POISSON INTEGRALS FOR HOMOGENEOUS, RANK 1 KOSZUL MANIFOLDS

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Section 0. *Introduction*

Let X be a complex, n-dimensional manifold. We shall say that X is homogeneous under the real analytic Lie group G if X is a homogeneous G-space for which the mapping $\nu : G \times X \to X$ is real analytic in the G variable and holomorphic in the X variable. Suppose that there is a G-invariant volume form ω on X. In local coordinates, we may express ω as

$$(i/2)^n K(z,\overline{z}) dz_1 \wedge \ldots \wedge dz_n \wedge d\overline{z}_1 \wedge \ldots \wedge d\overline{z}_n.$$

Homogeneity implies that K is strictly positive. In [Kl], Koszul introduced the following form which we refer to as the Koszul form:

$$g = (i/2)^n \sum \frac{\partial^2}{\partial z_i \partial \overline{z}_j} \log K dz_i d\overline{z}_j. \tag{1}$$

This form is bi-linear and satisfies

$$g(Z,W) = g(\overline{W},\overline{Z})$$

for all complex vector fields Z and W. We let H be the Hermitian form defined by

$$H(Z,W) = g(Z,\overline{W}).$$

The form g is invariant under any bi-holomorphic mapping which preserves the measure μ. We shall say that X is a Koszul manifold if g is non-degenerate. This gives X the structure of a pseudo-Kähler manifold for which the measure preserving bi-holomorphisms are isometries. The pseudo-Kähler form is defined by

$$\phi(U,V) = g(JU,V).$$

Examples of Koszul manifolds include all bounded homogeneous domains (and thus, all Hermitian symmetric spaces). In fact, all pseudo-Hermitian symmetric spaces are Koszul. There are also many lesser known examples. Even in rank one, the class of homogeneous Koszul manifolds seems to be so large as to defy precise classification, although much is known. (See [P2].) That rank one Koszul manifolds

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exist in such abundance is somewhat surprising, considering that of course, the only rank one Hermitian symmetric spaces are the unit balls in $\mathbf{C}^n$.

This work is based upon the belief that the class of homogeneous Koszul manifolds represents a natural generalization of the class of Hermitian symmetric spaces. It seems interesting to ask to what extent results obtained for symmetric spaces are true for general Koszul manifolds. Specifically, we are interested in attempting to understand to what extent harmonic functions, and their generalizations, may be described in terms of their boundary values. Ideally, we would like an analogue of the solution to the Helgason conjecture obtained in [KKMOOT].

At the moment, this ideal seems quite far away. Just to formulate a Helgason conjecture, one needs to define an appropriate concept of boundary for such manifolds. One must also define and understand the appropriate class of differential operators. In rank one, however, many of these technical difficulties disappear. The class of operators seems apparent. Corresponding to the pseudo-Kähler structure there is a canonical, second order differential operator Δ, the Laplace-Beltrami operator. (See below for the definition.) For a rank one symmetric space, this operator generates the algebra of invariant differential operators. The 'harmonic functions' one studies are the eigenfunctions for this differential operator.

There is also a natural concept of boundary for a rank one Koszul manifold X. In [P2], we showed that X could be realized as a domain Ω in $\mathbf{C}^n$. The specific realization will be described in §2 below. However, let us mention a few of the general features of this realization. Since X is rank one, there is a group $G = AN$ which acts transitively on X where N is nilpotent and A is a one parameter group which normalizes N. In the realization of [P2], N has one dimensional center and acts transitively on the topological boundary of Ω. Thus, the boundary may be identified with N/R where R is a closed subgroup of N. It will be seen below that the domain Ω may be identified with AN/R. Furthermore, there is an isomorphism of A with $\mathbf{R}^+$ such that approaching the boundary is equivalent with approaching 0 in $\mathbf{R}^+$. It turns out that relative to this boundary, Δ has regular singularities in the sense of [O] (or [OS]). This allows us to define the boundary value of any eigenfunction (or more generally, eigenhyperfunction) provided the eigenvalues stay away from a certain 'singular set'.

The goal of this work is to study the eigenvalue problem for Δ in the case that X is rank one. It turns out that off of the singular set, the boundary map is one-to-one. We would like to be able to describe its image and to compute an explicit inverse (the Poisson kernel). In this work we achieve this goal, modulo certain restrictions. The Poisson kernel is explicitly given in terms of the exponential of a certain pseudo-differential operator $\tilde{\Box}$ on N. Whenever these exponentials exist and are explicitly computable, and when the eigenvalues stay away from the 'singular set', then the Poisson kernel is explicitly computable as well. More precisely, suppose that X is identified with AN/R as in the above paragraph and A is appropriately identified with $\mathbf{R}^+$. Let us denote the general element of N/R by n and the general element of A by y. Let $\lambda \in \mathbf{C}$ have $\operatorname{re} \lambda \leq 0$. Consider the following integral:

$$T_\lambda(f)(y,n) = y^{(\lambda+n-2)/2} \int_{-\infty}^{\infty} e^{i\theta\tilde{\Box}} f((\exp(-xZ_o)n)((xy^{-1})^2+1)^{(\lambda-1)/2} dx \qquad (2)$$

where $\theta = \tan^{-1}(xy^{-1})$, Z_o is a basis for the center of the Lie algebra of N and where

