant. Lemma (6.5) now shows that $f$ is constant on $N_{>1}$. By the same argument all left translates of $f$ are also constant on cosets of $N_{>1}$ (as in the proof of Theorem (4.21)) proving the lemma.$\square$

Now we shall introduce $r + 1$ sets of functions. We define
\begin{equation}
\mathcal{F}_{r+1} = \{xF : x \in N\}.
\end{equation}
and
\begin{equation}
\mathcal{F}_{r+1} = \{xf : x \in N\}.
\end{equation}
Clearly $\mathcal{F}_{r+1}$ is the set of boundary values of functions from $\mathcal{F}_{r+1}$ considered as
functions on $N$. Indeed if
\begin{equation}
F(s) = \int_{N^+} f(\pi_N(x)y)P(y^+) dy^+
\end{equation}
then
\begin{equation}
F(xs) = \int_{N^+} f(\pi_N(xsy)P(y^+) dy^+
\end{equation}
so $f'(y^+) = f(\pi_N(xy^+))$ is the boundary function of $xF$ on $N^+$. Extending $f'$ to
$NT$ by (6.7) we have
\begin{equation}
f'(y^+y^-) = f(\pi_N(xy^+)) = f(xy^+) = f(xy^+ y^-)
\end{equation}
so $f' =_x f$.

We then define
\begin{equation}
\mathcal{F}_k = \{xF_k : F_k(s) = \lim_{t \to \infty} F_{k+1}((\exp tY_{kk})s), F_{k+1} \in \mathcal{F}_{k+1}\},
\end{equation}
for $1 \leq k \leq r$. We shall prove shortly that these limits converge in the $C_c^\infty$ topology
on $G$. Granted this $\Gamma \mathcal{F}_k$ is a set of $L$-harmonic (and Hua harmonic) functions on $G$
which is also invariant under left translation. Let $\mathbf{F}_k$ denote the set of $L$ boundary values of functions from $\mathcal{F}_k$. Then $\mathbf{F}_k$ is also invariant under the left action of $G$.

First we prove existence of the limits in (6.9). For $k = r\Gamma$ the limit exists in $C_c^\infty$ from Lemma (6.5) (applied to $-Y_{rr}$). It is constant on cosets of the normal subgroup $N_r = \exp N_r\Gamma$ where

$$N_r = \sum_{1 \leq i < r} S_{ir}.$$ 

The same is satisfied by the elements of $\mathbf{F}_r$. Note that $-Y_{(r+1)(r-1)}$ is contractive on $N/N_r$. Therefore Lemma (6.5) applied to $S/N_r$ proves the existence of $F_{r-1}$.

Let

$$N_k = \sum_{i \leq j, k \leq j} S_{ij}$$ 

and

$$N_k = \exp N_k.$$ 

$N_k$ is a normal subgroup of $N$. It similarly follows by induction applied to the quotient group $S/N_{k+1}$ that the limit in (6.9) exists in $C_c^\infty$ for all $k$ and elements of $\mathcal{F}_k$ are constant on cosets of the normal subgroup $N_k$. Furthermore the same holds for boundary functions from $\mathbf{F}_k$. Moreover suppose that $F_k \in \mathcal{F}_k$ and that $F_{k+1}$ is related to $F_k$ as in (6.9) and that the corresponding boundary functions are $f_k$ and $f_{k+1}$ respectively. Then $f_k$ equals $f_{k+1}$ on $N_k = \exp N_k \Gamma$ where

$$N_{<k} = \sum_{1 \leq i < j < k} S_{ij}.$$ 

From Lemma (6.8) $\Gamma f_k$ is also constant on cosets of $N_{>1}$.

Proposition (6.4) clearly follows from the following:

**Lemma.** Each $F_k \in \mathcal{F}_k$ is constant for $3 \leq k \leq r + 1$.

**Proof.** Our proof will be inductive. For $F_3 \in \mathcal{F}_3 \Gamma$ since $F_3$ is constant on cosets of $N_3$ we have:

(6.11) 

$$0 = c_1 HJK_1(F_3) = (2c_1^{-1} Y_{11}^2 + c_1^{-1} \sum_{\alpha} (Y_{12}^\alpha)^2 - \frac{d_1 + 2}{c_1} Y_{11}) F_3$$

(6.12) 

$$0 = c_2 HJK_2(F_3) = (2c_2^{-1} Y_{22}^2 + c_1^{-1} \sum_{\alpha} (Y_{12}^\alpha)^2 - \frac{d_2 + 2}{c_2} Y_{22}^2 - \frac{d_1}{c_1} Y_{11}) F_3$$

Both $\sum Y_{ii}$ and $Y_{ii} \Gamma i \geq 3\Gamma$ centralize $N_{<3}$. Hence by Lemma (6.6) $\Gamma f_i \geq 3\Gamma$

$$Y_{ii} F_3 = 0$$

$$(Y_{11} + Y_{22} + \cdots + Y_{rr}) F_3 = 0.$$
Hence $Y_{11}F_3 = -Y_{22}F_3$.

