THE PALEY-WIENER THEOREM FOR THE HUA SYSTEM

Section 1: Introduction

One of the more beautiful results in the harmonic analysis of symmetric spaces is the Helgason Theorem, which states that on a Riemannian symmetric space $X = G/K$, a function is annihilated by the algebra $D_G(X)$ of all G-invariant differential operators if and only if it is the Poisson integral of a hyperfunction over the "maximal" boundary. (See [KKMOOT].)

If X is a Hermitian symmetric space, then one is typically interested in complex function theory, in which case one is interested in functions whose boundary values are supported on the Shilov boundary rather than the maximal boundary. In this case, it turns out that the algebra of G invariant differential operators is not necessarily the most appropriate one for defining harmonicity. Johnson and Koryáni [JK], generalizing earlier work of Hua [Hu], Koryáni -Stein [KS], and Koryáni-Malliavin [KM], introduced an invariant system of second order differential operators (the HJK system) defined on any Hermitian symmetric space. They showed that any function that is annihilated by this system (i.e. any *Hua-harmonic* function) is the Poisson integral of a hyperfunction over the Shilov boundary. They also showed that in the special case that X is a tube domain, all Poisson integrals are Hua-harmonic. Thus, in the tube case, the Hua system plays the same role with respect to the Shilov boundary as the algebra $D_G(X)$ does with respect to the maximal boundary. (Later, Lassalle ([La1] and [La2]) showed the existence of a smaller *real* system with the same properties as the Jhonson-Koryáni system. This smaller system will not, however, play a role in the current work.)

In the general Hermitian symmetric case, it is not true that all Poisson integrals are Hua-harmonic. In [BV], Berline and Vergne commented that the boundary values of the Hua-harmonic functions should satisfy some "tangential" Hua equations. They also produced an invariant system of third order operators with the property that a function f is the Poisson integral of a hyperfunction over the Shilov boundary if and only if f is annihilated both by the Berline-Vergne system and by $D_G(X)$. Arguably, however, the Hua system is perhaps more appropriate for the study of analytic function theory than the Berline-Vergne system since it is simplier and defines a smaller class of boundary functions. (Both systems annihilate holomorphic functions.)

Every Hermitian symmetric space is, of course, a Kähler manifold. In [DHP2] it was noted that the HJK system is definable on any Kähler manifold X. This more general system is invariant under any bi-holomorphic isometry of the manifold. It seems interesting to ask to what extent the results of Johnson and Koryáni depend on the semi-simplicity of the the space and to what extent they are special cases of results valid for a larger class of Kählerian manifolds. Specifically, one is interested in the following questions.

(1) Given a Kählerian manifold X, is there a Poisson kernel on the Shilov boundary for X with the property that every function which is annihilated by the HJK system on X is the Poisson integral of a hyperfunction over the Shilov boundary?
(2) If the answer to the first question is affirmative, can we describe the space of boundary functions for HJK?

In the light of the Helgason theorem, it is natural to restrict initially to *homogeneous* Kähler manifolds. Then a result of Dorfmeister and Nakajima [DN] states that the general such manifold decomposes as a fiber bundle over a bounded homogeneous domain in $\mathbf{C}^n$ where the fibers are homogeneous Kähler manifolds of a particularly simple type. Thus, it is natural to restrict further to the class of bounded homogeneous domains in $\mathbf{C}^n$. Note that this class still contains all Hermitian symmetric manifolds.

Question (1) was studied in [DHP2] where it was shown that in the bounded-homogeneous case there is indeed a "Poisson" kernel on the Shilov boundary that reproduces the Hua-harmonic functions. In fact, it was shown that the Shilov boundary is a boundary (in the sense of [DH]) for the Laplace-Beltrami operator of the domain and that the Poisson kernel for this operator on the Shilov boundary suffices to reproduce the Hua-harmonic functions. It should be noted that the Laplace-Beltrami operator is a linear combination of operators from the Hua system so the Hua-harmonic functions are, in particular, harmonic for the Laplace-Beltrami operator. Typically, the maximal boundary for this operator is larger than the Shilov boundary ([DHP1]). Thus, the main content of the theorem for the HJK system just mentioned is that the boundary values for the HJK system, which initially exist only on the maximal boundary, are actually supported on the (smaller) Shilov boundary.

In the case of a symmetric domain, the Poisson kernel for the Laplace-Beltrami operator is easily computable in terms of the complex structure of the domain. Specifically, let $S(z,w)$ be the Szegö kernel function for the domain. (This is the reproducing kernel for $\mathcal{H}^2$.) Then, in this case, S extends almost every where in w to the Shilov boundary and the function

$$P(z,x) = \frac{|S(z,x)|^2}{S(z,z)}$$

where z belongs to the domain and x to the Shilov boundary, is the Poisson kernel for the Laplace-Beltrami operator. This function is called the *Cauchy-Szegö Poisson kernel*

For a non-symmetric domain, the Cauchy-Szegö Poisson kernel is not the Poisson kernel for the Laplace-Beltrami operator. In fact, it is known that the Cauchy-Szegö Poisson kernel is harmonic for the Laplace-Beltrami operator if and only if the domain is symmetric [Xu]. There is, to our knowledge, no general formula for the Laplace-Beltrami kernel outside of the symmetric case. This then tends to diminish the utility of the result mentioned above concerning the reproducibility of the Hua-harmonic functions from the boundary.

The first main result of this work is the remarkable statement that *the Cauchy-Szegö Poisson kernel also reproduces Hua-harmonic functions.* Thus, the two most natural candidates for a Poisson kernel, the Cauchy-Szegö Poisson kernel and the Laplace-Beltrami Poisson kernel, *both work equally well for the Hua-Harmonic functions.* This is all the more remarkable when one realizes that in the non-symmetric case, the Hua system does not annihilate the Cauchy-Szegö Poisson kernel. (Recall that the Laplace-Beltrami operator is a linear combination of operators from the Hua system.) Thus, there is no a priori reason to expect a connection between the Hua system and the Cauchy-Szegö Poisson kernel. It should also be noted that there is a considerable body of information relating to the Cauchy-Szegö Poisson kernel (See, for example, [DHP1].)