$\tilde{\Box}$ acts in the n-variable. We prove that (under the above mentioned restrictions) this formula makes sense for all f in a dense subset of $L^2(N/R)$ and defines an eigenfunction for Δ with eigenvalue $(\lambda^2 - n^2)/4$. Furthermore, the boundary value of $T_\lambda(f)$ turns out to be the function

$$\lambda D^{-\lambda}(f)$$

where

$$D^\lambda = \pi^{-1} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos t)^{\lambda-1} e^{-it\tilde{\Box}} dt.$$

(We describe how to define this integral for the values of λ of interest.) The Poisson kernel operator then is the operator

$$(\lambda D^{-\lambda})^{-1} T_\lambda.$$

We also describe how to compute $(\lambda D^{-\lambda})^{-1}$. There is also a description of the class of solutions so obtained (the so called 'moderately growing' solutions.) The singular set is described in formula25n below where σ may be taken to be the representation of $\mathbf{R}$ defined by exponentiating $-i\tilde{\Box}$.

Practical application of our results would require not only the ability to exponentiate the $\tilde{\Box}$ operators but also a detailed knowledge of their spectrum (in order to determine the singular set). Remarkably, it turns out that in most cases of interest, this information is available. In fact, these operators were studied in [P1] where the required information was worked out in considerable detail. Thus, the present work, together with [P1], amounts to a complete solution of the eigenvalue problem for Δ for large classes of groups. Even for groups for which these exponentials are not explicitly computable, our techniques yield considerable information.

Our results also, however, are considerably less precise and less general than what is obtained in the Hermitian symmetric case. The space of boundary values we work with is a space of rapidly decreasing C^∞ functions. ($T_N^{-1}(\mathcal{L}_+)$ in the notation of Corollary27n below.) Certainly, one can do much better. However, this is a topic for further investigations at some later date. It is probable, however, that some restriction on the boundary values is to be expected. The boundary we use is not a precise analogue of the boundary of a symmetric space. Our realization of X is always non-compact. Thus, for example, we study the upper-half plane instead of the unit disc. Our boundary would then be $\mathbf{R}$ instead of $\mathbf{T}$. This has the consequence that in the general theory, one must control the regularity at the infinite point. One can define compactifications for the spaces in question. However, the transformation defined in §3 below does not seem to exist on the compactification.

Our calculations are all based on a fascinating relationship between the Casimir operator on $\mathrm{Sl}(2,\mathbf{R})$ and the operator Δ. Explicitly, in §3 we define a unitary transformation which transforms Δ into the Casimir operator of the universal cover of $\mathrm{Sl}(2,\mathbf{R})$ acting on an infinite dimensional Hilbert bundle over the upper half plane. We then proceed to solve the eigenvalue problem for this operator by generalizing the methods of [O]. That such a transformation would exist is somewhat remarkable in that, in general, $\mathrm{Sl}(2,\mathbf{R})$ does not act in any natural way on X. In fact, there exist very simple examples of X with solvable isometry groups. Presumably, in higher rank, other semi-simple groups would play a role.

Section 1. *Vector Valued Boundary Values*

In this section, we describe the modifications of the boundary theory developed in [O] necessary to deal with solutions valued in infinite dimensional vector spaces. All of our arguments are motivated by the arguments given in [O]. We should note that we deal solely with the C^∞ category while [O] treats primarily the analytic category.

Let $\mathcal{V}$ be a countably normed, complex Frechet space. Let $\Omega \subset \mathbf{R}$ be an open set and let $\mathcal{D}(\Omega, \mathcal{V})$ be the space of compactly supported, $\mathcal{V}$-valued C^∞ functions on Ω, given its usual topology. If $K \subset \mathbf{R}$ is compact, we define $\mathcal{D}(K, \mathcal{V})$ to be the direct limit of the spaces $\mathcal{D}(\Omega, \mathcal{V})$ where $K \subset \Omega$. By a $\mathcal{V}$ valued distribution on Ω, we mean an element of $\mathrm{End}(\mathcal{D}(\Omega, \mathbf{C}), \mathcal{V})$ (The space of continuous, linear mappings of $\mathcal{D}(\Omega, \mathbf{C})$ into $\mathcal{V}$.) We denote this space by $\mathcal{D}'(\Omega, \mathcal{V})$. In general, we shall endow all such 'End' spaces with the strong topology (uniform convergence on bounded sets). This space is a module over both $C^\infty(\Omega)$ and $\mathrm{End}(\mathcal{V})$.

