HUA OPERATORS ON BOUNDED HOMOGENEOUS DOMAINS IN $\mathbb{C}^n$ AND ALTERNATIVE REPRODUCING KERNELS FOR HOLOMORPHIC FUNCTIONS.

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0. Introduction

A major theme in the study of function theory on bounded domains in $\mathbb{C}^n$ is the study of the "boundary values" of holomorphic functions on the Bergman-Shilov boundary. Often, the study of boundary values involves defining a suitable class of real valued "harmonic" functions. Ideally, such a class should:

(1) Contain all real and imaginary parts of bounded holomorphic functions.
(2) Be describable as "Poisson integrals" over the Bergman-Shilov boundary against a real kernel (the "Poisson" kernel).
(3) Be invariant under all bi-holomorphisms of the domain.
(4) Be describable as the nullspace $\mathbf{H}_{\mathcal{L}}$ of a degenerate-elliptic system $\mathcal{L}$ of second order differential operators. (We refer to $\mathbf{H}_{\mathcal{L}}$ as the space of $\mathcal{L}$-harmonic functions.)

In the literature, at least two classes of harmonic functions and their boundary behavior have been investigated: the Δ-harmonic functions, where Δ is the Laplace-Beltrami operator and the Poisson-Szegö integrals of functions on the Bergman-Shilov boundary, as defined in [H] and [K]. Neither of these classes is entirely satisfactory in that in general, the Δ-harmonic functions fail the second condition and the Poisson-Szegö integrals fail the fourth. ([BV]).

In this work, we study several different classes of functions which satisfy some (possibly weakened) form of the above conditions in the context of bounded *homogeneous* domains in $\mathbb{C}^n$. The study of this class of domains is already both interesting and challenging in that, in general, for such domains, the Bergman-Shilov boundary is much smaller than the topological boundary and the topological boundary is not smooth. (c.f. [S1] and [S2]).

We make heavy use of the fact that any such domain is realizable as a Siegel domain of type I or II. Explicitly, let $\mathcal{V} \subset \mathbb{R}^n$ be an open, convex cone which does not contain straight lines. We assume that the cone $\mathcal{V}$ is homogeneous, i.e. there is an algebraic subgroup S of $\mathrm{Gl}(n, \mathbb{R})$ which acts transitively on $\mathcal{V}$ via the usual representation of $\mathrm{Gl}(n)$ on $\mathbb{R}^n$. (We denote this representation by π.) S may be taken to be a triangular subgroup which acts simply transitively on $\mathcal{V}$. Suppose

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further that we are given a complex vector space $\mathcal{Z}$ and a Hermitian symmetric, bi-linear mapping $K : \mathcal{Z} \times \mathcal{Z} \to \mathbb{C}^n$. We shall assume that

$$(a)\, K(z,z) \in \overline{\mathcal{V}} \text{ for all } z \in \mathcal{Z}$$
$$(b)\, K(z,z) = 0 \text{ implies } z = 0$$

The Siegel domain $\mathcal{D}$ associated with this data is defined as

$$\mathcal{D} = \{(z_1, z_2) \in \mathcal{Z} \times \mathbb{C}^n : \Im z_2 - K(z_1, z_1) \in \mathcal{V}\}.$$

The domain is said to be type I or II, depending upon whether or not $\mathcal{Z}$ is non-trivial.

The Bergman-Shilov boundary $\mathcal{B}$ of $\mathcal{D}$ is defined as

$$\mathcal{B} = \{(z_1, z_2) \in \mathcal{Z} \times \mathbb{C}^n : \Im z_2 = K(z_1, z_1)\}.$$

Suppose further that we are given a complex linear algebraic representation σ of S in $\mathcal{Z}$ such that

$$K(\sigma(s)z, \sigma(s)w) = \rho(s)K(z,w) \text{ for all } z, w \in \mathcal{Z}.$$

The group S acts on $\mathcal{D}$ by

$$s(z,w) = (\sigma(s)z, \rho(s)w). \tag{0.1}$$

We let $\mathbb{R}^n$ act on $\mathcal{D}$ by translation:

$$x(z,w) = (z, w + x), \;\; x \in \mathbb{R}^n. \tag{0.2}$$

Finally, we let $\mathcal{Z}$ act by

$$z_0(z,w) = (z + z_0, w + 2iK(z, z_0) + iK(z_0, z_0)). \tag{0.3}$$

These actions generate a completely solvable group G which acts simply transitively on $\mathcal{D}$. The action of the group G extends to $\mathcal{B}$ and the nilpotent group N generated by transformations (0.2) and (0.3) acts simply transitively on $\mathcal{B}$.

Every bounded homogeneous domain in $\mathbb{C}^n$ is biholomorphic to a homogeneous Siegel domain on which the group G described above acts simply transitively. This group plays a fundamental role in our theory.[1]

In fact, in [D] and [DH], a general class of solvable lie groups which includes G were studied. These results apply to the space of bounded L-harmonic functions for a single, second order, degenerate-elliptic, G-invariant operator L which also satisfies the Hörmander condition. Following Furstenberg, Guivarc'h and Raugi, it was shown how to associate with every such operator, a class of boundaries and, on each boundary, a Poisson kernel P_L. It is also proved that every bounded

[1] The importance of the group G was noticed by Korányi and Stein almost thirty years ago in their study of the Hardy spaces $\mathcal{H}^p(\mathcal{D})$ cf. e.g. [KS1] and [KS2]. Later the group G and its representations played a fundamental role in the work of Rossi and Vergne, [RV1], [RV2].

L-harmonic is the integral over the maximal boundary of a function against the corresponding Poisson kernel. Using these results, we prove the following, which is one of the main results of the current work:

Let L be a G-invariant, real, second order operator which satisfies the Hörmander condition and annihilates holomorphic functions on a homogeneous Siegel domain $\mathcal{D}$. The Shilov-Bergman boundary $\mathcal{B}$ is one of the boundaries associated with L. Let P_L be the corresponding Poisson kernel on $\mathcal{B}$. Then every bounded holomorphic function F on $\mathcal{D}$ is the Poisson integral $F = P_L(f)$ of the boundary values f of F on $\mathcal{D}$.

Moreover,

For every homogeneous Siegel domain there exists an operator L as above for which the maximal boundary is $\mathcal{B}$.

In fact, for a given homogeneous domain there are many such operators.

Taken together, the above results imply that the space of L-harmonic functions satisfy conditions (1)-(4) stated above, except that in condition (3), invariance under the full automorphism group of the domain is replaced by the weaker condition of invariance under the transitive group G. On the other hand, condition (4) is strengthened–harmonicity is defined in terms of the nullspace of a *single* differential operator. This may be viewed as a characterization of the Bergman-Shilov boundary by means of a differential operator suggested by E. M. Stein many years ago.

If the operator L were invariant under all of Aut $(\mathcal{D})$, then of course the stronger form of condition (3) would follow. It general, however, it seems that the algebra of Aut $(\mathcal{D})$-invariant differential operators may be just the algebra generated by Δ, the Laplace-Beltrami operator for the Bergman metric on $\mathcal{D}$. In general, the Δ-harmonic, bounded functions are *not* reproducible from their boundary values on the Bergman-Shilov boundary, except if $\mathcal{D}$ is a product of balls. Thus, it seems that in order to retain condition (3), we are forced to consider invariant *systems* of differential operators. In this work, following an idea suggested to us by Nolan Wallach, we define a canonical system HJK (the Hua system) in terms of a contraction of $\partial\overline{\partial}$ against the curvature tensor. Our main result concerning this system is:

For every homogeneous Siegel domain, there exists a canonical system which we call the Hua system and denote HJK. The space of HJK harmonic functions $\mathbf{H}_{HJK}$ is Aut $(\mathcal{D})$-invariant and every bounded function F in $\mathbf{H}_{HJK}$ is a Poisson integral $\mathcal{P}_{HKJ}(f)$ of a bounded function f on $\mathcal{B}$.

For the Poisson kernel P_{HJK} one can take the Poisson kernel P_L on $\mathcal{B}$ corresponding to **any** *G-invariant operator L which is subelliptic and is a linear combination of the elements of the system HJK. In particular, we may use the Laplace-Beltrami operator Δ as L.*

Therefore,

Every bounded, HJK-harmonic function is the integral over the Bergman-Shilov boundary of a uniquely determined bounded function against the Poisson kernel for Δ.

In the case that $\mathcal{D}$ is a Hermitian symmetric tube domain, our HJK system is the one which Johnson and Korányi [JK], generalizing earlier work by Hua [H, defined. However, the kernel P_{HJK} is equal to the standard Poisson-Szegö kernel P_S on $\mathcal{B}$ (cf. [H] and [K]) iff $\mathcal{D}$ is a Hermitian-symmetric tube domain.

It might appear that the Laplace-Beltrami operator plays a special role in this theory. This, in fact, is not the case. It is possible to define a whole class of elliptic, second order, differential operators for which our results hold. In fact, there are cases where Δ does not provide the sharpest results. Explicitly, we show that

For the tube domain over the cone of real, positive definite $n \times n$ matrices there exists a single G-invariant elliptic second order differential operator Δ' such that the functions in $\mathbf{H}_{\Delta'}$ are precisely the Poisson-Szegö integrals of bounded functions on the Bergman-Shilov boundary. The operator Δ' is a linear combination of the "diagonal" elements of HJK.

For $n = 2$ Malliavin and Korányi [KM] (cf. also [J1] and [J2]) exhibited a system $\mathcal{L}$ of two G-invariant operators for which $\mathbf{H}_{\mathcal{L}}$ consists of Poison-Szegö integrals of L^∞ functions on the Bergman-Shilov boundary.

For an arbitrary *symmetric* domain Berline and Vergne [BV] exhibited a *third order* Aut $(\mathcal{D})$-invariant system $\mathcal{L}$ for which $\mathbf{H}_{\mathcal{L}}$ consists of Poison-Szegö integrals of L^∞ functions on the Bergman-Shilov boundary. In [D] and [DH] some probabilistic tools are used so restriction to the second order degenerate elliptic operators is necessary. This also explains why we are unable to go beyond second order systems in the present paper.

Our proofs are inductive, relying both on the characterization of bounded homogeneous domains as Siegel domains of type I and II due to [PS], as well as the structure theory of homogeneous cones due to [V]. We also, of course, use the results of [DH].

Section 1. *The Hua Operators*

In this section, we define the Hua operators in general and compute them in the context of a bounded homogeneous domain.

Let $\mathcal{D}$ be a Kählerian manifold and let T be the (real) tangent bundle for $\mathcal{D}$. (We shall not need to indicate its dependence on $\mathcal{D}$ in our notation.) We assume that the reader is familiar with the basic properties of Kählerian manifolds and their Riemannian connection. (See e.g. [He]). Let

$$T_c = T^{10} \oplus T^{01}$$

be the decomposition of T_c into holomorphic and anti-holomorphic vector fields. (T_c is, of course, the complex tangent bundle.) We have a similar decomposition

$$T_c^* = (T^*)^{10} \oplus (T^*)^{01}$$

where $(T^*)^{ij}$ is the annihilator of T^{ji} in T_c^*. Hence, $(T^*)^{ij}$ is the dual space of T^{ij}.

We define an operator $\overline{\partial}\partial : C^\infty(\mathcal{D}) \to {}, ((T^*)^{10} \otimes (T^*)^{01})$ in local holomorphic coordinates by

$$\overline{\partial}\partial f = \frac{\partial^2 f}{\partial \overline{z}_j \partial z_i} dz_i \otimes d\overline{z}_j$$

(It is easily seen that this is independent of the choice of coordinates.)

Now, let g be the Riemannian structure for $\mathcal{D}$ and let ∇ denote the corresponding Riemannian connection. For a C^∞ function f we define a 2-tensor by

$$\nabla^2 f(X,Y) = (XY - \nabla_X Y)f$$

Then, on a Kählerian manifold we have the following:

(1.1) Lemma. *For all* $f \in C^\infty(\mathcal{D})$, $\overline{\partial}\partial f = \nabla^2 f | T^{10} \times T^{01}$

Proof This follows immediately from the above formula for $\nabla^2 f$ and the observation that for all i and j,

$$\nabla_{\overline{Z}_j} Z_i = 0 \quad \text{and} \quad \nabla_{Z_j} \overline{Z}_i = 0$$

where $Z_j = \dfrac{\partial}{\partial z_j}$. (See the material below formula (12), p. 292 in [He]).

A usual, we define the curvature operator by

$$R(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$$

where X and Y are complex vector fields. We interpret R as a End (T_c, T_c) valued two form on $\mathcal{D}$. (We extend ∇ to T_c by complex linearity.) We also extend g to the complex tangent bundle by complex linearity. We shall let H be the Hermitian form on T_c defined by

$$H(Z,W) = \frac{1}{2} g(Z, \overline{W}).$$

Let $\{E_1, \ldots, E_n\} \subset T^{10}$ be a local orthonormal frame for T^{10} (orthonormal with respect to H). For $f \in C^\infty(\mathcal{D})$, we define

$$HJK(f) = -\sum \overline{\partial}\partial f(E_i, \overline{E}_j) R(\overline{E}_i, E_j) | T^{01} \tag{1.2}$$

It is easily seen that this is independent of the orthonormal frame.