The non-uniqueness of the reproducing kernel of course means that the space of boundary values of the Hua-harmonic functions cannot be dense in L^∞ of the boundary. Thus, a complete understanding of the Hua-harmonic functions requires describing the space formed by their boundary values. The second major result of this work is a characterization of the space of L^2 boundary values. To describe this result we must recall the definition of the homogeneous Siegel domains of type I. It should be noted that every symmetric tube domain has a realization as a Siegel domain of type I.

Let M be a finite dimensional real vector space and let $\mathcal{V} \subset M$ be an open, convex cone that does not contain straight lines. (Such cones are said to be *regular.*) Then the Siegel domain of type I defined by $\mathcal{V}$ is the domain $\mathcal{D} \subset M_c$

$$\mathcal{D} = M + i\mathcal{V}$$

It is known that $\mathcal{D}$ is bi-holomorphically equivalent with a bounded domain in $\mathbf{C}^n$.

We assume that the cone $\mathcal{V}$ is homogeneous, i.e. there is a real algebraic group S, an algebraic representation ρ of S on M, and a point $c \in \mathcal{V}$ for which $\mathcal{V} = \rho(S)c$. It is well known that in this case S may be chosen to be completely solvable and to act simply transitively on $\mathcal{V}$. ([Vin]) We shall assume that S has been so chosen.

Under these assumptions, S acts on $\mathcal{D}$ by means of ρ. The group M also acts on $\mathcal{D}$ by translation. In fact $\mathcal{D}$ is homogeneous under the semi-direct product $G = M \times_s S$ where S acts on M by means of ρ. This action makes $\mathcal{D}$ into a *homogeneous* Siegel domain of type I.

The set $M = M + 0i$ is referred to as the "Bergman-Shilov" boundary of $\mathcal{D}$–it is an open dense subset of the Shilov boundary. The second main result of this work is the statement that *a function in $L^2(M, dx)$, where dx is Lebesgue measure, is the boundary value of a Hua-harmonic function if and only if its Fourier transformation is supported in a certain open subset $\mathcal{O} \in M^*$. This set is invariant under the adjoint action of S on M^* and is a finite union of open S orbits.* Thus, describing the space of boundary functions comes down to determining which of the S orbits in M^* are contained in $\mathcal{O}$. Such orbits are said to be *harmonic.* It should be noted that the S orbits are cones that are typically non-convex.

We think of this result as an analogue of the classical Paley-Wiener theorem in that it describes the space of boundary values solely in terms of the support of their Fourier transformations. It is also analogous to results of Rossi and Vergne [RV] relating to the tangential Cauchy-Riemann equations. In fact, it occurs for much the same reasons as in [RV].

Determining the harmonic orbits is an important and, as yet, unsolved problem. However, there is a result that has some promise of yielding significant insight into this issue. To describe this result, let $P(z,x)$ denote the Cauchy-Szegö Poisson kernel for $\mathcal{D}$ where now z ranges over $\mathcal{D}$ and x ranges over M. For $\lambda \in M^*$ let

$$P^\wedge(z,\lambda) = \int_{-\infty}^{\infty} P(z,x) e^{-i<x,\lambda>} dx$$

be the Fourier transformation of P in the x variable. We show that if λ belongs to an open orbit of S, then *the corresponding orbit is harmonic if and only if $P^\wedge(z,\lambda)$ is Hua-harmonic as a function of z.*

Section 2: Homogeneous Cones

We continue the notation defined in the introduction. Specifically, we assume that M, S, ρ, c, and $\mathcal{V}$ are as defined at the end of the introduction. The 4-tuple (S, M, c, ρ) is referred to as "tube data." The following example plays an important role in this work.

Example (1.1) Let M^n be the space of all $n \times n$ real, symmetric matrices and let $\mathcal{V}^n$ be the cone of all positive definite elements of M^n. Let S^n be the group of $n \times n$ upper triangular matrices with positive diagonal. For $s \in S^n$ and $X \in M^n$, we define

$$\rho^n(s)X = sXs^t$$

where s^t is the transpose of s. Then, as is well known, (S^n, M^n, I, ρ^n) is tube data for a Hermitian symmetric tube domain.

It is classical that the domain $\mathcal{D}$ is biholomorphically equivalent with a bounded domain. As such, it has a canonical Riemannian structure defined from the Bergman metric. Since G acts simply transitively on $\mathcal{D}$, the tangent space at ic may be identified with the Lie algebra $\mathcal{G}$ of G.

In general, *we adopt the convention that Lie groups are denoted by upper case Roman letters and the corresponding Lie algebra is denotes by the corresponding upper case script letter.*

Since the Riemannian structure is G-invariant, it is defined by a scalar product g on the Lie algebra $\mathcal{G}$. Koszul ([Kl], Formula 4.5) proved the existence of a functional $\beta \in \mathcal{G}^*$ such that this scalar product is given by

$$g(X,Y) = \beta([JX,Y]). \tag{1}$$

where $J : \mathcal{G} \to \mathcal{G}$ defines the complex structure on $\mathcal{G}$. We shall not explicitly use any other information concerning β other than the fact that formula (1) defines a J-invariant, positive-definite, scalar product.

More explicitly, let $\mathcal{M}$ and $\mathcal{S}$ be the respective Lie algebras for S and M. Of course, since M is a vector space, we may identify $\mathcal{M}$ and M. The representation ρ defines a Lie algebra representation (also denoted ρ) of $\mathcal{S}$ on $\mathcal{M}$. Then $\mathcal{G}$ is $\mathcal{M} \times_s \mathcal{S}$ where the semi-direct product is defined from ρ. Since S acts simply transitively on $\mathcal{V}$, the mapping $X \to \rho(X)c$ defines a vector space isomorphism of $\mathcal{S}$ onto $\mathcal{M}$. In [DHP2], it is shown that there is s functional $\xi \in \mathcal{M}^*$ such that

$$g(\rho(X_1)c \times Y_1, \rho(X_2)c \times Y_2) = \xi(\rho(X_2)\rho(X_1)c) + \xi(\rho(Y_2)\rho(Y_1)c) \tag{2}$$

(See Lemma (2.5) in [DHP2].)