The space $C^\infty(\Omega, \mathcal{V})$ injects into $\mathcal{D}'(\Omega, \mathcal{V})$ in the same manner that C^∞ functions inject into the space of ordinary distributions. The following may be proved by a convolution argument in much the same way as in the standard case:

Lemma 1. *$C^\infty(\Omega, \mathcal{V})$ is dense in $\mathcal{D}'(\Omega, \mathcal{V})$.*

Let $\mathcal{W}$ be a complete, locally convex topological vector space. There is a natural mapping of $\mathcal{D}'(\Omega, \mathcal{V}) \times C^\infty(\Omega, \mathrm{End}(\mathcal{V}, \mathcal{W})) \to \mathcal{D}'(\Omega, \mathcal{W})$ which we would like to describe. Let $F \in \mathcal{D}'(\Omega, \mathcal{V})$. For functions of the form ϕA where $\phi \in C^\infty(\Omega, \mathbf{C})$ and $A \in \mathrm{End}(\mathcal{V}, \mathcal{W})$ we define

$$< \psi, (\phi A)F >= A < \phi\psi, F > .$$

for all $\psi \in \mathcal{D}(\Omega, \mathbf{C})$. To extend this definition to all of $C^\infty(\Omega, \mathrm{End}(\mathcal{V}, \mathcal{W}))$, we note that according to Theorems 50.1 and 44.1 of [T], $C^\infty(\Omega, \mathrm{End}(\mathcal{V}, \mathcal{W}))$ is isomorphic with $C^\infty(\Omega, \mathbf{C})\hat{\otimes}_\pi \mathrm{End}(\mathcal{V}, \mathcal{W})$. Furthermore, Proposition 43.4, loc. cit., guarantees that the map $\phi A \to \phi A F$ in fact extends to the whole tensor product and hence to all of C^∞, as desired.

It is a consequence of the above comments, that we may also 'apply' elements F of $\mathcal{D}'(\Omega, \mathcal{V})$ to elements ψ of $\mathcal{D}(\Omega, \mathrm{End}(\mathcal{V}, \mathcal{W}))$ producing elements of $\mathcal{W}$. Explicitly, we define

$$< \psi, F >=< \eta, \psi F >$$

where $\eta \in \mathcal{D}(\Omega, \mathbf{C})$ is one on the support of ψ. It is easily seen that this does not depend upon the choice of η.

Let $A \subset \Omega$ be relatively closed and let $u \in \mathcal{D}'(\Omega, \mathcal{V})$. We say that u is supported in A if u is zero on $\Omega \setminus A$. The smallest such A is called the support of u. The space of all distributions whose support is contained in a given set B is denoted by $\mathcal{D}'_B(\Omega, \mathcal{V})$. The following simple lemma is crucial to the definition of the boundary values.

Lemma 2. *Let $0 \in \Omega$ and let $u \in \mathcal{D}'_{\{0\}}(\Omega, \mathcal{V})$. Then, there is a finite sequence of vectors $v_i \in \mathcal{V}$ such that*

$$u = \sum v_i \otimes \delta_i$$

where δ_i is the i^{th} derivative of the Dirac delta function δ_0 at 0.

Proof We shall let t denote the coordinate function on $\mathbf{R}$. Let $\phi \in \mathcal{D}(\Omega, \mathbf{C})$ be equal to one on a neighborhood of 0. Define

$$v_i = (-1)^i < t^i\phi, u > /i!.$$

Let $w \in \mathcal{V}'$ be given. Since the lemma is known for $\mathbf{C}$ valued distributions, we can say that there is an N such that w is orthogonal to v_n for all $n \geq N$ and

$$<< \psi, u >, w >= \sum_1^N < \psi, \delta_i >< v_i, w > .$$

for all $\psi \in \mathcal{D}(\Omega, \mathbf{C})$. It follows from the Banach-Steinhaus theorem that the sequence of partial sums of $v_i < \psi, \delta_i >$ is bounded in $\mathcal{V}$. In particular, $\{v_i < \psi, \delta_i >\}$ is a bounded set. However, this implies that the set $\{c_i v_i\}$ is bounded for any choice of constants c_i. This is only possible if the sequence v_i is finite. This proves the lemma.

Let $f : (0, a) \to \mathcal{V}$ be given. We shall say that f has moderate growth if there is an $\alpha \in \mathbf{R}$ such that $t^\alpha f(t)$ is bounded as $t \to 0^+$.

Lemma 3. *Let $u : (0, a) \to \mathcal{V}$ be C^∞ and have moderate growth. Then there is a $v \in \mathcal{D}'((-\infty, a), \mathcal{V})$ which is supported in $[0, a)$ and which agrees with u on $(0, a)$.*

Proof There is an $n \in \mathbf{N}$ such that $v(t) = t^n u(t)$ is bounded near $t = 0$. It follows that the lemma is true for v. The desired distribution is obtained by division by t^n. This is possible since the image of the operator defined by multiplication by t^n is closed and complemented in $\mathcal{D}((-\infty, a), \mathcal{V})$. (The complement is spanned by functions of the form $t^k\phi v_k$ for integers $0 \leq k < n$ and $v_k \in \mathcal{V}$ and where ϕ is a fixed 'cut off' function as in the proof of Lemma2n.) This finishes the proof.

Let (b, a) be some open interval in $\mathbf{R}$ containing 0. By a differential operator on $\mathcal{D}((b, a), \mathcal{V})$ we mean an operator of the form

$$P = \sum_{k=0}^m A_k(t)(\frac{d}{dt})^k$$

where the $A_k \in C^\infty((b, a), \mathrm{End}(\mathcal{V}))$.