It is clear that HJK annihilates holomorphic functions. The next lemma will establish that HJK is real in the sense defined in the introduction.

(1.3) Lemma. *For all* Z *and* W *in* , (T^{01}) *and all* $f \in C^\infty(\mathcal{D})$,

$$H(HJK(f)Z, W) = H(Z, HJK(\overline{f})W).$$

Proof This follows from formula (1.2) along with the observations that, for all U, V, Z and W in , (T_c),

$$H(R(U,V)Z, W) = H(Z, R(\overline{V}, \overline{U})W).$$

and, for all Z and $W \in \Gamma(T^{01})$,

$$\overline{\overline{\partial}\partial f}(\overline{W}, Z) = \overline{\partial}\partial \overline{f}(\overline{Z}, W).\square$$

The next proposition establishes that every Hua-harmonic function is in the kernel of Δ, where Δ is the Laplace-Beltrami operator.

(1.4) Proposition. *For all* $f \in C^{\infty}(\mathcal{D})$,

$$\mathrm{Tr}\ HJK(f) = \Delta(f).$$

Proof We note first that for all Z and W in T_x^{01},

$$\mathrm{Tr}\ R_x(\overline{Z}, W)|T_x^{10} = -r_x(\overline{Z}, W).$$

where r is the Ricci curvature. (This formula follows easily from formula (5), p289 of [He] along with the identity $R^k_{lij^*} = R^k_{ilj^*} = -R^k_{ij^*l}$ where the notation is as in [He], *loc. cit.*)

From Proposition 3.6, p.300 of [He], along with formula (1.2) above, we see that

$$\text{(1.5)} \qquad \mathrm{Tr}\ HJK(f) = 2\sum \overline{\partial}\partial f(E_i, \overline{E}_j)H(E_i, E_j) = 2\sum \overline{\partial}\partial f(\overline{E}_i, E_i).$$

On the other hand, it is known that Δf is the contraction of $\nabla^2 f$ ([O], p. 86.) It is easily seen from Lemma (1.1) that this is exactly the quantity on the right. $\square$

Our next goal is to compute a formula for HJK in the case that $\mathcal{D}$ is a bounded homogeneous domain. Thus, in view of [PS] and [V] we may assume that there is a connected, simply connected Lie group G which acts simply transitively on $\mathcal{D}$ and that this action is real analytic in the G-variable and is holomorphic in the $\mathcal{D}$ variable. We let x_o be a fixed base point in $\mathcal{D}$.

Let $\mathcal{G}$ denote the Lie algebra of G. In general, we shall adopt the convention that upper case Roman letters will be used to denote Lie groups and that the corresponding upper case script letter will automatically denote the corresponding Lie algebra.

The complex tangent space $(T_c)_{x_o}$ may be identified with $\mathcal{G}_c$ and G-invariant vector fields on $\mathcal{D}$ with left-invariant vector fields on G. The set of elements X in $\mathcal{G}_c$ which annihilate holomorphic functions at x_o is denoted by $\mathcal{P}$. Clearly $\mathcal{P}$ is a complex subalgebra of $\mathcal{G}_c$. Since left translation preserves holomorphic functions, a vector field $X \in \mathcal{P}$ is a section of the bundle T^{01}. We let $\mathcal{Q} = \overline{\mathcal{P}}$. The vector fields valued in $\mathcal{Q}$ define the sections of the bundle T^{10}.

Note that since $T_c = T^{10} \oplus T^{01}$, we have

$$\mathcal{G}_c = \mathcal{P} \oplus \mathcal{Q}$$

Let π_Q be the projection to $\mathcal{Q}$ along $\mathcal{P}$. For each $Z \in \mathcal{P}$, we define an operator $M(Z) : \mathcal{Q} \to \mathcal{Q}$ by

$$M(Z)(X) = \pi_Q([Z, X]). \tag{1.6}$$

To compute the HJK operator, we must compute the connection. Notice that for $X \in , (T^{10})$, $Z \in , (T^{01})$, and f any (local) holomorphic function

$$\nabla_Z X(f) = \nabla_X Z(f) + [X, Z]f = [X, Z]f$$

since the torsion is zero and the conection preserves holomorphic type. It follows that

$$\nabla_Z X(f) = \pi([X, Z])(f) \tag{1.7}$$

where π is the projection to T^{10} along T^{01} in T_c. Hence $\nabla_Z X(f) = \pi([X, Z])$. In particular, for $Z \in \mathcal{P}$, $X \in \mathcal{Q}$ we have

$$\nabla_Z X = M(Z)(X).$$

Since the connection is real, we may also state that

$$\nabla_X Z = M(X)(Z),$$

where

$$M(X)Z = \overline{M(\overline{X})\overline{Z}}.$$

We will also need to know the connection on other types of forms. Since the Riemannian structure is invariant, the form g is defined by a scalar product g on $\mathcal{G}$ and H is defined, as above, by a Hermitian scalar product (still called H) on $\mathcal{G}_c$. For $Z \in \mathcal{Q}$, define an operator $M^*(Z) : \mathcal{Q} \to \mathcal{Q}$ by the identity

$$H(M^*(Z)X, Y) = H(X, M(\overline{Z})Y)$$

where X and Y range over $\mathcal{Q}$. Thus, $M^*(Z)$ is the adjoint in H of $M(\overline{Z})$.

(1.8) Proposition. *Let Z and X be elements of $\mathcal{Q}$. Then*

$$\nabla_Z X = -M^*(Z)(X).$$

Proof On a Kähler manifold, the connection preserves holomorphic types. Therefore, $\nabla_Z X$ is of type $(1, 0)$. Furthermore,

$$\overline{\nabla_Z X} = \nabla_{\overline{Z}} \overline{X}$$

We compute:

$$ZH(X, Y) = H(\nabla_Z X, Y) + H(X, \nabla_{\overline{Z}} Y).$$

for $Y \in \Gamma(T^{10})$. But for X, Y G-invariant, $ZH(X,Y) = 0$, and so

$$H(\nabla_Z X, Y) = -H(X, \nabla_{\overline{Z}} Y) = -H(X, M(\overline{Z})Y) = -H(M^*(Z)X, Y).\square$$

Again, since the connection is real, we may write

$$\nabla_Z \overline{X} = \overline{\nabla_{\overline{Z}} X} = -M^*(Z)\overline{X}$$

where $M^*(Z)\overline{X} = \overline{M^*(\overline{Z})X}$.

Next, we compute the curvature. Our result is:

(1.9) Theorem. *For X and Z in $\mathcal{Q}$ and $W \in \mathcal{P}$, the form R defined below is the curvature tensor at the identity e of G.*

$$R(Z,W)X = -[M^*(Z)M(W) - M(W)M^*(Z) + M^*(M(W)Z) + M(M(Z)W)]X$$

Proof It follows easily that at e

$$(\nabla_Z \nabla_W - \nabla_W \nabla_Z)X = (-M^*(Z)M(W) + M(W)M^*(Z))X$$

Also,

$$[Z,W] = \nabla_Z W - \nabla_W Z = M(Z)W - M(W)Z$$

It follows that at e

$$R(Z,W)X = (-M^*(Z)M(W) + M(W)M^*(Z) - M(M(Z)W) - M^*(M(W)Z)X.$$

This proves our formula.

Section 2. *Hua operators on type I Domains*

The Siegel domain of type I associated with a homogeneous regular cone $\mathcal{V}$ (as described in the introduction) is the domain in $\mathbb{C}^n$ defined by

$$\mathcal{E} = \mathbb{R}^n + i\mathcal{V}.$$

i.e. for such domains the space $\mathcal{Z}$ is trivial. Let S an algebraic subgroup S of $\mathrm{Gl}(n,\mathbb{R})$ which acts transitively on $\mathcal{V}$ via the usual action of $\mathrm{Gl}(n)$ on $\mathbb{R}^n$. It is a result of [V] that S may be taken to be a triangular subgroup which acts simply transitively on $\mathcal{V}$. We may also assume that S contains tI for all $t \in \mathbb{R}^+$. We shall let $c \in \mathcal{V}$ be a fixed base point.

The group S acts on $\mathcal{E}$ by matrix multiplication. We let $\mathcal{M} = \mathbb{R}^n$ thought of as a commutative Lie algebra. The corresponding Lie group M is $\mathbb{R}^n$ under addition. This group acts on $\mathcal{E}$ by translation. These two actions generate a simply transitive subgroup G of the automorphism group of $\mathcal{E}$. The group G is the semi-direct product $G = M \times_s S$ where the S action on M is matrix multiplication. We shall identify $\mathcal{M}$ and $\mathcal{S}$ with the corresponding subalgebras of $\mathcal{G}$ and hence, M and S with subgroups of G.

Let ρ be the representation of S on $\mathbb{R}^n$ defined by letting S act on $\mathbb{R}^n$ by matrix multiplication. We shall also let ρ denote the action of the Lie algebra $\mathcal{S}$ on $\mathbb{R}^n$ obtained by differentiating ρ. Since $\mathcal{S}$ acts simply transitively, the mapping σ of $\mathcal{S}$ into $\mathbb{R}^n$ defined by

$$\kappa(X) = \rho(X)c$$

is a vector space isomorphism. We extend ρ and κ to $\mathcal{S}_c$ by complex linearity. Then we have the following:

(2.1) Lemma. $\mathcal{P} = \{(\kappa(Y), iY) | Y \in \mathcal{S}_c\}$.

Proof We consider $\mathcal{E} \subset \mathbb{R}^n \times \mathbb{R}^n$. Then the tangent space at ic is $\mathbb{R}^n \times \mathbb{R}^n$. The tangent space is also identified with $\mathcal{G} = \mathcal{M} \times_s \mathcal{S}$. The identification is defined by mapping $(X, Y) \in \mathcal{G}$ into $(X, \kappa(Y)) \in \mathbb{R}^n \times \mathbb{R}^n$. Under this identification, the space defined in the statement of the lemma maps onto the Cauchy-Riemann operators, proving the lemma.□

It follows that the complex structure on the tangent space is defined by the mapping $J : (\kappa(X), Y) \to (-\kappa(Y), X)$.

There is an algebraic description of the general homogeneous cone which is due to Vindberg which we shall require. We define a product Δ on $\mathcal{S}$ by the equality

$$X \Delta Y = \kappa^{-1}(\rho(X)\rho(Y)c)$$

Since ρ is a Lie algebra representation, it is easily seen that for all X and Y in $\mathcal{S}$,

$$X \Delta Y - Y \Delta X = [X, Y].$$

The operation just introduced is useful in describing the operator M introduced above. Let $X = (i\kappa(B), B) \in \mathcal{Q}$ and $Z = (-i\kappa(A), A) \in \mathcal{P}$. Then

$$\begin{aligned}[Z, X] =& (i\rho(A)\kappa(B) + i\rho(B)\kappa(A), [A, B]) \\ =& (i\rho(A)\kappa(B), A\Delta B) - (-i\rho(B)\kappa(A), B\Delta A)\end{aligned}$$

In view of (1.7) it follows that

(2.2) $$M(Z)X = (i\kappa(A\Delta B), A\Delta B).$$

Our next goal is to explicitly compute the operator $M^*(Z)X$ for $Z \in \mathcal{Q}_o$. For this, we shall also require an algebraic description of the Riemannian structure of the domain. Assume, for the moment, that the Riemannian structure in question is that derived from the Bergman metric. Since this structure is G-invariant, it is defined by a scalar product g on the Lie algebra $\mathcal{G}$. Koszul ([Kl], Formula 4.5) proved the existence of a functional $\beta \in \mathcal{G}^*$ such that this scalar product is given by

(2.3) $$g(X, Y) = \beta([JX, Y]).$$

This functional has a very simple description in terms of the "normal decomposition" of $\mathcal{S}$, which will be explained after Proposition (2.6). Since g is J-invariant,

$$\beta([JX, JY]) = -\beta([J^2X, Y]) = \beta([X, Y]) \tag{2.4}$$

We shall not explicitly use any other information concerning β other than the fact that formula (2.3) defines a J-invariant, positive-definite, scalar product. Proving our results in this generality seems necessary in order to carry out the inductive portion of the proof (see Section 6).

Notice that for all A and B in $\mathcal{S}$

$$g((0,B),(0,A)) = \beta([(-\kappa(B),0),(0,A)]) = \beta((\kappa(A\Delta B),0)) = \xi(A\Delta B),$$

where $\xi \in \mathcal{S}^*$ is defined by

$$\xi(A) = \beta((\kappa(A),0)).$$

Note that one consequence of the above is that the expression

$$(A,B) = \xi(A\Delta B)$$

defines a scalar product on $\mathcal{S}$.