Note that we denote the general element of a product space $X \times Y$ as $x \times y$ rather than the more common (x, y).

Now let (S_1, M_1, c_1, ρ_1) and (S_2, M_2, c_2, ρ_2) be two sets of tube data. Then a homomorphisim from the first set of tube data to the second is a pair (τ, T) consisting of a homomorphism $\tau : S_1 \to S_2$ and a mapping $T : M_1 \to M_2$ such that,

(a) $T(c_1) = c_2$
(b) For all $s \in S_1$,

$$\rho_2(\tau(s))T = T\rho_1(s)$$

It follows that $T(\mathcal{V}_1) \subset \mathcal{V}_2$.

A homomorphism of a given set of tube data into the tube data of Example (1.1) is said to be a *representation* of the tube data in $\mathbf{R}^n$. Specifically, a representation of (S, M, c, ρ) is a pair (τ, T) where τ is a representation of S by $n \times n$ upper triangular matrices and T is a mapping of M into the space of $n \times n$ symmetric matrices where

(a) $T(c) = I$ where I is the $n \times n$ identity matrix.
(b) For all $s \in S$ and $m \in M$,

$$T(\rho(s)m)) = \tau(s)T(m)\tau(s)^t$$

Note that it follows that T maps $\mathcal{V}$ into the cone of positive definite matrices.

Representations are important in part because they provide an inductive procedure (due to Rothaus [Ro]) for constructing cones. To explain this, let (τ_o, T_o) be a representation of (S_o, M_o, c_o, ρ_o) in $\mathbf{R}^n$. Let S be the set of all matrices s of the form

$$s = \begin{bmatrix} a & v^t \\ 0 & s_o \end{bmatrix} \tag{3}$$

where $a \in \mathbf{R}^+$, $v \in \mathbf{R}^n$ (thought of as column vectors), $s_o \in S_o$ and 0 is the zero element of $\mathbf{R}^n$. We define the product of two such elements via

$$\begin{bmatrix} a & v^t \\ 0 & s_o \end{bmatrix} \begin{bmatrix} b & u^t \\ 0 & t_o \end{bmatrix} = \begin{bmatrix} ab & au + [\tau_o(t_o)v]^t \\ 0 & s_o t_o \end{bmatrix}$$

It is easily seen that S becomes a Lie group under this action.

Next let M be the vector space of all matrices of the form

$$m = \begin{bmatrix} b & w^t \\ w & m_o \end{bmatrix} \tag{4}$$

where $b \in \mathbf{R}$, $w \in \mathbf{R}^n$, and $m_o \in M_o$. For s as in formula (3) and m as above, we define

$$\rho(s)m = \begin{bmatrix} a^2 b + 2a(v, w) + (T_o(m_o)v, v) & [\tau_o(s)(aw + T_o(m_o)v)]^t \\ \tau_o(s)(aw + T_o(m_o)v) & \rho_o(s)m_o \end{bmatrix} \tag{5}$$

where $(\cdot,\cdot)$ is the Euclidian scalar product on $\mathbf{R}^n$. It is easily seen that ρ is a representation of S on M. (This formula results from formally expanding the matrix product

$$\begin{bmatrix} a & v^t \\ 0 & s_o \end{bmatrix} \begin{bmatrix} b & w^t \\ w & m_o \end{bmatrix} \begin{bmatrix} a & 0 \\ v & s_o^t \end{bmatrix}$$

using $(\cdot,\cdot)$, τ_o, and T_o to define the products of individual elements and where we interpret $s_o m_o s_o^t$ as $\rho_o(s_o)m_o$.)

Finally, let $c_o \in \mathcal{V}_o$ and set

$$c = \begin{bmatrix} 1 & 0 \\ 0 & c_o \end{bmatrix}$$

Let $\mathcal{V}$ be the S orbit of c in M. It is a result of Rothaus that $\mathcal{V}$ is an open, regular, convex cone which is independent of the choice of c_o in $\mathcal{V}_o$. We refer to (S, M, c, ρ) as the cone data induced from (S_o, M_o, c_o, ρ_o) using the representation (τ_o, T_o). Rothaus also showed that *every homogeneous cone is isomorphic to one induced from a lower dimensional cone using an appropriate representation.*

As an example of this construction, we note that in Example 1.1, the usual action of S^n on $\mathbf{R}^n$ defines a representation τ^n of S^n. The elements of M^n are symmetric matrices so the identity transformation defines a mapping to M^n into the space of symmetric matrices. The pair (τ^n, I) is a representation of (S^n, M^n, I, ρ^n). Then, as the reader may easily verify, the corresponding induced cone data is just $(S^{n+1}, M^{n+1}, I, \rho^{n+1})$.

The Lie algebra $\mathcal{G}$ is easily described. As a vector space $\mathcal{G} = \mathcal{M} \times \mathcal{S}$ where $\mathcal{M}$ and $\mathcal{S}$ are, respectively, the Lie algebras of M and S. Of course $\mathcal{M} = M$ since M is a vector space. The space $\mathcal{S}$ is the set of matrices s of the form

$$s = \begin{bmatrix} a & v^t \\ 0 & s_o \end{bmatrix}$$

where $s_o \in \mathcal{S}_o$.

As a Lie algebra, $\mathcal{G} = \mathcal{M} \times_s \mathcal{S}$ where the action of $s \in \mathcal{S}$ on $\mathcal{M}$ is found by differentiating formula (5) in the direction of s at the identity. It is given by

$$\rho(s) \begin{bmatrix} b & w^t \\ w & m \end{bmatrix} = \begin{bmatrix} 2ab + 2(v, w) & [aw + T_o(m_o)v + \tau_o(s_o)w]^t \\ aw + T_o(m_o)v + \tau_o(s_o)w & \rho_o(s)m_o \end{bmatrix}$$

where τ_o and ρ_o are, respectively, the Lie algebra representations obtained by differentiating τ_o and ρ_o

We shall also require a description of the scalar product on the induced cone. For this we note the following well known result. We sketch the proof for sake of completeness.