Let $\Theta = t\dfrac{d}{dt}$. Following [O], we shall say that P has regular singularities at $t = 0$ if it can be expressed in the form

$$P = \sum_{k=0}^m D_k(t)\Theta^k \tag{3}$$

where the $D_k \in C^\infty((b, a), \mathrm{End}(\mathcal{V}))$ and $D_k(0) = c_k I$ for all $0 \leq k \leq m$ where $c_k \in \mathbf{C}$ and $c_m \neq 0$. The indicial polynomial of P is the polynomial

$$\sigma(P)(s) = \sum_{k=0}^m c_k s^k.$$

The roots of $\sigma(P)(s)$ are denoted s_ν and are referred to as characteristic exponents. (Note: In [O], it is assume that the coefficients are analytic. Dealing with the more general case, however, adds essentially no additional complications.)

The following is a version of Lemma 3.2 of [O]. The proof is essentially the same as that in [O]. However, we shall include the proof for sake of completeness.

Theorem 4. *Let P be a differential operator on $\mathcal{D}((b,a),\mathcal{V})$ with regular singularities. Suppose that $s_\nu \notin -\mathbf{N}$. Then P defines a linear isomorphism of $\mathcal{D}'_{\{0\}}((b,a),\mathcal{V})$ onto itself.*

Proof We note that we may express P in the form

$$P = \sigma(P)(\Theta) + tR \tag{4}$$

where R is expressible in the form

$$R = \sum_{k=0}^{m} E_k(t)\Theta^k \tag{5}$$

with C^∞ coefficients E_k.

Now, let $u = \sum_0^n u_i \otimes \delta_i$ be an element of $\mathcal{D}'_{\{0\}}((b,a),\mathcal{V})$. Let $v = \sum_0^n v_i \otimes \delta_i$ be another such element and consider the equation $Pu = v$. Clearly,

$$P(u_n \otimes \delta_n) = \sigma(P)(-(n+1))(u_n \otimes \delta_n) + ...$$

where '....' refers to terms involving δ_k with $k < n$. For $Pu = v$, clearly we require $u_n = (\sigma(P)(-(n+1))^{-1} v_n$. Furthermore, for this choice of u_n, $v - P(u_n \otimes \delta_n)$ will only involve δ_k with $k < n$. Thus, we may continue by induction to uniquely determine u so that $P(u) = v$.

Corollary 5. *Let $u \in \mathcal{D}'((0,a),\mathcal{V})$ satisfy $Pu = 0$. Suppose that u extends to an element v of $\mathcal{D}'((b,a),\mathcal{V})$ which is supported in $[0,a)$. Then there is a unique such extension $\tilde{v}$ which satisfies $P\tilde{v} = 0$.*

Proof Pv is supported at $\{0\}$ and hence defines an element of $\mathcal{D}'_{\{0\}}((b,a),\mathcal{V})$. From the above theorem, there is a unique $h \in \mathcal{D}'_{\{0\}}((b,a),\mathcal{V})$ with $Ph = Pv$. The desired extension of u is $\tilde{u} = v - h$. The uniqueness follows from the injectivity in Theorem4n.

Now, to define the boundary value, suppose that P is some differential operator on $\mathcal{D}(b,a)$ with regular singularities at 0. We consider one particular characteristic exponent s_i and assume that $s_i - s_j \notin \mathbf{N}$ for all j. Let $u \in \mathcal{D}'((0,a),\mathcal{V})$ satisfy $Pu = 0$. Assume that $t^{-s_i}u$ has an extension v as in the hypothesis of the above corollary.

Since s_i are roots of $\sigma(P)$, there exists a differential operator Q_i (with C^∞ coefficients) which satisfies

$$t^{-s_i} P t^{s_i} = tQ_i. \tag{6}$$

Let $P_i = tQ_i$. Then P_i satisfies the hypotheses of Theorem4n. Moreover, on $(0,a)$, $P_i t^{-s_i} u = t^{-s_i} P_i u = 0$. The distribution $v_i = t^{-s_i} u$ satisfies the hypotheses of

Corollary5n above relative to P_i. Let $\tilde{v}_i$ be the solution to $P_i\tilde{v}_i = 0$ described in Corollary5n. Then $tQ_i\tilde{v}_i = 0$ implies that $Q_i\tilde{v}_i = v_0 \otimes \delta_0$ for some unique $v_0 \in \mathcal{V}$. We define

$$BV_i(u) = v_0$$

As a partial justification of this definition, we have the following version of Theorem 3.5 of [O].

Theorem 6. *Let P have regular singularities. Suppose that for all $i \neq j$, $s_i - s_j \notin \mathbf{Z}$. Let $u : (0, a) \to \mathcal{V}$ be a solution to $Pu = 0$ which is expressible as*

$$u(t) = \sum_{1}^{n} u_j(t)t^{s_j}$$

where the u_j are C^∞ on a neighborhood of 0 in $\mathbf{R}$. Then the boundary value of u corresponding to s_i is

$$BV_i(u) = \sigma(P)'(s_i)u_i.$$

Proof It of course suffices to assume that $i = 1$. Replacing u by $t^{-s_1}u$ and P by $t^{-s_1}Pt^{s_1}$, allows us to assume that $s_1 = 0$. In particular, there is a differential operator Q with C^∞ coefficients such that $P = tQ$.