Using formula (2.4) and the fact that $\mathcal{M}$ is abelian, it is easily seen that β is zero on $[\mathcal{S},\mathcal{S}]$. Moreover $\mathcal{M}$ and $\mathcal{S}$ are orthogonal. This easily implies:

(2.5) Lemma. *For $X = (i\kappa(A), A)$ and $Z = (i\kappa(B), B)$,*

$$H(Z,X) = \xi(A\Delta B).$$

To describe M^*, we shall require the 'dual' product on $\mathcal{S}$. We define a product '$\Box$' on $\mathcal{S}$ by the equality

$$(A\Delta B, C) = (B, A\Box C).$$

This product is, in fact, the 'Δ' product on $\mathcal{S}$ induced from the dual cone, although we shall not require this fact.

Now, let $X = (i\kappa(A), A) \in \mathcal{Q}$ and $Z = (i\kappa(B), B) \in \mathcal{Q}$. Then we have the following proposition which follows easily from formula (2.2).

(2.6) Proposition. $M^*(Z)X = (i\kappa(B\Box A), B\Box A)$.

To obtain more precise results, we shall need to use the structure theory of clans due to Vindberg. Let r be the rank of $\mathcal{S}$. (The dimension of the maximal torus in $\mathcal{S}$.) Vindberg proves (Proposition 8, p.374) that $\mathcal{S}$ has a 'normal decomposition'. This means that there is a direct sum decomposition

$$\mathcal{S} = \oplus \sum_{1\le i\le j\le r} \mathcal{S}_{ij}$$

where

(2.7) For each $1 \leq i \leq r$, $\mathcal{S}_{ii}$ is spanned by a single element e_{ii} such that $e_{ii}\Delta e_{ii} = e_{ii}$.

(2.8) For $1 \leq i \leq j \leq k$,

$$\begin{aligned}\mathcal{S}_{ij}\Delta\mathcal{S}_{jk} \subset \mathcal{S}_{ik} \text{ and}\\ \mathcal{S}_{jk}\Delta\mathcal{S}_{ik} + \mathcal{S}_{ik}\Delta\mathcal{S}_{jk} \subset \mathcal{S}_{ij}.\end{aligned}$$

(2.9) $\mathcal{S}_{ij}\Delta\mathcal{S}_{kl} = \{\mathbf{O}\}$ if $j \neq k$ and $j \neq l$.

(2.10) Let $i < j$ and let $s_{ij} \in \mathcal{S}_{ij}$. Then

$$\begin{aligned}e_{ii}\Delta s_{ij} =& \frac{1}{2}s_{ij} = e_{jj}\Delta s_{ij}\\ s_{ij}\Delta e_{jj} =& s_{ij}.\end{aligned}$$

(2.11) The functional ξ is zero on $\mathcal{S}_{ij}$ for $i < j$, and by definition, $\xi(e_{ii}) = g(e_{ii}, e_{ii})$. Therefore β is zero on $\sum_{i<j}\sigma(\mathcal{S}_{ij}) \times \mathcal{S}$.

We refer to the above properties as the 'properties of the normal decomposition'. In fact (2.11) can be derived easily from (2.7)-(2.10), the orthogonality of the decomposition $\mathcal{G} = \mathcal{M} \times \mathcal{S}$ and the invariance of g under J. To understand the meaning of (2.7)-(2.11), it helps to keep the following example in mind.

(2.12) Example: 1 Let $\mathcal{X}$ be the set of $n \times n$, real, symmetric matrices and let $\mathcal{V} \subset \mathcal{X}$ be the cone of positive definite matrices. The group S of all invertible upper-triangular matrices with positive diagonal acts simply transitively on $\mathcal{V}$ by means of the representation ρ defined by

$$\rho(S)X = SXS^t.$$

The differentiated representation of $\mathcal{S}$ then is given by

$$\rho(A)X = AX + XA^t.$$

We choose $c = I$ as our base point. Then

$$\kappa(A) = \rho(A)I = A + A^t.$$

If $B \in \mathcal{S}$, then $B\Delta A$ is defined by

$$B\Delta A = \kappa^{-1}(\rho(B)(A + A^t)) = \kappa^{-1}(B(A + A^t) + (A + A^t)B^t).$$

The space $\mathcal{S}_{ij}$ are just the space of matrices which are non-zero only in the (i, j) position. The elements e_{ii} are the diagonal matrices which have $1/2$ in the (i, i) entry and all other entries zero. The functional ξ may be taken to be the trace. The properties for the normal decomposition are easily verified in this case.

We shall also need information on how $\Box$ interacts with the normal decomposition. This is most easily stated in terms of the spaces

$$\mathcal{T}_{ij} = \mathcal{S}_{r-j\, r-i}.$$

The following is a simple consequence of the observation that the normal decomposition is an orthogonal decomposition.

(2.13) Proposition. *The operation $\Box$ satisfies the properties of the normal decomposition with respect to the spaces $\mathcal{T}_{ij}$.*

One requirement for the boundary theory which we utilize is a detailed knowledge of the root structure of $\mathcal{G}$. This too is readily obtained from the normal decomposition. Let $\mathcal{A} \subset \mathcal{S}$ be the span of the e_{ii} and let

$$\mathcal{N} = \sum_{i<j} \mathcal{S}_{ij}.$$

Then $\mathcal{N}$ is the unipotent radical for $\mathcal{S}$ and $\mathcal{A}$ is the maximal torus for both $\mathcal{S}$ and for $\mathcal{G}$. Let $\lambda \in \mathcal{A}^*$. Then λ is said to be a root if there are non-zero X such that for all $D \in \mathcal{A}$,

$$[D, X] = \lambda(D)X.$$

Such X are called root vectors. We shall let $\mathcal{M}_\lambda$ and $\mathcal{N}_\lambda$ denote, respectively, the spaces of the root vectors for λ in $\mathcal{M}$ and in $\mathcal{N}$. The set of all roots will be denoted R.

Let $\{\lambda_1, \lambda_2, \ldots, \lambda_r\} \subset \mathcal{A}^*$ be the dual basis to the e_{ii} basis. We shall leave the following to the reader:

(2.14) Proposition. *Let $i \leq j$. If $\mathcal{S}_{ij} \neq 0$, then both $(\lambda_i + \lambda_j)/2$ and $(\lambda_i - \lambda_j)/2$ are roots. The corresponding root spaces are, respectively, $\sigma(\mathcal{S}_{ij}) \subset \mathcal{M}$ and $\mathcal{S}_{ij} \subset \mathcal{S}$.*

Now we shall introduce some notation. Let $c_i = \xi(e_{ii})$. Also, for $i < j$, we let d_{ij} be the dimension of $\mathcal{S}_{ij}$. We choose a basis e_{ij}^α for $\mathcal{S}_{ij}$ such that

$$(e_{ij}^\alpha, e_{ij}^\gamma) = \delta_{\alpha,\gamma} c_i.$$

This basis turns out to be more convenient than an orthonormal basis due to the following:

(2.15) Lemma. *For $i < j$,*

$$e_{ij}^\alpha \Delta e_{ij}^\gamma = \delta_{\alpha\gamma} e_{ii}.$$

Proof Since $\mathcal{S}_{ii}$ is one dimensional, $e_{ij}^\alpha \Delta e_{ij}^\gamma = c e_{ii}$ for some scalar c. Furthermore

$$c c_i = \xi(e_{ij}^\alpha \Delta e_{ij}^\gamma) = (e_{ij}^\alpha, e_{ij}^\gamma) = c_i \delta_{\alpha,\gamma}.$$

This proves the lemma.$\Box$

Next, we shall require an orthonormal basis for $\mathcal{Q}$. For this, we define, for all $i \leq j$,

$$E^{\alpha}_{ij} = (Y^{\alpha}_{ij} + \sqrt{-1}X^{\alpha}_{ij})/\sqrt{c_i} \tag{2.16}$$

where

$$\begin{aligned} Y^{\alpha}_{ij} =& (0, e^{\alpha}_{ij}) \\ X^{\alpha}_{ij} =& (\kappa(e^{\alpha}_{ij}), 0). \end{aligned}$$

(If $i = j$, we interpret $\alpha = 1$ and $e^{\alpha}_{ii} = e_{ii}$) It follows easily from Lemmas (2.5) and (2.15) that the E^{α}_{ij} define an orthonormal basis for $\mathcal{Q}$.

Considered as vector fields on $\mathcal{E}$, the elements E^{α}_{ij} form an orthonormal frame field for T^{10}. We may therefore compute the HJK operators from formula (1.2). Actually, it turns out that we only require the 'strongly diagonal' HJK operators. These are the operators defined by

$$HJK_m f = H(HJK(f)E_{mm}, E_{mm}). \tag{2.17}$$

Our main result of this section is the following:

(2.18) Theorem.

$$HJK_m = c_m^{-1}(\Delta_m - \frac{d_m + 2}{c_m} Y_{mm} - \sum_{i<m} \frac{d_{im}}{c_i} Y_{ii})$$

where $d_m = \sum_{m<j} d_{mj}$ *and*

$$\Delta_m = \sum_{i \leq m, \alpha} c_i^{-1}((Y^{\alpha}_{im})^2 + (X^{\alpha}_{im})^2) + \sum_{m \leq j, \alpha} c_m^{-1}((Y^{\alpha}_{mj})^2 + (X^{\alpha}_{mj})^2).$$

Proof From Lemma (1.1) and Theorem (1.9),

$$HJK(f)_m = \sum f^{\alpha,\beta}_{ij,kl} C^{\alpha,\beta}_{ij,kl}$$

where

$$\begin{aligned} C^{\alpha,\beta}_{ij,kl} =& - H(R(E^{\alpha}_{ij}, \overline{E}^{\beta}_{kl})E_{mm}, E_{mm}) \\ =& H(M(\overline{E}^{\beta}_{kl})E_{mm}, M(\overline{E}^{\alpha}_{ij})E_{mm}) - H(M^*(E^{\alpha}_{ij})E_{mm}, M^*(E^{\beta}_{kl})E_{mm}) \\ &+ H(M(M(E^{\alpha}_{ij})\overline{E}^{\beta}_{kl})E_{mm}, E_{mm}) + H(M^*(M(\overline{E}^{\beta}_{kl})E^{\alpha}_{ij})E_{mm}, E_{mm}). \end{aligned} \tag{2.19}$$

and

$$f^{\alpha,\beta}_{ij,kl} = [\overline{E}^{\alpha}_{ij} E^{\beta}_{kl} - M(\overline{E}^{\alpha}_{ij})E^{\beta}_{kl}]f$$

The sum is over all indecies with $1 \leq i \leq j \leq r$, $1 \leq \alpha \leq d_{ij}$, $1 \leq k \leq l \leq r$ and $1 \leq \beta \leq d_{kl}$.

Our first observation is that if $(i,j,\alpha) \neq (k,l,\beta)$, then $C^{\alpha,\beta}_{ij,kl} = 0$. In fact, each term in formula (2.19) is zero. (This follows from the normal decomposition properties, Propositions (2.6), (2.14) and the observation that the $\mathcal{S}_{ij}$ spaces are mutually orthogonal.)