Lemma 2.1. *Let the cone data (S, M, c, ρ) be induced as described above. Then the functional ξ from formula (2) is zero on all elements of $\mathcal{M}$ of the form*

$$\begin{bmatrix} 0 & w^t \\ w & 0 \end{bmatrix}$$

Proof This follows very simply from the following formula which is a consequence of the symmetry of the scalar product. We leave the details to the reader.

$$\xi(\rho([X_2, X_1])c) = \xi(\rho(X_2)\rho(X_1)c) - \xi(\rho(X_1)\rho(X_2)c) = 0$$

As a direct consequence, we have the following:

Lemma 2.2. *Let $X \in \mathcal{G}$ with $X = m \times s$ where*

$$m = \begin{bmatrix} b & w^t \\ w & 0 \end{bmatrix} \qquad s = \begin{bmatrix} a & v^t \\ 0 & 0 \end{bmatrix}$$

Let

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Then

$$g(X, X) = 2\xi(E)(2a^2 + (v, v) + b^2/2 + (w, w))$$

From now on, we will assume that (S, M, c, ρ) is induced as described above. There are a number of subgroups of G which play an important role. Specifically, we define the named set on the left in the figure below to be the set of all elements of S of the form described on the right where e_o is the identity element of S_o, s_o ranges over S_o, v and w range over $\mathbf{R}^n$, a ranges over $\mathbf{R}^+$, and b ranges over $\mathbf{R}$.

S_H	typical element:	$\begin{bmatrix} a & v^t \\ 0 & e_o \end{bmatrix}$
A_H	typical element:	$\begin{bmatrix} a & 0 \\ 0 & e_o \end{bmatrix}$
N_H	typical element:	$\begin{bmatrix} 1 & v^t \\ 0 & e_o \end{bmatrix}$
M_H	typical element:	$\begin{bmatrix} b & w^t \\ w & 0 \end{bmatrix}$
M_H^o	typical element:	$\begin{bmatrix} 0 & w^t \\ w & 0 \end{bmatrix}$

We also identify the groups S_o and M_o respectively with the subgroups of S and M described below:

S_o	typical element:	$\begin{bmatrix} 1 & 0 \\ 0 & s_o \end{bmatrix}$
M_o	typical element:	$\begin{bmatrix} 0 & 0 \\ 0 & m_o \end{bmatrix}$

Finally, we define the following subgroups of $G = M \times_s S$.

$$\begin{aligned} G_o &= M_o S_o \\ G_H &= M_H S_H \\ H &= M_H N_H \end{aligned}$$

It is easily seen that H is a normal subgroup of G. The reason for calling this subgroup H is that it is a Heisenberg group–i.e it is a two step nilpotent Lie group with one dimensional center. In fact, its center is the set of elements in M such that $w = 0$ and $m_o = 0$.

The orbit of ic in M_c under G_H is the set of elements of M of the form

$$\begin{bmatrix} b + i(a^2 + |v|^2) & w^t + iv^t \\ w + iv & ic_o \end{bmatrix}$$

This set is identifiable with the domain $\mathcal{B}$ in $\mathbf{C} \times \mathbf{C}^n$ consisting of all points (W, Z) such that $\operatorname{im} W > |\operatorname{im}(Z)|^2$.

The domain $\mathcal{B}$ is, in fact, equivalent with the unit ball in $\mathbf{C}^{n+1}$. The simplest way to prove this is to note that the transformation

$$(W, Z) \to (2W - i \sum Z_i^2, Z)$$

transforms $\mathcal{B}$ into the domain described by $\operatorname{im} W > |Z|^2$, which is well known to be equivalent with the unit ball. Since G_H acts simply transitively on $\mathcal{B}$, we may identify G_H with the unit ball in $\mathbf{C}^{n+1}$.

There is a representation of (S, M, c, ρ) that plays an important role. For s as in formula (3), we define $\tau(s)$ to be the operator on $\mathbf{R}^{n+1} = \mathbf{R} \times \mathbf{R}^n$ defined by the matrix

$$\begin{bmatrix} a & v^t \\ 0 & \tau_o(s_o) \end{bmatrix}$$

Similarly, for m as in formula (4) we define $T(m)$ to be the operator on $\mathbf{R}^{n+1} = \mathbf{R} \times \mathbf{R}^n$ defined by the matrix

$$\begin{bmatrix} b & w^t \\ w & T_o(m_o) \end{bmatrix}$$

It is easily seen that (τ, T) is a representation of (S, M, c, ρ) on $\mathbf{R}^{n+1}$. We refer to this representation as the representation induced from the representation (τ_o, T_o) of (S_o, M_o, c_o, ρ_o). Notice that the mapping

$$T \times \tau : M \times_s S \to M^n \times_s S^n$$

is a group homomorphism that restricts to an isomorphism of G_H onto G_H^n. Lemma (2.2) tells us that the corresponding Lie algebra isomorphism is a scalar multiple of an isometry of $\mathcal{G}_H$ onto $\mathcal{G}_H^n$.

If $\mathcal{V}$ is a regular cone in a vector space $\mathcal{M}$, then we define the dual cone $\mathcal{V}^*$ be the set of elements $\lambda \in \mathcal{M}^*$ that are strictly positive on $\overline{\mathcal{V}} - 0$. The group S acts on $\mathcal{V}^*$ via the adjoint representation which is defined by the formula

$$\rho^*(s) = \rho(s^{-1})^*$$

It is known that $\mathcal{V}^*$ is homogeneous under this action.

For each $m \in \mathcal{V}$, we define

$$D(m) = \int_{\mathcal{V}^*} e^{-<m,\lambda>}\, d\lambda \tag{6}$$

where $d\lambda$ denotes Lebesgue measure on $\mathcal{V}^*$, *which we normalize so that* $D(c) = 1$. This function is referred to as the *characteristic function* for the cone. The absolute convergence of this integral for all $v \in \mathcal{V}$ is proved in Vindberg [V].