We substitute this relation into formula (6.12) and subtract the result from formula (6.11) getting

$\left(2(c_1^{-1} - c_2^{-1})Y_{11}^2 - \left(\frac{d_1 + 2 + f_1 - d_{12}}{c_1} + \frac{d_2 + 2 + f_2}{c_2}\right)Y_{11}\right)F_3 = 0.$

If $c_1 = c_2$ then $Y_{11}F_3 = 0$. (Note that $d_1 \geq d_{12}$.) In particular $Y_{11}F_3 = 0.$

Then according to [DH] the maximal boundary for $Y_{11}^2 + \sum \alpha (Y_{12}^\alpha)^2 + Y_{11}$ on $S/N_3$ is trivial so $F_3$ is constant on $N_{<3}$ and hence $\Gamma$ on $S$.

If $c_1 \neq c_2$ then we conclude that there is a nonzero constant $\rho$ such that

$$(Y_{11}^2 + \rho Y_{11})F = 0$$

(Note that $\frac{d_1 + 2 + f_1 - d_{12}}{c_1} + \frac{d_2 + 2 + f_2}{c_2} > 0$). Hence solving a simple differential equation we see that there are constants $\tau$ and $\eta$ such that for all $s \in S$.

$$F_3(g(\exp tY_{11})) = (\tau + \eta e^{-\rho t})F_3(g).$$

Boundedness forces $\eta = 0$ and hence $Y_{11}F_3 = 0$. We see as above that $F_3$ is constant.

Now suppose by induction that we have shown that each $F_k \in F_k$ is constant. It follows that each $f_k \in F_k$ is constant on $N_{<k}$. But since $F_k \Gamma F_k$ are closed under left translations $\Gamma f_{k+1}$ is constant on right cosets of $N_{<k}$. Thus the boundary function $f_{k+1}$ for $F_{k+1}$ reduces to $N/N'$ where $N'$ is some homogeneous subgroup containing $N_{>1} \Gamma N_{<k}$ and $N_{k+1}$. We may choose a homogeneous compliment to $N'$ contained in

$N_{1k}.$

From Lemma (6.6)$\Gamma$

$$YF_{k+1} = 0$$

for any $Y \in N_{>1} + N_{<k} + N_{k+1} \Gamma$ which centralizes $N_{1k}$. In particular $\Gamma$

$$Y_{ij}F_{k+1} = 0$$

for all $1 \leq i \leq j < k \Gamma j \neq 1$. The above formula is also true for $k + 1 \leq j \leq r$ since $F_{k+1}$ is constant on cosets of $N_{k+1}$. For $1 < i < k \Gamma$ the equation $c_i H J K_i(F_{k+1}) = 0$ says exactly that

$$\left(c_i^{-1} \sum \alpha (Y_{i1}^\alpha)^2 - \frac{d_{1i}}{c_1}Y_{11}\right)F_{k+1} = 0.$$

(6.13)
For \( i = k \) we obtain:

\[
(6.14) \quad (2c_k^{-1}Y_{kk}^2 + \sum_{1 \leq i < k, \alpha} c_i^{-1}(Y_{ik}^\alpha)^2 - \frac{d_k + f_k + 2}{c_k}Y_{kk} - \frac{d_{1k}}{c_1}Y_{11})F_{k+1} = 0
\]

As before \( \Gamma(\sum Y_{ii})F_{k+1} = 0 \). Hence

\[
Y_{11}F_{k+1} = -Y_{kk}F_{k+1}.
\]

Then from formula (6.13) and formula (6.14) \( \Gamma F_{k+1} \) is annihilated by the operator:

\[
(6.15) \quad 2c_k^{-1}Y_{11}^2 + c_1^{-1} \sum_\alpha (Y_{1k}^\alpha)^2 + \left( \frac{d_k}{c_1} + \frac{d_k + f_k + 2}{c_k} - \sum_{i < k} \frac{d_{1i}}{c_1} \right)Y_{11}
\]

Finally from \( HJK_1 \Gamma \) we see that \( F_{k+1} \) is also annihilated by

\[
(6.16) \quad 2c_k^{-1}Y_{11}^2 + c_1^{-1} \sum_\alpha (Y_{1k}^\alpha)^2 - \frac{d_1 + f_1 + 2}{c_1}Y_{11}.
\]

Subtracting (6.16) from (6.15) we see that

\[
(6.17) \quad (2(c_k^{-1} - c_1^{-1})Y_{11}^2 + \left( \frac{d_k + f_k + 2}{c_k} + \frac{d_k}{c_1} - \sum_{i < k} \frac{d_{1i}}{c_1} + \frac{d_1 + f_1 + 2}{c_1} \right)Y_{11})F_{k+1} = 0
\]

Moreover since \( d_1 > \sum_{i < k} d_{1i} \) the coefficient by \( Y_{11} \) in (6.17) is strictly positive.

As in the \( F_3 \) case \( \Gamma \) it follows that \( F_{k+1} \) is constant on right cosets of the group whose lie algebra is generated by \( Y_{11} \) and \( Y_{i1}^\alpha \). Since this function is also constant on right cosets of \( N' \) we see that \( f_{k+1} \Gamma \) and hence \( F_{k+1} \) is constant \( \Gamma \) as desired. \( \square \)

This finishes the proof of Proposition (6.4) and hence of Theorem (6.1).

References


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