Next, we shall record a series of formulae which the reader may readily verify. We set

$$Z_i = E_{ii}/\sqrt{c_i} = c_i^{-1}(Y_{ii} + \sqrt{-1}X_{ii}).$$

For $i < j$

$$\begin{aligned}
M(\overline{E}^{\alpha}_{ij})E_{mm} &= \frac{\delta_{j,m}}{\sqrt{c_m}} E^{\alpha}_{im} \\
M^*(E^{\alpha}_{ij})E_{mm} &= \frac{\delta_{i,m}}{\sqrt{c_m}} E^{\alpha}_{mj} \\
M(\overline{E}^{\alpha}_{ij})E^{\alpha}_{ij} &= \frac{1}{\sqrt{c_i}} E_{ii} = Z_i \\
M(M(E^{\alpha}_{ij})\overline{E}^{\alpha}_{ij})E_{mm} &= \frac{\delta_{i,m}}{c_m} E_{mm} \\
M^*(M(\overline{E}^{\alpha}_{ij})E^{\alpha}_{ij})E_{mm} &= \frac{\delta_{i,m}}{c_m} E_{mm}
\end{aligned}$$

It follows that for $i \leq j$

$$C^{\alpha,\alpha}_{ij,ij} = \frac{1}{c_m}(\delta_{j,m} - \delta_{i,m} + 2\delta_{i,m}) = \frac{1}{c_m}(\delta_{j,m} + \delta_{i,m})$$

and

$$f^{\alpha,\alpha}_{ij,ij} = (\overline{E}^{\alpha}_{ij} E^{\alpha}_{ij} - Z_i) f. \tag{2.20}$$

We sum the terms with indecies (m,j), $j \geq m$ and (i,m), $i \leq m$ separately. Note also that for each pair (i,j) there are d_{ij} possible values of α. Note also that $d_{ii} = 1$. We get

$$c_m HJK_m = \tilde{\Delta}_m - \frac{d_m + 2}{c_m}(Y_{mm} + \sqrt{-1}X_{mm}) - \sum_{i \leq m} \frac{d_{im}}{c_i}(Y_{ii} + \sqrt{-1}X_{ii})$$

where $d_m = \sum_{m<j} d_{mj}$ and

$$\tilde{\Delta}_m = \sum_{i \leq m, \alpha} \overline{E}^{\alpha}_{im} E^{\alpha}_{im} + \sum_{m \leq j, \alpha} \overline{E}^{\alpha}_{mj} E^{\alpha}_{mj}.$$

From Lemma (1.3), $HJK_m(\overline{f}) = \overline{HJK_m(f)}$. Thus, HJK_m is a real operator. Taking real parts proves the desired formula.□

Our proof of our main theorem will be an inductive argument based upon the fact that every bounded, homogeneous domain may be built up from a lower dimensional domain. To explain this, we introduce two subalgebras of $\mathcal{S}$. We define,

$$\mathcal{S}_{1*} = \sum_{1 \le m \le d} \mathcal{S}_{1m}$$

$$\mathcal{S}_{>1} = \sum_{2 \le i \le j \le d} \mathcal{S}_{ij}$$

Clearly, $\mathcal{S}_{1*}$ is a Lie ideal in $\mathcal{S}$ and $\mathcal{S}_{>1}$ is a complimentary Lie subalgebra. We define subspaces of $\mathbb{R}^n$ by

$$\mathcal{M}_{1*} = \kappa(\mathcal{S}_{1*}) \text{ and } \mathcal{M}_{>1} = \kappa(\mathcal{S}_{>1}).$$

Then, $\mathcal{M}_{1*}$ is $\mathcal{S}$ invariant under ρ. We identify $\mathcal{M}_{>1}$ with the quotient $\mathbb{R}^n/\mathcal{M}_{1*}$. The image $\mathcal{V}_{>1}$ in $\mathcal{M}_{>1}$ of the cone $\mathcal{V}$ is a cone which is homogeneous under $S/S_{1*} = S_{>1}$. (See [V].) It follows that

$$G_{>1} = M_{>1} S_{>1} \subset G$$

acts simply transitively on the tube domain over $\mathcal{V}_{>1}$. We use the functional $\beta_{>1} = \beta | \mathcal{G}_{>1}$ to define the Riemannian structure on $G_{>1}$. Let $HJK_{>1}$ be the corresponding Hua system for $G_{>1}$.

We shall identify $G_{>1}$ with the quotient G/G_{1*} where

$$G_{1*} = M_{1*} S_{1*}.$$

Note that G_{1*} is normal in G. This identification allows us to consider functions on $G_{>1}$ as functions on G which are constant on cosets of G_{1*}. Under these identifications, the strongly diagonal Hua operators on G reduce to those on $G_{>1}$ in the sense of the lemma below. This lemma is a direct consequence of Theorem (2.18).

(2.21) Lemma. *Let $HJK_{>1}$ be the Hua system for $G_{>1}$ under the Riemannian structure defined above. Then, for all $f \in C^\infty(G)$ which are constant on G_{1*} cosets,*

$$(HJK_{>1})_m f = HJK_{m+1}(f)$$

for all $r \ge m \ge 2$. $\square$

Section 3. *Hua Operators on Type II Domains*

Let $\mathcal{D}$ be a Siegel domain of type II as described in the introduction (nontrivial $\mathcal{Z}$). The group G generated by the actions (0.1), (0.2), (0.3) may be algebraically described as follows. Let $\phi = \Im K$. We let $\mathcal{M} = \mathcal{Z} \times \mathbb{R}^n$ with the Lie structure

$$[(z_1, t_1), (z_2, t_2)] = (0, 4\phi(z_1, z_2)).$$

The corresponding group is $\mathcal{M}$ with the product

$$(z_1, t_1) \cdot (z_2, t_2) = (z_1 + z_2, t_1 + t_2 + 2\phi(z_1, z_2)).$$

However, following our convention of denoting Lie groups by upper case Roman letters, we shall denote this space by M when it is considered as a group. This matches with the notation for tube domains, which correspond to the case $\mathcal{Z} = 0$. From now on M will be understood in this larger sense.

Let $G = M \times_s S$ where $s(z,t)s^{-1} = (\sigma(s)z, \rho(s)t)$. Then, G is a completely solvable group which acts simply transitively on $\mathcal{D}$. The corresponding identification of G with $\mathcal{D}$ is defined by

$$((z,t), s) \to (z, t + i\rho(s)c + iK(z,z)). \tag{3.1}$$

We shall let $T = \mathbb{R}^n \times_s S \subset G$. Note that T is the group of the Type I domain $\mathcal{E} = \mathbb{R}^n + i\mathcal{V}$. The Lie algebra of T will be denoted by $\mathcal{T}$.

We identify the tangent space of $\mathcal{D}$ at ic with $\mathcal{G}$. Let $J : \mathcal{G} \to \mathcal{G}$ define the complex structure. From formula (3.1), it is easily seen that $J : \mathcal{T} \to \mathcal{T}$ and on this set acts as described below Lemma (2.1). It also follows from formula (3.1) that on $\mathcal{Z}$, J is just multiplication by i. Next, we assume that the Riemannian structure may be defined by a formula such as formula (2.3) above where $\beta \in \mathcal{G}^*$. Notice that then $\beta|\mathcal{T}$ defines a Riemannian structure for $\mathcal{T}$.

As before, we shall also let σ denote the representation of $\mathcal{S}$ in $\mathcal{Z}$ obtained by differentiating σ. Since (by assumption) σ is algebraic, we know that $\sigma(\mathcal{A})$ is diagonalizable over $\mathbb{R}$. Thus, we may decompose $\mathcal{Z}$ into a direct sum of root spaces for $\mathcal{A}$ under σ. Let $\{\tau_1, \tau_2, \ldots, \tau_k\}$ be the set of root functionals in $\mathcal{A}^*$. The following is well known. We include the proof for sake of completeness.

(3.2) Lemma. *Let λ_i be as above Proposition (2.14) Then*

$$\{\tau_1, \tau_2, \ldots, \tau_k\} \subset \{\lambda_1/2, \lambda_2/2, \ldots, \lambda_r/2\}.$$

Proof Let $Z \in \mathcal{Z}$ be a root vector for $\mathcal{A}$ under σ corresponding to the root functional $\tau \in \mathcal{A}^*$. Then, $U = K(Z,Z)$ is a non-zero root vector for ρ corresponding to 2τ. It follows from Proposition (2.14) that $\tau = (\lambda_i + \lambda_j)/4$ for some choice of $i \le j$. We need to show that necessarily, $i = j$. Suppose that $i < j$. Let $X = \kappa^{-1}(U) \in \mathcal{S}_{ij}$. Then for all $A \in \mathcal{A}$,

$$\sigma(A)\sigma(X)Z = \sigma([A,X])Z + \sigma(X)\sigma(A)Z = \gamma(A)\sigma(X)Z$$

where

$$\gamma = (\lambda_i - \lambda_j)/2 + \tau = \frac{3}{4}\lambda_i - \frac{1}{4}\lambda_j.$$

We know from the previous paragraph that such a functional cannot be a root for σ. Hence $\sigma(X)Z = 0$. But then $\rho(X)U = 0$. We obtain our contradiction by noting that then $X\Delta X = \kappa^{-1}(\rho(X)\rho(X)c) = \kappa^{-1}(\rho(X)U) = 0$, proving that $X = U = 0$ and hence that $Z = 0$. $\square$

From now on R, $\mathcal{M}_\lambda$, $\mathcal{N}_\lambda$ defined before Proposition (2.14) will be understood in this more general situation i.e. when $\mathcal{M} = \mathcal{Z} \oplus \mathbb{R}^n$. We assume that our Riemannian structure is defined via a functional $\beta \in \mathcal{G}^*$ as in formula (2.3) above.

(3.3) Corollary. *The functional β is zero on $\mathcal{Z}$.*

Proof Let $A \in \mathcal{A}$ and $Z \in \mathcal{Z}$. Then $JA \in \sigma(\mathcal{S})$. Hence

$$\beta([A, Z]) = \beta([JA, JZ]) = 0.$$

Our corollary follows since, from Lemma (3.2), Ad $\mathcal{A}$ maps $\mathcal{Z}$ onto $\mathcal{Z}$.□

We let $\mathcal{Z}_i$ denote the root space corresponding to $\lambda_i/2$ in $\mathcal{Z}$. Then

$$\mathcal{Z} = \sum \mathcal{Z}_i.$$

Furthermore, in $\mathcal{L}$,

$$[\mathcal{Z}_i, \mathcal{Z}_j] \subset \mathcal{M}_{ij}. \tag{3.4}$$

(Recall that $\mathcal{M}_{ij} = \kappa(\mathcal{S}_{ij})$).

The subalgebra $\mathcal{Q}$ is the set of all elements

$$X - \sqrt{-1}JX$$

where $X \in \mathcal{G}_c$. For each i, we define $\mathcal{Q}_i$ to be the set of all elements $X - \sqrt{-1}JX$ as above where $X \in (\mathcal{Z}_i)_c$. For any pair of indecies (i, j), we define $\mathcal{Q}_{ij}$ to be the set of such elements where $X \in (\mathcal{M}_{ij})_c$. Clearly, the spaces $\mathcal{Q}_{ij}$ and $\mathcal{Q}_i$ together span $\mathcal{Q}$. Furthermore, the space

$$\mathcal{Q}_T = \sum \mathcal{Q}_{ij}$$

is the algebra which we called $\mathcal{Q}$ in the last section relative to the domain $\mathcal{E}$. Let f_i be the complex dimension of $\mathcal{Q}_i$.

For any subscript α, we define $\mathcal{P}_\alpha = \overline{\mathcal{Q}}_\alpha$. Then, from formula (3.4),

$$[\mathcal{Q}_i, \mathcal{P}_j] \subset (\mathcal{M}_{ij})_c = \mathcal{P}_{ij} + \mathcal{Q}_{ij}. \tag{3.5}$$

Indeed, let $X \in \mathcal{Q}_i$, $Y \in \mathcal{Q}_j$. Then

$$H(X, Y) = \frac{1}{2}g(X, \overline{Y}) = \frac{1}{2}\beta([JX, \overline{Y}]) = 0.$$

One immediate conclusion is that the spaces $\mathcal{Q}_i$ are mutually H-orthogonal, because β is zero on $\mathcal{M}_{ij}$ for $i < j$.

We choose a basis for $\mathcal{Q}$ consisting of

(a) The basis E_{ij}^α for $\mathcal{Q}_T$ defined in formula (2.16).

(b) An H-orthonormal basis $Z_j^\alpha = X_j^\alpha - \sqrt{-1}Y_j^\alpha$ for each $\mathcal{Z}_i$ where $1 \le \alpha \le f_i$ and X_j^α and Y_j^α are real.

It is clear that this defines an orthonormal basis for $\mathcal{Q}$. We shall use this basis to compute the Hua operators. Again, though, we are only interested in the strongly diagonal Hua operators. These are still defined by formula (2.17). The analogue of Theorem (2.18) for a Siegel II domain is the following

(3.6) Theorem. *Let HJK_m^T be the operator defined as HJK_m in Theorem (2.18). Then, for the case at hand,*

$$HJK_m = HJK_m^T + c_m^{-1}(\sum_\alpha (X_m^\alpha)^2 + (Y_m^\alpha)^2 - \frac{f_m}{c_m} Y_{mm}).$$

Proof It is clear that from formula (2.19),

$$HJK_m = HJK_m^T + \sum f_{i,j}^{\alpha,\gamma} C_{i,j}^{\alpha,\gamma}$$

where

$$\begin{aligned} C_{i,j}^{\alpha,\gamma} =& H(M(\overline{Z}_j^\gamma)E_{mm}, M(\overline{Z}_i^\alpha)E_{mm}) - H(M^*(Z_i^\alpha)E_{mm}, M^*(Z_j^\gamma)E_{mm}) \\ &+ H(M(M(Z_i^\alpha)\overline{Z}_j^\gamma)E_{mm}, E_{mm}) + H(M^*(M(\overline{Z}_j^\gamma)Z_i^\alpha)E_{mm}, E_{mm}). \end{aligned} \tag{3.7}$$

and

$$f_{i,j}^{\alpha,\gamma} = [\overline{Z}_i^\alpha Z_j^\gamma - M(\overline{Z}_i^\alpha)Z_j^\gamma]f$$

To get (3.7) one has to prove that $H(R(E_{ij}^\alpha, \overline{Z}_l^\beta)E_{mm}, E_{mm}) = 0$, which is an easy but a tedious calculation based on two facts:

$$M(\overline{Z}_l^\beta)E_{ij}^\alpha = 0 \tag{3.8}$$

and

$$M(\overline{E}_{ij}^\alpha)Z_k^\beta \subset (\mathcal{Z}_k)_c. \tag{3.9}$$

Indeed, since $\mathcal{P}$ is a subalgebra, we have

$$M(\overline{Z}_l^\beta)E_{ij}^\alpha = \pi_{\mathcal{Q}}[\overline{Z}_l^\beta, Y_{ij}^\alpha + iX_{ij}^\alpha] = \pi_{\mathcal{Q}}[\overline{Z}_l^\beta, Y_{ij}^\alpha - iX_{ij}^\alpha] = 0.$$

Analogously

$$\pi_{\mathcal{Q}}[\overline{E}_{ij}^\alpha, Z_k^\beta] = \pi_{\mathcal{Q}}[E_{ij}^\alpha, Z_k^\beta] = [E_{ij}^\alpha, Z_k^\beta],$$

which for $i = j$ is included in $(\mathcal{Z}_k)_c$ and for $i < j$ belongs to the root space $\frac{\lambda_i - \lambda_j}{2} + \frac{\lambda_k}{2}$, which is zero.