We would like to describe D inductively. To this end, we note that a simple change of variables in formula (6) shows that for all $s \in \mathcal{S}$ and all $v \in \mathcal{V}$,

$$D(\rho(s)m) = \chi(s)^{-1} D(m) \tag{7}$$

where

$$\chi(s) = \det(\rho(s))$$

Since $D(c) = 1$, it follows that D maps $\rho(s)c$ into $\chi(s)^{-1}$.

For $s_o \in S_o$, let

$$\chi_o(s_o) = \det(\rho_o(s_o)) \quad \text{and} \quad \kappa(s_o) = \det(\tau_o(s_o))$$

Lemma 2.3. *For s as in formula (3),*

$$\chi(s) = a^{n+2}\kappa(s_o)\chi_o(s_o)$$

Proof We note that

$$\begin{bmatrix} a & v^t \\ 0 & s_o \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & s_o \end{bmatrix} \begin{bmatrix} 1 & a^{-1}v^t \\ 0 & e_o \end{bmatrix}$$

Let us call the first matrix on the right δ and the second u. Since u is unipotent, $\chi(u) = 1$. Hence $\chi(s) = \chi(\delta)$. Our lemma follows easily by taking $v = 0$ in formula (5).

Now, suppose that $m = \rho(s)c$. Then, from formula (5), $m_o = \rho_o(s_o)c_o$. Hence

$$\chi_o(s_o) = D_o(m_o)^{-1}$$

where D_o is the characteristic function for $\mathcal{V}_o$. Furthermore

$$T_o(m_o) = \tau_o(s_o)T_o(c_o)\tau_o(s_o)^t = \tau_o(s_o)\tau_o(s_o)^t$$

Hence

$$\kappa(s_o) = \det(T_o(m_o))^{1/2}$$

Finally, we note that a similar argument proves that

$$\det(T(m)) = \det(\tau(s))^2 = a^2 \det(\tau_o(s_o))^2 = a^2 \det(T_o(m_o))$$

Thus, substitution into the formula from Lemma (2.3) yields

$$\begin{aligned} D(m) &= \chi(m)^{-1} \\ &= \left(\frac{\det(T(m))}{\det(T_o(m_o))}\right)^{-(n+2)/2} \det(T_o(m_o))^{-1/2} D_o(m_o) \\ &= \det(T(m))^{-(n+2)/2} \det(T_o(m_o))^{(n+1)/2} D(m_o) \end{aligned}$$

The following (well known) corollary follows from the above formula by induction. (see [KF], p.11)

Corollary 2.4. *Let D^n be the characteristic function for the cone $\mathcal{V}_n$ of Example (1.1). Then for all $m \in \mathcal{V}_n$,*

$$D^n(m) = \det(m)^{-(n+1)/2}$$

We note the following consequence of this corollary which will be used in the next section:

$$\frac{D(m)}{D(m_o)} = \frac{D^{n+1}(T(m))}{D^n(T_o(m_o))} \tag{8}$$

Section 3. *The Poisson Kernel for the HJK system.*

We continue the notation established in §2. Specifically, we assume that $\mathcal{D}$ is the tube domain defined by the cone data (S, M, c, ρ) which is induced from the representation (τ_o, T_o) of (S_o, M_o, c_o, ρ_o).

Our goal in this section is to prove that a bounded Hua-harmonic function F is reproducible from its boundary value function by integration against the Cauchy-Szegö Poisson kernel. Our proof will rely heavily on one of the main results of [DHP2]–namely that there is a Poisson kernel on the Bergman-Shilov boundary that reproduces Hua harmonic functions from their boundary values. Actually, in [DHP2] we proved for F to be reproducible using the stated kernel, it sufficed that F be harmonic for a smaller system, called the *strongly diagonal Hua system.* Functions harmonic for this system are referred to as *diagonally harmonic.* It is this stronger result that we use. We will not need to recall the definitions of either the Hua system or of the strongly diagonal Hua operators since we will only require a few of their general properties from [DHP2].

Now, let F be a bounded, diagonally-harmonic function on $G = M \times_s S$. In [DHP2] we used one particular strongly diagonal operator denoted HJK_1 (cf. , Theorem (2.18), *loc. cit.*). This operator had the form

$$HJK_1 = \xi(E)^{-2}\left[2(Y^2+X^2)-(n+2)Y+\sum_1^n Y_j^2+X_j^2\right]$$

where E and ξ are as in Lemma 2.2, $\xi(E)^{-1/2}Y_i$ is an orthonormal basis for $\mathcal{N}_H$, $\xi(E)^{-1/2}X_i$ is an orthonormal basis for $\mathcal{M}_H^o$ and $Y = 0\times E/2$ and $X = E\times 0$. (Note that under the obvious identifications of $\mathcal{N}_H$ and $\mathcal{M}_H^o$ with $\mathbf{R}^n$, X_i and Y_i are orthonormal bases with respect to $2(\cdot,\cdot)$. Thus, HJK_1 is independent of the choice of ξ, up to scalar multiples.)

Note that HJK_1 is defined by an element Δ_H in the enveloping algebra of G_H. In fact, it is easily seen that Δ_H is just the Laplace-Beltrami operator for the unit ball in $\mathbf{C}^{n+1}$ under the identification of G_H with the unit ball described in Section 2. It follows from [DHP1] that the maximal boundary for Δ_H on G_H is H, which we identify with G_H/A_H. Let P_H be the Poisson kernel function for Δ_H on H. Since $F|G_H$ is Δ_H-harmonic, there is a function f_H on G_H which is constant on cosets of A_H in G_H such that

$$F(e) = \int_H f_H(h)P_H(h)\,dh \tag{9}$$

Let $\delta(t)$ be the element of G defined by

$$\delta(t) = 0\times\begin{bmatrix} t & 0\\ 0 & I\end{bmatrix}$$

Lemma 3.1. *Let notation be as above. Assume that f_H is continuous on H. Then for all $k\in H$,*

$$\lim_{t\to 0} F(k\delta(t)) = f_H(k)$$

Proof For all $g\in G$, we set $g(t) = \delta(t)g\delta(t)^{-1}$. Thus, if

$$h = \begin{bmatrix} b & w^t\\ w & 0\end{bmatrix}\times\begin{bmatrix} 1 & v^t\\ 0 & I\end{bmatrix}$$

then

$$h(t) = \begin{bmatrix} t^2b & tw^t\\ tw & 0\end{bmatrix}\times\begin{bmatrix} 1 & tv^t\\ 0 & I\end{bmatrix}$$

As $t\to 0$, $h(t)$ ends to the identity element. Then, for all $k\in H$,

$$\begin{aligned} F(k\delta(t)) &= \int_H f_H(k\delta(t)h)P_H(h)\,dh\\ &= \int_H f_H(kh(t))P_H(h)\,dh \end{aligned}$$

Our result follows easily from this and a standard approximate identity argument based on the observation that P_H is positive and has integral 1. We leave the details to the reader.