Let $P_j = t^{-s_j}Pt^{s_j}$. Then P_j has C^∞ coefficients and on $(0, a)$,

$$0 = Pu = \sum_{1}^{n} t^{s_j}P_ju_j.$$

Lemma 7. *Each P_ju_j vanishes to infinite order at $t = 0$.*

Proof Assume that each P_ju_j has been shown to vanish up to order k_j. Then there are C^∞ functions v_j such that $t^{s_j}P_ju_j = t^{r_j}v_j$ where $r_j = k_j + s_j$. Let r be the minimum of the real parts of the r_j. We claim that if $\operatorname{re} r_l = r$, then $v_l(0) = 0$. This will allow us to write $v_l = t\tilde{v}_l$ where $\tilde{v}_l$ is C^∞. This increases the value of r. Thus, our result will follow by induction on r.

To show our claim, let $\operatorname{re} r_l = r$. We note that on $(0, a)$,

$$0 = \sum t^{r_j - r_l}v_j.$$

Let $\alpha_j = r_j - r_l$. Letting t tend to zero, we see that

$$\lim_{t \to 0} \sum_{\operatorname{re} \alpha_j = 0} t^{\alpha_j}v_j(0) = 0.$$

The α_j are distinct and purely imaginary. Such a limit can be zero only if all of the $v_j(0)$ in the sum are zero. (This is a simple 'uniform distribution' argument which we leave to the reader.) This finishes the proof of Lemma7n.

To continue with the proof of Theorem6n, let $v : [0, a) \to \mathcal{V}$ be C^∞ on some open set containing $[0, a)$. For $\operatorname{re} s > -1$, we may define a $\mathcal{V}$-valued distribution vt_+^s on $(-\infty, a)$ and supported in $[0, a)$, by integration against $v(t)t^s$ over $[0, a)$.

By analytic continuation, this distribution may be defined for all s which are not equal to a negative integer. (See [GS], vol. 1, (1.3)). Let $\tilde{u}_j = u_j t_+^{s_j}$ and set

$$\tilde{u} = \sum_1^n \tilde{u}_j.$$

Then $P\tilde{u}$ is supported at 0. Furthermore, since $s_1 = 0$, each s_j is non-integral, $j > 0$. It follows that $\Theta\tilde{u}_j = t_+^{s_j}(\Theta + s_j)u_j$. More generally, $P\tilde{u}_j = t_+^{s_j}(P_j u_j)$. Thus, from Lemma7n, $P\tilde{u}$ is C^∞ and hence vanishes identically. It follows that $BV_1(u)$ is computable from $Q\tilde{u}$ where $P = tQ$.

To compute this, we apply formula4n. Since $s = 0$ is a root of $\sigma(P)$, we may write $\sigma(P)(s) = sb(s)$ for some polynomial b. Then

$$Q = \frac{d}{dt} b(\Theta) + R.$$

Note that for all s not equal to a negative integer, $\Theta t_+^s = s t_+^s$. Note also that t_+^0 is the Heavyside function. It follows that $Q\tilde{u}_1 = b(0)u_1(0) \otimes \delta_0 + h_1 t_+^0$ where h_1 is C^∞ near 0. Also, it is easily verified that $Q\tilde{u}_j = h_j t_+^{s_j - 1}$ for some C^∞ function h_j. Then $tQ\tilde{u} = 0$ shows that

$$h_1 t_+^1 + \sum h_j t_+^{s_j} = 0.$$

It follows (from the argument in Lemma7n)that

$$h_1 t_+^0 + \sum h_j t_+^{s_j - 1} = 0.$$

Thus, $Q\tilde{u} = b(0)u_1(0) \otimes \delta_0$, proving the theorem.

In addition to the above results we shall require a 'change of variables' formula. Let P be as in Theorem 5. Assume that $\mathcal{V} = C^\infty(\Omega_1, \mathcal{V}_1)$ where Ω_1 is an open subset of $\mathbf{R}^n$ and $\mathcal{V}_1$ is a countably normed Frechet space.

Suppose now that we are given a diffeomorphism τ mapping $\tilde{X} = (-\epsilon, \epsilon) \times \Omega_1$ onto an open subset of $\mathbf{R} \times \Omega_2$ where $\Omega_2 \subset \mathbf{R}^n$ is open. Let the coordinates of $\tilde{X}$ be denoted (t, x) and let

$$\tau(t, x) = (\tau_1(t, x), \tau_2(t, x)). \tag{7}$$

We assume:

(1) There is a strictly positive C^∞ function b on $\tilde{X}$ such that $\tau_1 = tb$.
(2) For all $x \in \Omega_1$, the function $\tau_x = \tau_1(\cdot, x)$ is injective on $(-\epsilon, \epsilon)$.
(3) There is an interval $(-c, c)$ contained in the images of all of the τ_x.
(4) There is a $0 < a \le \epsilon$ such that $\tau : (-a, a) \times \Omega_1 \to (-c, c) \times \Omega_2$.