We claim that $C_{ij}^{\alpha,\gamma} = 0$ unless $i = j = m$ and $\alpha = \gamma$. Furthermore, in this case we get c_m^{-1}.

To prove this, let $X = A - \sqrt{-1}JA$ be an element of $\mathcal{Q}_T$ where $A \in \mathcal{A}$. Then, $JA \in \mathcal{M}$ and hence

$$[X, \overline{Z}_i^\alpha] = [A - iJA, \overline{Z}_i^\alpha] = [A + iJA, \overline{Z}_i^\alpha] \in \mathcal{P}.$$

It follows that

$$M(\overline{X})Z_i^\alpha \text{ and } M(\overline{Z}_i^\alpha)X = 0,$$

In particular, the first term to the right of the equality in formula (3.7) is zero.

Next, from formula (3.5), we note that $M(\overline{Z}_i^\alpha)Z_j^\alpha$ belongs to $\mathcal{Q}_{ij}$. Hence, the third term on the right in formula (3.7) will be zero unless $i = j = m$. The same is true for the fourth term since this term is just the conjugate of the third. The following lemma clearly finishes the proof of our claim. In fact, this will also finish the proof of Theorem (3.6).□

(3.10) Lemma.

$$M(\overline{Z}_m^\alpha)Z_m^\gamma = \frac{\delta_{\alpha,\gamma}}{\sqrt{c_m}}E_{mm}$$

$$M^*(Z_i^\alpha)E_{mm} = \frac{\delta_{im}}{\sqrt{c_m}}Z_m^\alpha.$$

Proof For the first equality, we note that $\kappa(\mathcal{S}_{mm})$ is one dimensional. Hence, from formula (3.4) there is a complex constant $C^{\alpha,\gamma}$ such that

(3.11) $$[\overline{Z}_m^\alpha, Z_m^\gamma] = C^{\alpha,\gamma}X_{mm}.$$

Computing this constant is simple. If we apply β to both sides of the above, we find that

$$2H(Z_m^\gamma, Z_m^\alpha) = \beta([JZ_m^\gamma, \overline{Z}_m^\alpha]) = -iC^{\alpha,\gamma}c_m$$

Thus,

$$C^{\alpha,\gamma} = \frac{2i}{c_m}\delta_{\alpha,\gamma}.$$

On the other hand, from formula (2.16),

$$X_{mm} = \frac{-i\sqrt{c_m}}{2}(E_{mm} - \overline{E}_{mm}).$$

The first equality follows by applying π_Q to formula (3.11).

For the second equality, recall that, by (3.8), $M(\overline{Z}_i^\alpha)$ is zero on $\mathcal{Q}_T$. It follows that $M^*(Z_i^\alpha)E_{mm}$ is H-orthogonal to $\mathcal{Q}_T$ and hence belongs to $\mathcal{Z}_c$. Also

$$H(M^*(Z_i^\alpha)E_{mm}, Z_j^\gamma) = H(E_{mm}, M(\overline{Z}_i^\alpha)Z_j^\gamma).$$

This is zero unless $i = j = m$, in which case, the first part of the lemma proves our result.□

Section 4. *Alternative reproducing kernels for holomorphic functions*

In this section we consider G-invariant real second order elliptic degenerate operators L on $\mathcal{D}$, which annihilate holomorphic functions. We are going to apply the boundary theory of [DH] in order to show that there are many real kernels on M, which reproduce holomorphic functions.

Let L be a real (i.e. $L\overline{f} = \overline{Lf}$) second order operator which annihilates holomorphic functions and $x_0 \in \mathcal{D}$ a fixed base point. In local coordinates around x_0 we have

$$L = \sum c_{kj} \frac{\partial}{\partial \overline{z}_k} \frac{\partial}{\partial z_j}, \tag{4.1}$$

with $c_{jk} = \overline{c}_{kj}$. Therefore writing L in terms of partial derivatives $\dfrac{\partial}{\partial x_k}, \dfrac{\partial}{\partial y_k}$ we obtain an operator with real coefficients, and being elliptic degenerate means that the second order symbol of L is positive semi-definite.

Since, additionally, L is G-invariant, we may write it in terms of left-invariant vector fields on G. The identification (3.1) of G with $\mathcal{D}$ defines global coordinates for G. Let $\mathcal{X}_k$ be the left-invariant vector field on G which equals $\dfrac{\partial}{\partial z_k}$ at e in these coordinates. In view of Lemma (1.1) we have

$$L = \sum c_{kj}(\overline{\mathcal{X}}_k \mathcal{X}_j - \nabla_{\overline{\mathcal{X}}_k} \mathcal{X}_j) = \sum c_{kj}(\overline{\mathcal{X}}_k \mathcal{X}_j - M(\overline{\mathcal{X}}_k)\mathcal{X}_j),$$

where We choose a basis of $\mathcal{Q}$ as in the previous section. Let $\{E_{ij}^{\alpha}\}$, $1 \le i \le j \le r$, $1 \le \alpha \le d_{ij}$ be the basis for $\mathcal{Q}_T$ and $\{Z_j^{\alpha}\}$, $1 \le j \le k$, $1 \le \alpha \le f_j$, the basis for $\mathcal{Z}$. Therefore,

$$\begin{aligned} L = &\sum C_{ij,kl}^{\alpha,\beta}(\overline{E}_{ij}^{\alpha} E_{kl}^{\beta} - M(\overline{E}_{ij}^{\alpha})E_{kl}^{\beta}) + \sum C_{i,j}^{\alpha,\beta}(\overline{Z}_i^{\alpha} Z_j^{\beta} - M(\overline{Z}_i^{\alpha})Z_j^{\beta}) \\ &+ \sum C_{ij,k}^{\alpha,\beta}(\overline{E}_{ij}^{\alpha} Z_k^{\beta} - M(\overline{E}_{ij}^{\alpha})Z_k^{\beta}) + \sum C_{i,kl}^{\alpha,\beta}(\overline{Z}_i^{\alpha} E_{kl}^{\beta} - M(\overline{Z}_i^{\alpha})E_{kl}^{\beta}). \end{aligned} \tag{4.2}$$

The condition $L\overline{f} = \overline{Lf}$ implies that L belongs to the enveloping algebra of $\mathcal{G}$ i.e can be written as

$$L = \mathcal{Y}_1^2 + ... + \mathcal{Y}_m^2 + \mathcal{Y}_0$$

for some $\mathcal{Y}_0, ...\mathcal{Y}_m \in \mathcal{G}$.

For the rest of the paper we assume that L satisfies the Hörmander condition i.e

$$\mathcal{Y}_1, ..., \mathcal{Y}_m \text{ generate } \mathcal{G} \text{ as a Lie algebra} \tag{4.3}$$

The same condition is satisfied by $\pi_A(L)$, where $\pi_A(L)$ is the image of L under the canonical homomorphism $\pi : S \to A = G/MN$, i.e $\pi_A(L)$ is elliptic. Since the second order part of $\pi_A(L)$ is equal to

$$\sum_{i,j=1}^{r} C_{ii,jj}^{\alpha,\alpha} Y_{ii}^{\alpha} Y_{jj}^{\alpha} = \sum_{i,j=1}^{r} C_{ii,jj} Y_{ii} Y_{jj}$$

we have

$$C^{\alpha,\alpha}_{ii,ii} = C_{ii,ii} > 0. \tag{4.4}$$

In order to determine the Poisson boundary for L we need to know the $\mathcal{A}$-component Y_0 of the first order part of L. For this, we compute the contribution of each term in (4.2). In view of (3.8) and (3.9), $M(\overline{Z}^{\alpha}_{i})E_{kl} = 0$ and $M(\overline{E}^{\alpha}_{ij})Z^{\beta}_{k} \in (\mathcal{Z}_k)_c$. Hence

$$H(M(\overline{E}^{\alpha}_{ij})Z^{\beta}_{k}, E_{mm}) = 0$$

and

$$H(M(\overline{Z}^{\alpha}_{i})Z^{\beta}_{j}, E_{mm}) = 0.$$

Moreover $M(\overline{Z}^{\alpha}_{i})Z^{\beta}_{j} \in \mathcal{Q}_{ij}$ and by Lemma (3.10)

$$M(\overline{Z}^{\alpha}_{i})Z^{\beta}_{i} = \frac{\delta_{\alpha,\beta}}{\sqrt{c_i}}E_{ii}.$$

By orthogonality of $\mathcal{P}$ and $\mathcal{Q}$, Lemma (3.10) and (2.2)

$$H(M(\overline{E}^{\alpha}_{ij})E^{\beta}_{kl}, E_{mm}) = \xi((Y^{\alpha}_{ij}\Delta Y^{\beta}_{kl})\Delta Y_{mm}),$$

which, in view of the properties of Δ, is nonzero if and only if $\alpha = \beta$, $i = k$, $j = l$ and $m = i$. Then, we have,

$$M(\overline{E}^{\alpha}_{ij})E^{\alpha}_{ij} = \frac{1}{\sqrt{c_i}}E_{ii}.$$

Therefore

$$L = L_0 - Y_0, \tag{4.5}$$

where

$$Y_0 = \sum_{i=1}^{r} \frac{1}{c_i}\Big(\sum_{j\geq i,\alpha} C^{\alpha,\alpha}_{ij,ij} + \sum_{\alpha} C^{\alpha,\alpha}_{i,i}\Big)Y_{ii}.$$

We claim that for every $1 \leq i \leq r$

$$\sum_{j\geq i,\alpha} C^{\alpha,\alpha}_{ij,ij} + \sum_{\alpha} C^{\alpha,\alpha}_{i,i} > 0. \tag{4.6}$$

Since L has a nonnegative second order symbol, $Lf(x_0) \geq 0$ for f having minimum at x_0. Let, in local coordinates around x_0, $f = \mid z_j \mid^2$. Then in the notation of (4.1)

$$Lf(x_0) = c_{jj}\frac{\partial}{\partial \overline{z}_j}\frac{\partial}{\partial z_j} \mid z_j \mid^2 = c_{jj}.$$

Hence $C^{\alpha,\alpha}_{ij,ij}, C^{\alpha,\alpha}_{i,i} \geq 0$ and so, (4.4) implies (4.6).

The following proposition sums up our considerations

(4.7) Proposition. *A G-invariant, real, second order operator L on $\mathcal{D}$, which annihilates holomorphic functions and satisfies the Hörmander condition, can be written in the form*

$$L = L_0 - \sum_{m=1}^{r} b_m Y_{mm}, \tag{4.8}$$

where $b_m > 0$ and L_0 is a left-invariant second order operator with the first order part contained in $\mathcal{M} \oplus \mathcal{N}$.$\square$

Notice that both the Laplace-Beltrami operator Δ and the 'diagonal Laplacian'

$$\Delta_{\text{diag}} = \sum_m HJK_m$$

belong to the class of the operators described in Proposition (4.7). The first statement follows from formula (1.5) and for the second from (2.17) and Lemma (1.3).

The vector Y_0 will play a special role in our discussion. We wish to apply the boundary theory of [DH] to operators described in Proposition (4.7). We let R^+ denote the set of roots λ such that

$$\lambda(Y_o) > 0$$

and $R^- = R \setminus R^+$. (This set corresponds to $\Delta_1(L)$ on p.8 of [DH]). Note that our Y_o is $-Z_o$ in the notation of [DH].)

It is important to notice that R^+ is non-empty. In fact, it is clear from Proposition (2.14) that for all $i \le j$ such that $\mathcal{S}_{ij} \neq 0$,

$$(\lambda_i + \lambda_j)/2 \in R^+.$$

Also

$$\lambda_i/2 \in R^+.$$

The root spaces corresponding to these functionals span $\mathcal{M}$. We let

$$\mathcal{N}^+ = \sum_{\lambda \in R^+} \mathcal{N}_\lambda$$

and

$$\mathcal{N}^- = \sum_{\lambda \in R^-} \mathcal{N}_\lambda.$$

In the notation of [DH], *loc. cit.*, $\mathcal{N}_0(L) = \mathcal{N}^-$ and $\mathcal{N}_1(L) = \mathcal{M} \oplus \mathcal{N}^+$. Note that both $\mathcal{N}^\pm$ are subalgebras of $\mathcal{S}$.