In [DHP2] we proved the existence of a Poisson kernel function for the Hua system on Siegel domains. Thus, there is a positive function P on $G/S = M$, with integral 1, such that for all bounded, HJK-harmonic functions F and all $g \in G$,

$$F(g) = \int_M f(gm)P(m)\,dm$$

where f is the boundary function of F.

The function P may not be unique. However, we assume that one specific P has been chosen. From the above lemma, for all $k \in H$,

$$\begin{aligned} f_H(k) &= \lim_{t\to 0} F(k\delta(t)) \\ &= \lim_{t\to 0} \int_M f(k\delta(t)m)P(m)\,dm \\ &= \lim_{t\to 0} \int_M f(km(t))P(m)\,dm \end{aligned}$$

Now, for $m \in M$, we may write $m = m_H + m_o$ where $m_H \in M_H$ and $m_o \in M_o$. Then, since $\delta(t)$ centralizes M_o,

$$m(t) = m_H(t) + m_o$$

Noting that $m_H(t)$ tends to 0 as $t \to 0$, we see that

$$f_H(k) = \int_{M_o} f(km_o)P_o(m_o)\,dm_o \tag{10}$$

where

$$P_o(m_o) = \int_{M_H} P(m_o + m_H)\,dm_H \tag{11}$$

Putting formulas (9) and (10) together, we see that

$$F(e) = \int_H \int_{M_o} f(km_o)P_H(k)P_o(m_o)\,dm_o\,dk \tag{12}$$

The function P_o has an important interpretation. As commented earlier, the group G_o acts transitively on the domain $\mathcal{D}_o$. Let HJK_o be the corresponding Hua system. From Lemma (2.21) in [DHP2], a function F_o on G_o is diagonally harmonic for HJK_o, if and only if it is the restriction to G_o of an HJK diagonally-harmonic function $\tilde{F}$ that is constant on G_H cosets in G. The boundary function $\tilde{f}$ of $\tilde{F}$ will be constant on cosets of M_H in M. But then

$$\begin{aligned} \tilde{F}(e) &= \int_M \tilde{f}(m)P(m)\,dm \\ &= \int_{M_H}\int_{M_o} \tilde{f}(m_H + m_o)P(m_H + m_o)\,dm_H\,dm_o \\ &= \int_{M_o} \tilde{f}(m_o)P_o(m_o)\,dm_o \end{aligned}$$

It follows that the restriction of P_o to M_o is a Poisson kernel function for the diagonally harmonic functions on G_o. Actually, *in formula (12) we may replace P_o by any Poisson Kernel function for the strongly diagonal Hua operators on G_o.* To see this, it suffices to show that for all $k \in H$, the function

$$m_o \to f(km_o)$$

is the boundary function for a diagonally harmonic function on G_o, since then the integral in formula (10) will be independent of the particular kernel chosen. Since our differential operators commute with left translation, it in fact suffices to assume that $k = e$.

However, for $g \in G$, let

$$F_o(g) = \lim_{t\to 0} F(\delta(t)g)$$

Lemma 3.2. *The limit defining F_o exists for all $g \in G$ and defines a diagonally harmonic function that is constant on cosets of G_H on G. The corresponding boundary function equals f on M_o.*

Proof Let $g \in G$. We may write

$$g = g_o g_H$$

where $g_H \in G_H$ and $g_o \in G_o$. Then,

$$\begin{aligned} F_o(g) &= \lim_{t\to 0} \int_M f(\delta(t) g_o g_H m) P(m)\, dm \\ &= \lim_{t\to 0} \int_M f(g_o g_H(t) m(t)) P(m)\, dm \end{aligned}$$

Reasoning as in the proof of Lemma 1, we see that

$$F_o(g) = \int_M f(g_o m_o) P_o(m)\, dm \tag{13}$$

Thus, in particular, the limit defining F_o exists and defines a function that is constant on cosets of G_H. Furthermore, since the strongly diagonal operators are left invariant, the function

$$F_t(g) = F(\delta(t)g)$$

is diagonally harmonic for all $t > 0$. The system of strongly diagonal operators has an elliptic operator in its span. Hence, the limit defining F_o converges in the C^∞ topology and F_o is diagonally-harmonic. It follows from formula (13) that the boundary function for F_o is $f|G_o$, proving our lemma.

From this point on, P_o represents any Poisson kernel function for the diagonally harmonic functions on G_o, not just the P_o defined by formula (11).

Since f is constant on cosets of S in G, we may reduce the integral in formula (12) to an integral over M. Specifically, let $k = m_H h$ where $h \in N_H$ and $m_H \in M_H$. Then

$$f(km_o) = f(m_H h m_o) = f(m_H h m_o h^{-1}) = f(m_H [h, m_o] m_o)$$

where

$$[h, m_o] = h m_o h^{-1} m_o^{-1}$$

But $[h, m_o] \in M_H$. Hence, changing variables in (12), yields

$$\begin{aligned} F(e) &= \int_{M_o} \int_{N_H} \int_{M_H} f(m_H m_o) P_H(m_H [h, m_o]^{-1} h) P_o(m_o)\, dm_o\, dh\, dm_H \\ &= \int_{M_o} \int_{M_H} f(m_H m_o) Q(m_H m_o) P_o(m_o)\, dm_o\, dm_H \end{aligned}$$

where

$$Q(m_H m_o) = \int_{N_H} P_H(m_H m_o h m_o^{-1})\, dh \qquad (15)$$

The above computations may be summarized in the following theorem.