We define a C^∞ function $h : (-c, c) \times \Omega_1 \to \mathbf{R}$ by

$$h(y, x) = \tau_x^{-1}(y). \tag{8}$$

We let $X = (-a, a) \times \Omega_1$ and $Y = (-c, c) \times \Omega_2$. Let X^+ be defined as the set of elements in X where $t > 0$. We denote the general element of Y by (y, z) and define Y^+ to be the subset defined by $y > 0$. From (1) above, τ maps X^+ into Y^+.

Let $\mathcal{W} = C^\infty(\Omega_2, \mathcal{V}_1)$. Observe that $C^\infty((-c,c),\mathcal{W}) = C^\infty(Y,\mathcal{V}_1)$ and $C^\infty((-a,a),\mathcal{V}) = C^\infty(X,\mathcal{V}_1)$. ([T], Theorem 40.1 and Corollary 1, p.415). Thus, composition with τ defines a continuous linear transformation T from $C^\infty((-c,c),\mathcal{W})$ to $C^\infty((-a,a),\mathcal{V})$. Clearly, T has dense image since, for example, $\mathcal{D}(X,\mathcal{V}_1)$ is contained in the image. Similar comments hold with $(-c,c)$ replaced by $(0,c)$ and $(-a,a)$ replaced by $(0,a)$. We refer to the resulting transformation as T^+. Finally, we shall let T_0 denote the operator from $\mathcal{W}$ into $\mathcal{V}$ defined by composition with $x \to \tau_1(x,0)$. The following proposition is crucial.

Proposition 8. *T has a unique extension to a continuous mapping of $\mathcal{D}'((-c,c),\mathcal{W})$ into $\mathcal{D}'((-a,a),\mathcal{V})$. This extension maps distributions supported in $[0,c)$ onto distributions supported in $[0,a)$.*

Proof

Let $u \in C^\infty(Y,\mathcal{V}_1))$ and let $\psi \in \mathcal{D}((-a,a),\mathbf{C})$. We extend ψ to all of $\mathbf{R}$ by requiring ψ to be 0 off of $(-a,a)$. Then

$$< \psi, T(u) > (x) = \int_{-a}^{a} u(\tau_1(t,x),\tau_2(t,x))\psi(t)dt.$$

We make the change of variables $t = h(y,x)$ where h is as in formula8n. Then, the integral may be expressed

$$< \psi, T(u) > (x) = \int_{-c}^{c} u(y,\tau_2(t,x))\psi(h(y,x))h_y(y,x)dy.$$

Now, we define a function $A \in C^\infty((-c,c),\mathrm{End}(\mathcal{W},\mathcal{V}))$ by setting, for $y \in (-c,c)$ and $f \in \mathcal{W}$,

$$A(y)f(x) = f(\tau_2(h(y,x),x)).$$

We also define a mapping $B_\psi \in \mathcal{D}((-c,c),\mathrm{End}(\mathcal{V}))$ by

$$B_\psi(y)g(x) = \psi(h(y,x))h_y(y,x)g(x).$$

where $g \in \mathcal{V}$.

Then, using the notation defined after Lemma1n,

$$< \psi, T(u) > = < B_\psi, Au > . \tag{9}$$

This formula makes sense for any $u \in \mathcal{D}'((-c,c),\mathcal{W})$ and defines a $\mathcal{V}$ valued distribution. The extension is unique due to Lemma1n. The statement about supports is clear. This proves the proposition.

Now, suppose that there is a differential operator P' on $\mathcal{D}((-c,c),\mathcal{W})$ with regular singularities such that

$$PT = TP'.$$

Lemma 9. $\sigma(P) = \sigma(P')$.

Proof Let the coordinates of Y be (y, z) where $y \in (-c, c)$. Let $\Theta' = y\dfrac{\partial}{\partial y}$. It is easily verified that

$$T\Theta' = tbT\frac{d}{dy} = (tbb^{-1}\frac{d}{dt} + tR)T = (\Theta + tR)T$$

where R is given by an expression such as formula5n with $m = 1$. Hence, expressing P' in a form similar to formula4n,

$$PT = TP' = T(\sigma(P')(\Theta') + yR') = (\sigma(P')(\Theta) + tR'')T$$

where again R'' is as in formula4n. Our result follows from the density of the image of T.

Now we come to the change of variables theorem. Let P and P' be as above and let s be a fixed characteristic exponent for P' (and hence for P). We shall denote the respective boundary maps by BV' and BV.

Theorem 10. *Let $P'u = 0$ where $u \in C^\infty((0, c), \mathcal{W})$ has moderate growth. Then T^+u has a well defined boundary value and*

$$BV(T^+u) = b_0^s T^0 BV'(u)$$

where b_0^s denotes multiplication by the function $b_0^s(x) = (b(0, x))^s$ in $C^\infty(\Omega_1, \mathcal{V}_1)$.