Since

$$S = N^+ N^- A$$

the homogeneous space $B = G/N^-A$ is identifiable with the nilpotent Lie group MN^+. We refer to B as the maximal boundary for L. With this identification M is contained in it. Let dx be Haar measure on MN^+. According to the main result of [DH], there exists a bounded, positive, C^∞ function P on B such that

(4.9)
$$\int P(x)dx = 1.$$

(4.10) For all L^∞ functions f on B, the function

$$F(g) = \int_B f(gx)P(x)\,dx$$

satisfies $LF = 0$. (Here, $x \to gx$ denotes the action of G on the coset space G/N^-A.)

(4.11) If F is a bounded solution to $LF = 0$, then there is a unique L^∞ function f on B which expresses F as above. This function is called the boundary value of F.

Finally we have

(4.12) Proposition. *There is a choice of constants in (4.2) such that the maximal boundary for L is M. In particular, the Poisson kernel for such L reproduces holomorphic functions in the sense of (4.9)-(4.11).*

Proof We consider L of the form

$$L = \sum C^{\alpha,\alpha}_{ij,ij}(\overline{E}^\alpha_{ij}E^\alpha_{ij} - M(\overline{E}^\alpha_{ij})E^\alpha_{ij}) + \sum C^{\alpha,\alpha}_{i,i}(\overline{Z}^\alpha_i Z^\alpha_i - M(\overline{Z}^\alpha_i)Z^\alpha_i),$$

which is clearly elliptic if all $C^{\alpha,\alpha}_{ij,ij}$ and $C^{\alpha,\alpha}_{i,i}$ are greater then 0. Hence we have to find positive $C^{\alpha,\alpha}_{ij,ij}$ and $C^{\alpha,\alpha}_{i,i}$ such that

(4.13)
$$(\lambda_i - \lambda_j)(Y_0) =$$
$$-\frac{1}{c_i}(\sum_{k\geq i,\alpha} C^{\alpha,\alpha}_{ik,ik} + \sum_\alpha C^{\alpha,\alpha}_{i,i}) + \frac{1}{c_j}(\sum_{k\geq j,\alpha} C^{\alpha,\alpha}_{jk,jk} + \sum_\alpha C^{\alpha,\alpha}_{j,j}) \geq 0,$$

for $i < j$, which is very easy. Assume we can satisfy (4.13) for $i < i_0$ and all $j > i$. To get (4.13) for $i = i_0$ and $j > i_0$ we increase $C^{\alpha,\alpha}_{jk,jk}$ and $C^{\alpha,\alpha}_{j,j}$ sufficiently, which does not change positivity of $(\lambda_i - \lambda_j)(Y_0)$ for $i < i_0$. $\square$

It turns out that all L from Proposition (4.7) give rise to reproducing kernels on M, although their maximal boundaries may be larger. For that we have to explain the idea of a boundary for L and consider not only the maximal boundary but also the smaller ones. We shall say that a subalgebra $\mathcal{N}^0$ of $\mathcal{N}$ is homogeneous if it is normalized by $\mathcal{A}$. For any such algebra, there is an $\mathcal{A}$-invariant subspace $\mathcal{N}^1$ of $\mathcal{N}$ such that

$$\mathcal{N} = \mathcal{N}^1 \oplus \mathcal{N}^0.$$

Let $\mathcal{N}^0 \supset \mathcal{N}^-$ be some homogeneous subalgebra of $\mathcal{N}$. Then, according to [DH], Theorem (4.7), the homogeneous space $\tilde{B} = G/N^0A$ is a boundary for L. This means that there is a probability measure $\tilde{P}$ on G/N^0A such that the functions

$$F(g) = \int_{G/N^0A} f(gx)\tilde{P}(x)dx, \;\; f \in L^\infty(G/N^0A) \tag{4.14}$$

are L-harmonic. $\tilde{P}$ is closely related to P.

Clearly $M = G/S$ is a boundary for L. Let P_M be the corresponding Poisson kernel. We are going to prove that P_M reproduces bounded holomorphic functions. This is not totally obvious unless $B = G/S$ and $P_M = P$.

The boundary $\tilde{B}$ can be realised as MN^1, $N^1 = \exp \mathcal{N}^1$, with an appropriate action of G. Indeed, the mapping

$$MN^1 \times N^0 \ni (x_1, x_0) \to x_1x_0 \in MN \tag{4.15}$$

is a diffeomorphism between $MN^1 \times N^0$ and MN and so, every $x \in MN$ can be written in a unique way as

$$x = x_1x_0, \quad x_1 \in MN^1, x_0 \in N^0.$$

We have a well defined projection $\pi_{N_1} : MN \to MN^1$, given by

$$\pi_{MN^1}(x) = x_1.$$

In this terms (4.14) becomes

$$F(g) = \int_{MN^1} f(\pi_{MN^1}(gx))\tilde{P}(x)dx, \;\; f \in L^\infty(MN^1) \tag{4.16}$$

and $x \to \pi_{MN^1}(gx)$ is the action of G on MN^1 corresponding to the action $x \to gx$ in $\tilde{B} = G/N^0A$ realization.

(4.15) follows from a more general fact. One can obtain this kind of decomposition of a connected and simply connected nilpotent Lie group as far as the assumptions of the following lemma are satisfied.

(4.17) Lemma. *Let a nilpotent Lie algebra $\mathcal{N} = \mathcal{N}_1 \oplus \mathcal{N}_0$ be a sum of two linear subspaces $\mathcal{N}_1$ and $\mathcal{N}_0$. Assume that we can find a basis $E_1, ..., E_n$ of $\mathcal{N}$ such that every E_j belongs either to $\mathcal{N}_1$ or to $\mathcal{N}_0$ and, in coordinates $x = \exp(x_1E_1 + ... + x_nE_n)$, the multiplication in $N = \exp \mathcal{N}$ is given by*

$$(xy)_i = x_i + y_i + T_i(x_1, ..., x_{i-1}, y_1, ..., y_{i-1})$$

with $T_i \in C^\infty(N)$ independent of $x_i, ..., x_n, y_i, ..., y_n$. Then

$$\exp \mathcal{N}_1 \times \exp \mathcal{N}_0 \ni (x_1, x_0) \to x_1x_0 \in N$$

is a diffeomorphism. $\square$

For the proof of Lemma (4.17), which by all means is standard, see e.g. the preliminaries of [DH]. Although Lemma (4.17) is not formulated there in the above form, the proof is essentially the same as the proof of Lemmas (1.21), (1.22), (1.25) there. In every such situation we are going to consider the corresponding projections π_N^1 and π_N^0.

In view of Lemma (4.17) we can decompose N^+ as

$$N^+ = N^1 N^2,$$

where $N^2 = \exp \mathcal{N}_2$ and $\mathcal{N}_2 = \mathcal{N}^+ \cap \mathcal{N}^0$. Notice that N^2 is a subgroup. Moreover, in view of (4.16) we have

$$\tilde{P}(x) = \int_{N^2} P(xy)dy, \quad x \in MN^1. \tag{4.18}$$

Indeed, identifying $f \in C_b(MN^1)$ with a continuous bounded function on MN constant on the right cosets of N^0, we have

$$F(g) = \int_{MN^1 \times N^2} f(\pi_{MN^1N^2}(gxy))P(xy)dxdy,$$

(4.19)

$$= \int_{MN^1 \times N^2} f(\pi_{MN^1}(gxy))P(xy)dxdy = \int_{MN^1} f(\pi_{MN^1}(gx))\Big(\int_{N^2} P(xy)dy\Big)dx.$$

On the other hand,

$$F(g) = \int_{MN^1} f(\pi_{MN^1}(gx))\tilde{P}(x)dx,$$

which proves (4.18). In particular,

$$P_M(x) = \int_{N^+} P(xy)dy, \quad x \in M. \tag{4.20}$$

Now we are ready to formulate the main result of this section

(4.21) Theorem. *Let L, L_1 be G-invariant (not necessarily distinct), real, second order operators on $\mathcal{D}$, which annihilate holomorphic functions and satisfy the Hörmander condition. Assume that the maximal boundary for L_1 is M. Let P_M be the L-Poisson kernel on M, and F a bounded function, which is at the same time L and L_1 harmonic. Then there is $f \in L^\infty(M)$ such that*

$$F(g) = \int_M f(\pi_M(gx))P_M(x)dx. \tag{4.22}$$

In particular, bounded holomorphic functions are reproducible from their boundary values on M via the kernel P_M.

Remark. The last chapter of the paper will be devoted to the proof of an analogous theorem with the Hua diagonal operators playing the role of L_1. This involves somewhat more work because, except for the case of the tube over the cone of symmetric real $r \times r$ matrices (see the next section), it is not known whether or not there is a linear combination of Hua diagonal operators which satisfies the hypotheses of L_1.

For the proof of both theorems we need a technical lemma, which will be formulated and proved below. Before that we must introduce some notation. Elements of $C_b(B)$ are defined by right N^--invariant continuous functions on MN, while elements of $C_b(\tilde{B})$ are defined by right N^0-invariant functions. We say that an element of $C_b(B)$ 'reduces' to $\tilde{B}$ if it is defined by a right N^0-invariant function on MN. In this case, as in (4.19), we may write

$$\begin{aligned} F(g) &= \int_{MN^+} f(\pi_{MN^+}(gx))P(x)\,dx \\ &= \int_{MN^1 \times N^2} f(\pi_{MN^+}(gx_1x_2))P(x_1x_2)dx_1\,dx_2 \\ &= \int_{MN^1} f(\pi_{MN^1}(gx_1))\tilde{P}(x_1)dx_1. \end{aligned} \tag{4.23}$$

Let $Y \in \mathcal{A}$. We say that Y is contractive on $\mathcal{G}$ if $\operatorname{ad} Y$ has only non-negative eigenvalues. In this case, we let $\mathcal{N}_Y^0$ be the span of the positive eigenspaces in $\mathcal{M} + \mathcal{N}$ and $\mathcal{N}_Y^1$ be the centralizer of Y in $\mathcal{M} + \mathcal{N}$. Note that $\mathcal{N}_Y^0$ is an ideal in $\mathcal{M} + \mathcal{N}$.

(4.24) Lemma. *Let F be a bounded, L-harmonic function. Assume that the L-boundary value f is continuous on the maximal boundary $B = MN^+$. Let $Y \in \mathcal{A}$ be contractive. Then*

$$\lim_{t \to -\infty} F((\exp tY)g) = F_Y(g)$$

converges uniformly on compact sets in G and defines an L-harmonic function with continuous boundary function f_Y. Both F_Y and f_Y are constant on right cosets of N_Y^0 in G and in MN respectively. Additionally, f_Y and f agree on $N_Y^1 \cap MN^+$. If f reduces to G/AN^0, then f_Y will reduce to G/AN', where N' is the subgroup generated by N_Y^0 and N^0.

Proof Given $g \in G$, we write

$$g = an_Y^1 n_Y^0$$

relative to the decomposition $G = AN_Y^1 N_Y^0$. (The assumptions of Lemma (4.17) are clearly satisfied because $\mathcal{N}_Y^1$, $\mathcal{N}_Y^0$ together contain all of the eigenspaces of $\operatorname{ad} Y$.) We define

$$g(t) = (\exp tY)g(\exp -tY).$$

Then

$$g(t) = an^1_Y n^0_Y(t)$$

Let

$$I^1 = N^1_Y \cap MN^+ \text{ and } I^0 = N^0_Y \cap MN^+.$$

Notice that also $\mathcal{I}^1$, $\mathcal{I}^0$ are composed of the whole eigenspaces and so

$$MN^+ = I^1 I^0.$$

Then

$$F(\exp tY) = \int_{I^1} \left(\int_{I^0} f(x^1 x^0(t)) P(x^1 x^0)\, dx^1 \right) dx^0. \tag{4.25}$$

When $t \to -\infty$ then $x^0(t) \to e$ and (4.25) converges to

$$\int_{I^1} f(x^1) \tilde{P}(x^1)\, dx^1, \tag{4.26}$$

where

$$\tilde{P}(x^1) = \int_{I^0} P(x^1 x^0)\, dx^0.$$

More generally, for $g \in G$,

$$F((\exp tY)g) = F(an^1_Y n^0_Y(t)(\exp tY))$$

$$= \int_{I^1} \left(\int_{I^0} f(\pi_{MN^+}(an^1_Y n^0_Y(t) x^1 x^0(t))) P(x^1 x^0)\, dx^1 \right) dx^0.$$

As $t \to -\infty$, $n^0_Y(t) \to e$, $x^0(t) \to e$ and we see that

$$F_Y(g) = \int_{I^1} f(\pi_{MN^+}(an^1_Y x^1)) \tilde{P}(x^1)\, dx^1 = \int_{I^1} f(\pi_{I^1}(an^1_Y x^1)) \tilde{P}(x^1)\, dx^1. \tag{4.27}$$

The convergence of the limit as well as the fact that F_Y is constant on right cosets of N^0_Y follows.