Theorem 3.3. *Let Q be defined as in formula (15) where P_H is the Poisson kernel function for HJK_1 on G_H. Let P_o be a Poisson kernel function for the strongly diagonal HJK system on G_o. Then the function P on G defined by*

$$P(m) = Q(m) P_o(m_o) \qquad (16)$$

is a Poisson kernel function for the diagonally harmonic functions on G.

At first glance, it might appear that the integral in formula (15) would be difficult to evaluate. Actually, there is a trick that evaluates it quite simply. Consider first the special case where we are inducing from the cone data (S^n, M^n, I, ρ^n) defined in Example (1.1) relative to the canonical representation. In this case, we obtain $(S^{n+1}, M^{n+1}, I, \rho^{n+1})$. The Poisson kernel functions for the corresponding domains are unique and well known. It follows from formula (16) that

$$Q^{n+1}(m) = \frac{P^{n+1}(m)}{P^n(m_o)}$$

where P^n and P^{n+1} are the Poisson kernel functions for the domains defined by the cones $\mathcal{V}^n$ and $\mathcal{V}^{n+1}$ respectively and Q^{n+1} the function corresponding to Q on M^{n+1}.

The computation of Q in the general case may be reduced to that just done using the induced representation (τ, T) described in §2. Specifically, we noted in §2 that the mapping $T \times \tau$ restricts to an isomorphism ν of G_H onto G_H^{n+1}. Furthermore, at the Lie algebra level, this mapping is (up to a scalar) an isometry. It follows that the HJK_1 operator on G_H^n is a scalar multiple of the image of the corresponding operator on G_H. In particular,

$$P_H = P_H^n \circ \nu$$

where P_H^n is the Poisson kernel function for HJK_1 on G_H^n.

It follows easily from this and formula (15) that

$$\frac{P(m)}{P_o(m_o)} = Q(m) = Q^{n+1}(T(m)) = \frac{P^{n+1}(T(m))}{P^n(T_o(m_o))} \tag{17}$$

The reader should note the similarity between this formula and formula (8). In fact, it is known ([KF], p. 181) that for the domain defined by the cone in Example (1.1)

$$P^n(m) = \pi^{-n(n+1)/2}|D^n(I+im)|^2$$

Using formulas (17), (8) and mathematical induction, we prove the following theorem, which is our first major result.

Theorem 3.4. *Let $P(m) = \pi^{-n}|D(c+im)|^2$ where n is the dimension of M. Suppose that F is a bounded C^∞ function on $\mathcal{D}$ which is annihilated by HJK and has continuous boundary function f on M. Then*

$$F(ic) = \int_M f(m)P(m)\,dm$$

Proof We may assume by induction that

$$P_o(m_o) = \pi^{-n_o}|D_o(c+im_o)|^2$$

where n_o is the dimension of M_o. Then $n = n_o + k + 1$ where T_o acts on $\mathbf{R}^k$. Then

$$\begin{aligned} P(m) &= \frac{P^{k+1}(T(m))}{P^k(T_o(m_o))}P_o(m_o) \\ &= \pi^{-(k+1)}\frac{|D^{k+1}(I+iT(m))|^2}{|D^k(I+iT_o(m_o))|^2}|D_o(c_o+im_o)|^2 \\ &= \pi^{-n}|D(c+im)|^2 \end{aligned}$$

(We used formula (8) in the last equality.) This proves the theorem.

Of course, once we can compute $F(ic)$ from f, we can compute $F(z)$ for any $z \in \mathcal{D}$. Let $z = x + iy \in \mathcal{D}$. Write $z = g(ic)$ where $g = x \times s$. Then,

$$\begin{aligned} F(z) &= \int_M f(xsm)P(m)\,dm \\ &= \int_M f(xsms^{-1})P(m)\,dm \\ &= \int_M f(x+\rho(s)m)P(m)\,dm \\ &= \int_M f(m)P(\rho(s)^{-1}(m-x))\det(\rho(s)^{-1})\,dm \end{aligned} \tag{18}$$

The Poisson kernel is, then, the function

$$\begin{aligned} P(z,m) &= P(\rho(s)^{-1}(m-x))\det(\rho(s)^{-1}) \\ &= \pi^{-n}|D(ic+\rho(s)^{-1}(m-x))|^2(\det(\rho(s)^{-2})\det(\rho(s)) \\ &= \pi^{-n}|D(iy+m-x)|^2/D(y) \end{aligned}$$

From [KF], p.181, this is exactly the Cauchy-Szegö Poisson kerned for $\mathcal{D}$. It is known that this function is Harmonic for the Laplace-Beltrami operator if and only if $\mathcal{D}$ is symmetric. On the other hand, if the boundary value functions for the Hua harmonic functions (or, more generally, the diagonally harmonic functions) are dense in $L^\infty(M)$, then this kernel would have to be harmonic in z. Thus, we arrive at the following theorem: (Note that the density of the boundary values is known in the semi-simple case by [JK].)

Theorem 3.5. *The space of boundary functions for the Hua harmonic functions on $\mathcal{D}$ is dense in $L^\infty(M)$ if and only if $\mathcal{D}$ is symmetric.*

Section 4: L^2 Boundary Values

In this section, we continue the notation from the previous sections. We let $\mathcal{H}(\mathcal{D})$ denote the space of bounded Hua-harmonic, functions on $\mathcal{D}$. We will generally denote the elements of $\mathcal{H}(\mathcal{D})$ by upper case Roman letters and their boundary functions on M by the corresponding lower case Roman letter. We define

$$\mathcal{H}^2_o(\mathcal{D}) = \{F \in \mathcal{H}(\mathcal{D}) \,|\, f \in L^2(M)\}$$

where we use Lebesgue measure on M. Since the mapping $F \to f$ is one-to-one, we may put a norm on $\mathcal{H}^2_o(\mathcal{D})$ by declaring

$$||F|| = ||f||_2$$

We define $\mathcal{H}^2(\mathcal{D})$ to be the completion of $\mathcal{H}^2_o(\mathcal{D})$ in this norm. This space is identifiable with a closed subspace $B^2(M)$ of $L^2(M)$.