Proof We begin by applying formula6n to P', concluding that there is a differential operator Q' such that

$$y^{-(s+1)}P'y^s = Q'.$$

Then

$$TQ' = b^{-(s+1)}t^{-(s+1)}Pt^s b^s T.$$

We conclude that there is a differential operator $\tilde{Q}$ such that $TQ' = \tilde{Q}T$. We repeat the above sequence of arguments, beginning with the equality $P' = y^{s+1}Q'y^s$, concluding that

$$P = t^{s+1}b^{s+1}\tilde{Q}b^{-s}t^{-s}.$$

Comparing with formula6n, we see that

$$Q = b^{s+1}\tilde{Q}b^{-s}.$$

Now, let $P'u = 0$ where u has moderate growth. From Lemma3n and Corollary5n, the function $y^{-s}u$ extends to a unique $\mathcal{W}$ valued distribution $\tilde{u}$ supported in $[0, c)$ satisfying

$$yQ'\tilde{u} = 0.$$

Then,

$$0 = b^s T(yQ'\tilde{u}) = tb^{s+1}\tilde{Q}T\tilde{u} = tQb^s T\tilde{u}.$$

Of course, $b^sT^+y^{-s}u = t^{-s}T^+u$. Hence $b^sT\tilde{u}$ extends t^sT^+u. We conclude that T^+u has a well defined boundary value which is defined by the equality

$$Qb^sT\tilde{u} = BV(T^+u)\otimes\delta_0.$$

On the other hand

$$T(BV'(u)\otimes\delta_0) = T(Q'\tilde{u}) = b^{-(s+1)}Qb^sT\tilde{u} = b^{-(s+1)}(BV(T^+u)\otimes\delta_0).$$

Our result follows from this and the observation that for all $u_o \in \mathcal{W}$

$$T(u_o\otimes\delta_0) = b_o^{-1}T^0u_o\otimes\delta_0.$$

(See formula9n.)

Section 2. *The Laplace-Beltrami operator*

In this section, unless otherwise stated, we shall adopt the convention that Lie groups will be denoted by upper case Roman letters and the corresponding Lie algebra will be denoted by the corresponding upper case script letter. We shall also assume that all Lie groups explicitly stated as being nilpotent are realized so that the exponential mapping is the identity mapping.

We continue the notation established in the introduction. We begin by recalling the general definition of the Laplace Beltrami operator. We consider g as a real two-tensor on the bundle $T(X)$. Let $T^*(X)_c$ be the complex cotangent bundle for X. By duality, H gives rise to a Hermitian scalar product H^* on this bundle. Let

$$T^*(X)_c = T^*(X)^{0,1}\oplus T^*(X)^{1,0}$$

be the decomposition of the complex cotangent bundle into types. Let $\mathcal{E}$ (resp. $\mathcal{E}^{0,1}$, $\mathcal{E}^{1,0}$) be the spaces of compactly supported C^∞ sections of these bundles. If σ and τ belong to $\mathcal{E}$, we define

$$<\sigma,\tau> = \int_{G/R} H^*(\sigma,\tau)(g)\,dg.$$

where dg is the invariant measure on G/R. (This exists because R is nilpotent, and hence, unimodular.) It is easily verified that this is a non-degenerate pairing. Furthermore, this form remains non-degenerate when restricted to sections of $\mathcal{E}^{0,1}$. Let $\overline{\partial} : C_c^\infty(G/R)\to\mathcal{E}^{0,1}$ be the usual antiholomorphic differentiation and let $\overline{\partial}^*$ be its formal adjoint. We define $\Delta : C^\infty\to C^\infty$ by $\Delta = -\overline{\partial}^*\overline{\partial}$. More explicitly,

$$-<\Delta\sigma,\tau> = <\overline{\partial}\sigma,\overline{\partial}\tau> \tag{10}$$

Then Δ is the 'Laplace-Beltrami' operator for the manifold. In general, this operator will be non-elliptic and non-hypoelliptic, due to the fact that H, while non-degenerate, will usually be non-definite.

We wish to explicitly compute Δ in the rank one context. We shall require the basic structure theory of rank one Koszul manifolds as developed in [P2]. Explicitly, we showed in [P2] that every rank one, homogeneous Koszul domain is realizable as a homogeneous nil-ball. To recall the definition, let N be a nilpotent Lie group with Lie algebra $\mathcal{N}$. Let $\lambda\in\mathcal{N}^*$. A complex subalgebra $\mathcal{P}'\subset\mathcal{N}_c$ is a totally complex polarization for λ if it satisfies the following properties:

(a) $Z\in\mathcal{P}'$ if and only if $[Z,\mathcal{P}']\subset\ker\lambda$.
(b) $\mathcal{P}'+\overline{\mathcal{P}'} = \mathcal{N}_c$.

Given a totally complex polarization, there is a canonical way of associating a domain with it. Explicitly, we let $\mathcal{P} = \mathcal{P}' \cap \ker\lambda$. By forming a quotient, we may assume that the kernel of λ contains no non-trivial ideals. We refer to this condition as 'effectivity'. In this case, the center $\mathcal{Z}$ of $\mathcal{N}$ is one dimensional and λ is non-trivial on $\mathcal{Z}$. Let Z_o be a basis for the center, normalized by the condition that $\lambda(Z_o) = 1$. Then, clearly,

$$\mathcal{N}_c = \mathcal{N} + i\mathbf{R}Z_o + \mathcal{P}.$$

Let $\mathcal{X} = N_c/P$. From the nilpotency of N_c, this space is bi-holomorphic with $\mathbf{C}^n$ for some n. The above equality implies that the image of N in $\mathcal{X}$ under the quotient map is a real co-dimension one submanifold which divides $\mathcal{X}$ into two connected components

$$\Omega^{\pm} = N(\exp\mathbf{R}^{\pm}Z_o)P/P.$$

We refer to $\Omega^+ = \Omega$ as the domain associated to the polarization. (Note that Ω^- is the domain associated to $(\mathcal{P}, -\lambda)$.) Such domains are what we refer to as the nil-balls. The boundary of each of these domains is smooth and equals N/R where $\mathcal{R} = \mathcal{P} \cap \mathcal{N}$. In fact, we may identify Ω with $N/R \times \mathbf{R}^+$ in an N-equivariant fashion. In particular, N does not act transitively.