Each of the functions $g \to F((\exp tY)g)$ is L-harmonic since L is left invariant. Our limit will converge in the C^∞_c topology due to the hypoellipicity of L [B].

From formula (4.26) and formula (4.27), f_Y is the function on MN defined by

$$f_Y(x^1 x^0 n^-) = f(x^1)$$

for all $x^1 \in I^1$, $x^0 \in I^0$ and $n^- \in N^-$, so the agreement of f_Y and f is proved. Both functions will be considered as functions on MN constant on N^- right cosets. Writing $x \in MN$ as

$$x = x^1 x^0 n^- \quad x^1 \in I^1,\ x^0 \in I^0,\ n^- \in N^-,$$

for $y \in N^0_Y$ we have

$$f_Y(x^1x^0n^-y) = f_Y(x^1x^0n^-y(n^-)^{-1}n^-)$$

$$= f_Y(x^1x^0n^-y(n^-)^{-1}) = f(x^1) = f_Y(x^1x^0n^-),$$

which proves that f_Y is right N^0_Y invariant as a function on MN. Finally, suppose that f is constant on right cosets of N^0 where N^0 is a homogeneous subgroup of N. Since N^0 is homogeneous, we see that

$$N^0 = (N^0 \cap (MN^+))(N^0 \cap N^-) = (N^0 \cap I^1)(N^0 \cap I^0)(N^0 \cap N^-).$$

Now since N^0_Y is a normal subgroup of N and f_Y is right N^0_Y and N^- invariant, for $y \in N^0$, we have

$$f_Y(x^1_Yx^0_Yy) = f_Y(x^1_Yyy^{-1}x^0_Yy) = f_Y(x^1_Yy).$$

Now decomposing y as

$$y = y^1y^0y^- \quad y^1 \in I^1,\ y^0 \in I^0,\ y^- \in I^-$$

we obtain

$$f_Y(x^1_Yy) = f_Y(x^1_Yy^1) = f(x^1_Yy^1) = f_Y(x^1_Yx^0_Y).$$

We see that f_Y is constant on all N^0 right cosets. Since f_Y is also constant on N^0_Y cosets, we see that it is constant on N' cosets, proving the lemma.□

Proof of Theorem (4.21). Let

$$Y = \sum Y_{ii}.$$

Then $N^1_Y = S$ and $N^0_Y = M$. Let F be a bounded L and L_1 harmonic function with continuous L-boundary value f on MN^+ (MN^+ being the maximal boundary for L). Hence F_Y is an L_1-harmonic function which is constant on cosets of M in G. Therefore F_Y may be considered as a function on S harmonic with respect to $\pi_S(L_1)$, which by (4.9)-(4.11) and (4.13) has a trivial maximal boundary and so F_Y is constant. Lemma (4.24) then says that f_Y is constant and so f is constant on $N^1_Y = S$. Applying the same argument to left translates of F shows that f is constant on right cosets of S. Indeed, let $\tilde{F}(g) = F(g_1g)$. Then by (4.16) (with $N^1 = N^+$) the boundary value $\tilde{f}$ of $\tilde{F}$ satisfies

$$\tilde{f}(x) = f(\pi_{MN^+}(g_1x)), \quad x \in MN^+. \tag{4.28}$$

But when both f, $\tilde{f}$ are considered as N^- right invariant functions on N, (4.28) becomes

$$\tilde{f}(x) = f(g_1x), \quad x \in MN.$$

Therefore, by (4.23), f reduces to M. This proves Theorem (4.21) in the case where the boundary value is continuous.

Actually, the general case of Theorem (4.21) also follows. If F is an arbitrary L and L_1-harmonic function and $\phi \in C_c^\infty(MN^+)$ then the convolution $F_\phi(g) = \phi * F(g) = \int_{MN^+} \phi(x)F(x^{-1}g)\, dx$ is L, L_1-harmonic with continuous boundary value $\phi * f$. (See Lemma (4.9) of [DH].) Hence, $\phi * f$ treated as a function on MN is constant on S-cosets. Letting ϕ range over an approximate identity proves Theorem (4.21).□

Section 5. *Tube domain over the cone of symmetric positive definite $r \times r$ matrices.*

In this section let $\mathcal{D}$ be the tube domain over the cone of symmetric positive definite $r \times r$ matrices. By Δ we denote the Laplace-Beltrami operator on $\mathcal{D}$. For a sequence of strictly positive numbers $\mathbf{a} = (a_1, ..., a_m)$ let

$$L^{\mathbf{a}} = \sum_{m=1}^{r} a_m HJK_m$$

be a linear combination of strongly diagonal Hua operators. We are going to prove that the operators $L^{\mathbf{a}}$ having M as the maximal boundary play a special role on $\mathcal{D}$ –they characterize the classical Poisson-Szegö integrals from the Shilov boundary. This means that a bounded function is a Poisson-Szegö integral if and only if it is $L^{\mathbf{a}}$-harmonic. Unfortunately this nice characterization is not true for other symmetric tube domains because then there are no $L^{\mathbf{a}}$ having M as the maximal boundary.

We begin by proving the existence of $L^{\mathbf{a}}$ which have the M as their maximal boundary. In the case of the cone of symmetric, positive definite, $r \times r$ matrices, $c_j = \frac{1}{2}$, $d_{ij} = 1$ and $d_m = r - m$, and so the strongly diagonal Hua operators have the form

$$HJK_m = 4(\frac{1}{2}\Delta_m - (r + 2 - m)Y_{mm} - \sum_{i<m} Y_{ii}).$$

(5.1) Lemma. *There is a choice of $a_1, ..., a_r$ such that the maximal boundary for $L^{\mathbf{a}}$ is M.*

Proof Let $Y = -\sum_{m=1}^{r} a_m((r + 2 - m)Y_{mm} + \sum_{i<m} Y_{ii})$. We need $a_1, ..., a_r$ such that

$$(\lambda_i - \lambda_j)(Y) \geq 0 \quad \text{for } i < j. \tag{5.2}$$

We start with $\lambda_i - \lambda_r$, $i < r$. Then

$$(\lambda_i - \lambda_r)(Y) = -(\lambda_i - \lambda_r)(2a_r Y_{rr} + a_r Y_i + \sum_{i<m<r} a_m Y_i) = a_r - \sum_{i<m<r} a_m.$$

Whenever $a_m < \frac{a_r}{r}$, $m = 1, ..., r-1$, (5.2) is true. Assume we can satisfy (5.2) for $j > m$. We have

$$(\lambda_i - \lambda_m)(Y) = (-\sum_{j>m} a_j(Y_i + Y_m) - a_m((r+2-m)Y_m + Y_i) - \sum_{i<j<m} a_j Y_i - a_i(r+2-i)Y_i)$$

$$= a_m(r+2-m) - a_m - \sum_{i<j<m} a_j - a_i(r+2-i) = a_m(r+1-m) - \sum_{i<j<m} a_j - a_i(r+2-i).$$

Clearly making a_j, $i \leq j < m$ small enough we can satisfy (5.2) for $j = m$.$\Box$

Remark. The above lemma is not true for other symmetric tube domains i.e. when $d_{ij} = 2, 4, 8$.

Let P_M^{Δ} be the Δ-Poisson kernel on M while $P_M^{\mathbf{a}}$ be $L^{\mathbf{a}}$-Poisson kernel on M. The main theorem of this section is

(5.3) Theorem. *Let $L^{\mathbf{a}}$ be as above with the maximal boundary being M. For a bounded function F the following are equivalent*

$$\text{There is } f \in L^{\infty}(M) \text{ such that } F(g) = \int_M f(\pi_M(gx)) P_M^{\Delta}(x)\ dx \tag{5.4}$$

$$L^{\mathbf{a}} F = 0. \tag{5.5}$$

Moreover, for all such $L^{\mathbf{a}}$ the kernel $P_M^{\mathbf{a}}$ is equal to P_M^{Δ}.

Remark. The proof relays heavily on the Johnson-Korányi result [JK] saying that for a bounded F, (5.4) is equivalent to be Hua harmonic.

Proof Implication (5.4)$\rightarrow$(5.5) follows directly from the result of Johnson-Korányi mentioned above. For the converse we first prove that $P_M^{\mathbf{a}} = P_M^{\Delta}$. For a function $f \in C_c(M)$ let

$$P_M^{\Delta} f(g) = \int_M f(\pi_M(gx)) P_M^{\Delta}(x)\ dx. \tag{5.6}$$

Since $P_M^{\Delta} f$ is $L^{\mathbf{a}}$-harmonic, in view of Theorem (3.8) of [DH] there is $h \in L^{\infty}(M)$ such that

$$P_M^{\Delta} f(g) = P_M^{\mathbf{a}} h(g) = \int_M h(\pi_M(gx)) P_M^{\mathbf{a}}(x)\ dx. \tag{5.7}$$

Convolving (5.6) and (5.7) from the left by $\phi \in C_c(M)$ we have

$$P_M^{\Delta}(\phi * f)(g) = P_M^{\mathbf{a}}(\phi * h)(g).$$

Indeed,

$$P_M^{\Delta}(\phi * f)(g) = \int_M \int_M \phi(y^{-1}) f(y \pi_M(gx)) P_M^{\Delta}(x)\ dy dx$$

$$= \int_M \phi(y^{-1}) P_M^{\Delta} f(yg)\ dy = \int_M \phi(y^{-1}) P_M^{\mathbf{a}} h(yg)\ dy = P_M^{\mathbf{a}}(\phi * h)(g).$$

Let now $g(t) = \exp t \sum Y_{ii}$. Letting $t \to -\infty$ we see that

$$\phi * f(e) = \phi * h(e).$$

This proves $f = h$ and

$$P_M^{\Delta} = P_M^{\mathbf{a}}. \tag{5.8}$$

Now if $L^{\mathbf{a}} F = 0$ holds for a bounded function F then by Theorem (3.8) [DH]

$$F = P_M^{\mathbf{a}} f$$

for an $f \in L^\infty(M)$, which, in view of (5.8), implies (5.4).□

Section 6. *Hua harmonic functions*

The main result of this section is the following

(6.1) Theorem. *Let F be a bounded function on G annihilated by strongly diagonal Hua operators and harmonic with respect to an operator L satisfying the assumptions of Proposition (4.7). Then there is $f \in L^\infty(M)$ such that*

$$F(g) = \int_M f(\pi_M(gx)) P_M(x)\, dx, \tag{6.2}$$

where P_M is the L-Poisson kernel on M.

Remark. Clearly, we could use Δ_{diag} as L in (6.1). In this case, any F which is annihilated by the Hua system is automatically annihilated by L since Δ_{diag} is the sum of the strongly diagonal Hua operators. Thus, we produce a single Poisson kernel which is capable of representing Hua harmonic functions. We find the more general formulation of Proposition (6.3) remarkable, however, in that the maximal boundary of L would typically be considerably larger than M. The above proposition says that just being Hua harmonic forces the L-boundary function to reduce to a smaller boundary.

To prove Theorem (6.1) we need the following proposition:

(6.3) Proposition. *Suppose that a bounded function F is annihilated by the strongly diagonal Hua operators and, together with its L-boundary function f, is constant on cosets of M in G. Then F is a constant function.*

Once Proposition (6.3) has been proved, then Theorem (6.1) will follow as in the proof of Theorem (4.21).

To prove Proposition (6.3), we let f be the (continuous) L-boundary function for F. By assumption, F and f are now constant on cosets of M. Effectively, we may ignore M and consider all the functions and the operators as functions and operators on S. This means that F is $\pi_S(L)$-harmonic and instead of the Hua operators Δ_m, $m = 1, ..., r$ we have

$$\pi_S(\Delta_m) = c_m^{-1}(\sum_{i \leq m, \alpha} c_i^{-1}(Y_{im}^\alpha)^2 + \sum_{m \leq j, \alpha} c_m^{-1}(Y_{mj}^\alpha)^2 - \frac{d_m + f_m + 2}{c_m} Y_{mm} - \sum_{i<m} \frac{d_{im}}{c_i} Y_{ii}).$$

Therefore, now our goal is to prove

(6.4) Proposition. *Suppose that bounded $\pi(L)$-harmonic function F is annihilated by $\pi(\Delta_m)$, $m = 1, ..., r$. Then F is constant.*

To prove Proposition (6.4) we have to formulate Lemma (4.24) in a more general situation. Let $S = NA$ be a semi-direct product of a connected and simply connected nilpotent Lie group N and the group $A = \mathbb{R}^r$ with a diagonal action of A on N. Let $\mathcal{L} = \mathcal{Y}_1^2 + ... + \mathcal{Y}_m^2 + \mathcal{Y}_0$ be a left-invariant operator on S satisfying the Hörmander condition with the maximal boundary N/N^- identified, as in section 4, with N^+, $N^+ = \sum_{\lambda \in R^+} \mathcal{N}_\lambda$, $N^- = \sum_{\lambda \in R^-} \mathcal{N}_\lambda$. Let $Y \in \mathcal{A}$. We say that Y is contractive on S if ad Y has only non-negative eigenvalues as an automorphism of $\mathcal{N}$. We let $\mathcal{N}_Y^0$ be the span of the positive eigenspaces in $\mathcal{N}$ and $\mathcal{N}_Y^1$ be the centralizer of Y in $\mathcal{N}$. In this setting Lemma (4.24) is true and we formulate it here again for the readers convenience.