We note that from Theorem 3.4,

$$1 = \int_M P(m)\,dm$$

In particular, P is in $L^1(M)$. It also follows from formula (6) that $P \in L^\infty(M)$. Thus, $P \in L^2(M)$. It follows that the Poisson integral defines a continuous mapping of $L^2(M)$ into a space of continuous functions on $\mathcal{D}$. The ellipticity of the Hua system then tells us that $\mathcal{H}^2(\mathcal{D})$ is also identifiable with a space of C^∞ functions on $\mathcal{D}$. We refer to $B^2(M)$ as the space of boundary values of elements of $\mathcal{H}^2(\mathcal{D})$. Our goal is to describe $B^2(M)$.

We begin with the observation that $\mathcal{H}^2(\mathcal{D})$ is invariant under the action of G on $\mathcal{D}$. In fact, the argument leading up to formula (18) shows that if F is diagonally harmonic, then the boundary function for $z \to F(g^{-1}z)$ is $\pi(g)f$ where

$$\pi(g)f(m) = f(\rho(s^{-1})(m-x))$$

which will be in $L^2(M)$ if f is. In fact, the representation

$$\pi_o(g) = \det(\rho(s))^{-1/2}\pi(g)$$

is unitary on $L^2(M)$.

We will describe $B^2(M)$ by describing the irreducible decomposition of the restriction of π_o to this subspace. The irreducible decomposition of π_o itself is easily described. For $f \in L^2(M)$ and $\lambda \in M^*$, let

$$f^\wedge(\lambda) = \int_M f(x)e^{-i<x,\lambda>}\,dx$$

It is easily computed that $f \to f^\wedge$ intertwines π_o and the representation $\tilde{\pi}_o$ on $L^2(M^*)$ defined by

$$\tilde{\pi}_o(x \times s)h(\lambda) = \det(\rho(s)^{1/2})e^{-i<x,\lambda>}h(\rho^*(s^{-1})\lambda) \tag{19}$$

Since ρ^* has open orbits (e.g. $\mathcal{V}^*$), we know that the union of the open orbits is dense in M^*. Let $\mathcal{V}_i$, for $i = 1, 2\ldots, k$, be the set of open ρ^* orbits. Let $\mathcal{L}_i$ be the space of functions in $L^2(M^*)$ that are supported in $\mathcal{V}_i$. Then the $\mathcal{L}_i$ are closed, invariant subspaces for $\tilde{\pi}_o$.

Lemma 4.1. *The restriction π_i of $\tilde{\pi}_o$ to $\mathcal{L}_i$ is irreducible and the π_i are mutually inequivalent.*

Proof For each i, let λ_i be a fixed base point in $\mathcal{V}_i$ and let χ_i be the character of M defined by

$$\chi_i(m) = e^{i<m,\lambda_i>}$$

It is easily seen that π_i is equivalent with the representation of G induced from χ_i. From Mackey theory, the irreducibility of π_i is equivalent with proving that the stabilizer of χ_i under the adjoint action of G on M is just M itself. This, in turn, will follow if we can show that the ρ^* stabilizer of λ_i is trivial. However, since the r^* orbit of λ_i is open, this stabilizer must have zero dimension. The triviality follows from the complete solvability of S.

To prove the mutual inequivalence of the π_i, it suffices to show that the restrictions of these representations to M are inequivalent, which is clear from formula (19).

It follows from Lemma 4.1 that there is a set $i_1, i_2, \ldots i_k$ of indecies for which

$$(B^2(M))^\wedge = \oplus \sum_j \mathcal{L}_{i_j}$$

This decomposition defines the irreducible decomposition of π_o. The following theorem follows from these comments. We call this the "Paley-Wiener" theorem because it characterizes the boundary values of the space of L^2 harmonic functions in terms of the support of their Fourier transforms.

Paley-Wiener Theorem. *Let $f \in L^2(M)$. Then $f \in B^2(M)$ if and only if the Fourier transformation $f^\wedge$ is supported in the union of the $\mathcal{V}_{i_j}$.*

The orbits $\mathcal{V}i_j$ are called *harmonic*. If we can characterize the harmonic orbits, then we have an essentially complete picture of the boundary values of the elements of $\mathcal{H}^2(\mathcal{D})$. At first glance, it appears that obtaining such a characterization should not be difficult. Let $P^\wedge(z,\cdot)$ be as defined at the end of §1. If $f \in \mathcal{L}_i$, then $f^\wedge$ is supported in $\mathcal{V}_i$. Thus

$$\begin{aligned} F(z) &= \int_M f(m)P(z,m)\,dm \\ &= \int_{M^*} f^\wedge(\lambda)P^\wedge(z,\lambda)\,dl \\ &= \int_{\mathcal{V}_i^*} f^\wedge(\lambda)P^\wedge(z,\lambda)\,dl \end{aligned}$$

This defines a Hua-harmonic function for all f in $\mathcal{L}_i$ if and only if

$$z \to P^\wedge(z,\lambda)$$

is Hua-harmonic for all $\lambda \in \mathcal{V}_i^*$. Actually, it is easily seen that for all $s \in S$

$$P^\wedge(\rho(s)z,\lambda) = P^\wedge(z,\rho^*(s)\lambda)$$

Thus, the orbit will be harmonic if $P^\wedge$ is harmonic for any single λ in the orbit. Hence, we arrive at the following theorem.

Theorem 4.2. *The orbit $\mathcal{V}_i$ is harmonic if and only if there is a $\lambda_i \in \mathcal{V}_i$ such that $z \to P^\wedge(z,\lambda_i)$ is harmonic for the Hua system.*

Unfortunately, there does not seem to be a simple formula for $P^\wedge$. Hence, we do not yet have a simple criterion ot the harmonicity of a given orbit. This, hopefully, will be the subject of further work on this problem.

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