(6.5) Lemma. *Let F be a bounded, $\mathcal{L}$-harmonic function on S. Assume that the $\mathcal{L}$-boundary value f is continuous on the maximal boundary N^+. Let $Y \in \mathcal{A}$ be contractive. Then*

$$\lim_{t \to -\infty} F((\exp tY)g) = F_Y(g)$$

converges uniformly on compact sets in S and defines an $\mathcal{L}$-harmonic function with continuous boundary function f_Y. Both F_Y and f_Y are constant on right cosets of N_Y^0 in S and in N respectively. Additionally, f_Y and f agree on $N_Y^1 \cap N^+$. If f reduces to N/N^0, then f_Y will reduce to S/N', where N' is the subgroup generated by N_Y^0 and N^0.$\square$

Still in the above general setting, we have.

(6.6) Lemma. *Let N/N^0 be a boundary for $\mathcal{L}$ identified, as in section 4, with N^1 being a complement to N^0 in the sense of $N = N^1 N^0$. Assume that*

$$F(s) = \int_{N^1} f(\pi_{N^1}(sx)) \ \tilde{P}(x) \ dx$$

and $Y \in \mathcal{A} + \mathcal{N}^0$ centralize $\mathcal{N}^1$. Then F is constant under right translation by $\exp tY$ *for all $t \in \mathbb{R}$.*

Proof The proof is immediate, because

$$\pi_{N^1}(s \exp tYx) = \pi_{N^1}(sx \exp tY) = \pi_{N^1}(sx).\square$$

Let N^+ be the maximal boundary for $\pi(L)$ and F a bounded $\pi(L)$-harmonic function. Its boundary function f defined originally on N^+ is extended to N by

$$f(x^+x^-) = f(x^+) \tag{6.7}$$

and considered as a function on N.

We assume by induction that Proposition (6.4) is known for all domains of rank less than r.

(6.8) Lemma. *The function f is constant on cosets of $N_{>1}$ in S, where $\mathcal{N}_{>1} = \mathcal{S}_{>1} \cap \mathcal{N}$.*

Proof We apply Lemma (6.5) with $Y = Y_{11}$. Then N_Y^1 is exactly $N_{>1}$ and N_Y^0 is $N_{1*} = S_{1*} \cap N$. The function F_Y is annihilated by $\pi_S(\Delta_m)$, $m = 1, ..., r$ and is constant on cosets of N_{1*} in S. From Lemma (2.21), F_Y is annihilated by the image under π_S of strongly diagonal $HJK_{>1}$ operators, and hence, by induction, is constant. Lemma (6.5) now shows that f is constant on $N_{>1}$. By the same argument, all left translates of f are also constant on cosets of $N_{>1}$ (as in the proof of Theorem (4.21)), proving the lemma.□

Now, we shall introduce $r+1$ sets of functions. We define

$$\mathcal{F}_{r+1} = \{{}_xF : x \in N\}.$$

and

$$\mathbf{F}_{r+1} = \{{}_xf : x \in N\}.$$

Clearly $\mathbf{F}_{r+1}$ is the set of boundary values of functions from $\mathcal{F}_{r+1}$ considered as functions on N. Indeed, if

$$F(s) = \int_{N^+} f(\pi_{N^+}(sy^+))P(y^+)\, dy^+$$

then

$$F(xs) = \int_{N^+} f(\pi_{N^+}(xsy^+))P(y^+)\, dy^+$$

so $f'(y^+) = f(\pi_{N^+}(xy^+)$ is the boundary function of ${}_xF$ on N^+. Extending f' to N, by (6.7) we have

$$f'(y^+y^-) = f(\pi_{N^+}(xy^+)) = f(xy^+) = f(xy^+y^-)$$

so $f' =_x f$.

We then define

$$\mathcal{F}_k = \{{}_xF_k : F_k(s) = \lim_{t\to\infty} F_{k+1}((\exp tY_{kk})s), F_{k+1} \in \mathcal{F}_{k+1}\}, \tag{6.9}$$

for $1 \le k \le r$. We shall prove shortly that these limits converge in the C_c^∞ topology on G. Granted this, $\mathcal{F}_k$ is a set of L-harmonic (and Hua harmonic) functions on G

which is also invariant under left translation. Let $\mathbf{F}_k$ denote the set of L boundary values of functions from $\mathcal{F}_k$. Then $\mathbf{F}_k$ is also invariant unde the left action of G.

First we prove existence of the limits in (6.9). For $k = r$, the limit exists in C_c^∞ from Lemma (6.5) (applied to $-Y_{rr}$). It is constant on cosets of the normal subgroup $N_r = \exp \mathcal{N}_r$, where

$$\mathcal{N}_r = \sum_{1 \leq i < r} \mathcal{S}_{ir}.$$

The same is satisfied by the elements of $\mathbf{F}_r$. Note that $-Y_{(r-1)(r-1)}$ is contractive on N/N_r. Therefore Lemma (6.5) applied to S/N_r proves the existence of F_{r-1}. Let

$$\mathcal{N}_k = \sum_{i<j, k \leq j} \mathcal{S}_{ij}$$

and

$$N_k = \exp \mathcal{N}_k.$$

N_k is a normal subgroup of N. It similarly follows by induction applied to the quotient group S/N_{k+1} that the limit in (6.9) exists in C_c^∞ for all k and elements of $\mathcal{F}_k$ are constant on cosets of the normal subgroup N_k. Furthermore, the same holds for boundary functions from $\mathbf{F}_k$. Moreover, suppose that $F_k \in \mathcal{F}_k$ and that F_{k+1} is related to F_k as in (6.9) and that the corresponding boundary functions are f_k and f_{k+1} respectively. Then f_k equals f_{k+1} on $N_{<k} = \exp \mathcal{N}_{<k}$, where

$$\mathcal{N}_{<k} = \sum_{1 \leq i < j < k} \mathcal{S}_{ij}.$$

From Lemma (6.8), f_k is also constant on cosets of $N_{>1}$.

Proposition (6.4) clearly follows from the following:

(6.10) Lemma. *Each $F_k \in \mathcal{F}_k$ is constant for $3 \leq k \leq r+1$.*

Proof Our proof will be inductive. For $F_3 \in \mathcal{F}_3$, since F_3 is constant on cosets of N_3 we have:

(6.11)
$$0 = c_1 HJK_1(F_3) = (2c_1^{-1} Y_{11}^2 + c_1^{-1} \sum_\alpha (Y_{12}^\alpha)^2 - \frac{d_1 + 2 + f_1}{c_1} Y_{11}) F_3$$

(6.12)
$$0 = c_2 HJK_2(F_3) = (2c_2^{-1} Y_{22}^2 + c_1^{-1} \sum_\alpha (Y_{12}^\alpha)^2 - \frac{d_2 + 2 + f_2}{c_2} Y_{22} - \frac{d_{12}}{c_1} Y_{11}) F_3$$

Both $\sum_1^r Yii$ and Y_{ii}, $i \geq 3$, centralize $\mathcal{N}_{<3}$. Hence, by Lemma (6.6), for $i \geq 3$,

$$Y_{ii} F_3 = 0$$
$$(Y_{11} + Y_{22} + \cdots + Y_{rr}) F_3 = 0.$$

Hence,

$$Y_{11}F_3 = -Y_{22}F_3.$$

We substitute this relation into formula (6.12) and subtract the result from formula (6.11), getting

$$(2(c_1^{-1} - c_2^{-1})Y_{11}^2 - (\frac{d_1 + 2 + f_1 - d_{12}}{c_1} + \frac{d_2 + 2 + f_2}{c_2})Y_{11})F_3 = 0.$$

If $c_1 = c_2$, then $Y_{11}F_3 = 0$. (Note that $d_1 \geq d_{12}$.) In particular,

$$(Y_{11}^2 + \sum_\alpha (Y_{12}^\alpha)^2 + Y_{11})F_3 = 0.$$

Then according to [DH] the maximal boundary for $Y_{11}^2 + \sum_\alpha (Y_{12}^\alpha)^2 + Y_{11}$ on S/N_3 is trivial, so F_3 is constant on $N_{<3}$ and, hence, on S.

If $c_1 \neq c_2$, then we conclude that there is a nonzero constant ρ such that

$$(Y_{11}^2 + \rho Y_{11})F = 0$$

(Note that $\frac{d_1+2+f_1-d_{12}}{c_1} + \frac{d_2+2+f_2}{c_2} > 0$). Hence, solving a simple differential equation we see that there are constants τ and η such that for all $s \in S$.

$$F_3(g(\exp tY_{11})) = (\tau + \eta e^{-\rho t})F_3(g).$$

Boundedness forces $\eta = 0$ and, hence, $Y_{11}F_3 = 0$. We see as above that F_3 is constant.

Now, suppose by induction that we have shown that each $F_k \in \mathcal{F}_k$ is constant. It follows that each $f_k \in \mathbf{F}_k$ is constant on $N_{<k}$. But since $\mathcal{F}_k$, $\mathbf{F}_k$ are closed under left translations, f_{k+1} is constant on right cosets of $N_{<k}$. Thus, the boundary function f_{k+1} for F_{k+1} reduces to N/N' where N' is some homogeneous subgroup containing $N_{>1}$, $N_{<k}$ and N_{k+1}. We may choose a homogeneous compliment to $\mathcal{N}'$ contained in

$$\mathcal{N}_{1k}.$$

From Lemma (6.6),

$$YF_{k+1} = 0$$

for any $Y \in \mathcal{N}_{>1} + \mathcal{N}_{<k} + \mathcal{N}_{k+1}$, which centralizes $\mathcal{N}_{1k}$. In particular,

$$Y_{ij}F_{k+1} = 0$$

for all $1 \leq i \leq j < k$, $j \neq 1$. The above formula is also true for $k+1 \leq j \leq r$ since F_{k+1} is constant on cosets of N_{k+1}. For $1 < i < k$, the equation $c_i HJK_i(F_{k+1}) = 0$ says exactly that

$$(c_i^{-1} \sum_\alpha (Y_{ik}^\alpha)^2 - \frac{d_{1i}}{c_1} Y_{11})F_{k+1} = 0. \tag{6.13}$$

For $i = k$ we obtain:

$$(6.14) \qquad (2c_k^{-1}Y_{kk}^2 + \sum_{1\le i<k,\alpha} c_i^{-1}(Y_{ik}^\alpha)^2 - \frac{d_k + +f_k + 2}{c_k}Y_{kk} - \frac{d_{1k}}{c_1}Y_{11})F_{k+1} = 0$$

As before, $(\sum Y_{ii})F_{k+1} = 0$. Hence

$$Y_{11}F_{k+1} = -Y_{kk}F_{k+1}.$$

Then, from formula (6.13) and formula (6.14), F_{k+1} is annihilated by the operator:

$$(6.15) \qquad 2c_k^{-1}Y_{11}^2 + c_1^{-1}\sum_\alpha (Y_{1k}^\alpha)^2 + (\frac{d_{1k}}{c_1} + \frac{d_k + f_k + 2}{c_k} - \sum_{i<k}\frac{d_{1i}}{c_1})Y_{11}$$

Finally, from HJK_1, we see that F_{k+1} is also annihilated by

$$(6.16) \qquad 2c_1^{-1}Y_{11}^2 + c_1^{-1}\sum_\alpha (Y_{1k}^\alpha)^2 - \frac{d_1 + f_1 + 2}{c_1}Y_{11}.$$

Subtracting (6.16) from (6.15) we see that

$$(6.17)\ \ (2(c_k^{-1} - c_1^{-1})Y_{11}^2 + (\frac{d_k + f_k + 2}{c_k} + \frac{d_{1k}}{c_1} - \sum_{i<k}\frac{d_{1i}}{c_1} + \frac{d_1 + f_1 + 2}{c_1})Y_{11})F_{k+1} = 0$$

Moreover, since $d_1 > \sum_{i<k} d_{1i}$, the coefficient by Y_{11} in (6.17) is strictly positive.

As in the F_3 case, it follows that F_{k+1} is constant on right cosets of the group whose lie algebra is generated by Y_{11} and Y_{1k}^α. Since this function is also constant on right cosets of N', we see that f_{k+1}, and hence F_{k+1} is constant, as desired.$\square$

This finishes the proof of Proposition (6.4) and hence of Theorem (6.1).

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