However, suppose that $t \to \delta(t)$, $t \in \mathbf{R}^+$, is a one parameter group of semi-simple, $\mathbf{R}$-split automorphisms of $\mathcal{N}$. We shall say that the pair $(\lambda, \mathcal{P}')$ is homogeneous if $\delta(t)$ preserves $\mathcal{P}'$ and $\delta(t)^*(\lambda) = t\lambda$ for all t. In this case, the action of $\delta(t)$ on N_c projects to an action on N_c/P. Furthermore, $\delta(t)(Z_o) = tZ_o$. It is follows that under these assumptions, each of the nil-balls is homogeneous under the group $G = N_s \times \mathbf{R}^+$ where $\mathbf{R}^+$ acts on N by means of δ. In this case, we shall choose z_oP where $z_o = \exp iZ_o$ as our base point. Then the isotropy subgroup is just R. This allows us to identify Ω with $Y = G/R$. The Lie algebra $\mathcal{G}$ of G is $\mathcal{N}_s \times \mathbf{R}$. Note that $\mathbf{R}^+$ normalizes R. The basic structural fact concerning rank one Koszul domains is the following:

Theorem 11. *Let X be a rank one Koszul domain on which a completely solvable Lie group acts transitively. Then X is bi-holomorphic with a dilated nil-ball.*

In passing from algebraic data on $\mathcal{G}$ to geometric data on Y, it will be useful to use the well known description of $T(Y)$ as a homogeneous G-bundle. Explicitly, let τ be any representation of R in a vector space $\mathcal{V}$. We let R act on $G \times \mathcal{V}$ by

$$(g, v)k = (gk, \tau(k^{-1})v)$$

We let π denote the mapping of $G \times \mathcal{V} \to G/R$ defined by the quotient mapping on G. We then define $G \times_\tau \mathcal{V} = (G \times \mathcal{V})/R$. This defines a homogeneous vector bundle over G/R with the projection mapping induced from π. The sections of $G \times_\tau \mathcal{V}$ may be identified with mappings $F : G \to \mathcal{V}$ which satisfy

$$F(gk) = \tau(k^{-1})F(g)$$

for all $g \in G$ and $k \in R$.

The real tangent bundle $T(Y)$ is obtained in this manner using the adjoint representation of R in $\mathcal{Y} = \mathcal{G}/\mathcal{R}$. Clearly, the sections are defined by vector fields on G which satisfy the above formula, modulo $\mathcal{R}$. We may describe the action of such a section on $C^\infty(Y)$. Let us agree that if f is a C^∞ function on G and X is a lie algebra element, then $R(X)f$ is the action of X on f as a left invariant differential operator and $L(X)f$ is the right invariant action of X on f. Hence R is the derivative of the right regular representation and L is the derivative of the left regular representation. Any C^∞ function f on Y may be identified with a right R-invariant function on G. Then the action of F on f is given by

$$(Ff)(g) = R(F(g))f(g).$$

Every Lie algebra element X in $\mathcal{G}$ defines a section of this bundle by the formula

$$X(g) = \text{ ad } g^{-1}X.$$

The corresponding vector field on Y is again denoted by $L(X)$. Now L is the derivative of the left action of G on Y.

The complex tangent bundle is similarly obtained by projection from $\mathcal{G}_c$. Generally speaking, the requirement that a algebraic concept on $\mathcal{G}_c$ define a G-invariant geometric structure on Y is that it project to an ad R invariant object on $\mathcal{Y}_c$.

Our first application of these ideas is to describe the complex structure on $\mathcal{G}/\mathcal{R}$. Since Z_o is central, the real tangent space to Ω at z_o is $\mathcal{N}_c/\mathcal{P}$, thought of as a real vector space. The complex structure is defined by the natural complex structure on $\mathcal{N}_c/\mathcal{P}$. The identification of $\mathcal{G}/\mathcal{R}$ with $\mathcal{N}_c/\mathcal{P}$ is obtained by mapping A into iZ_o and $\mathcal{N}$ into $\mathcal{N}_c/\mathcal{P}$ by means of the identification

$$\mathcal{N}/\mathcal{R} = (\mathcal{N} + \mathcal{P})/\mathcal{P}.$$

From this, the complex structure J on $\mathcal{G}/\mathcal{R}$ may be described as the inverse image of that on $\mathcal{N}_c/\mathcal{P}$. Rather than describe it as an operator, however, it is easier to describe the corresponding decomposition of $\mathcal{Y}_c = \mathcal{G}_c/\mathcal{R}_c$ into the $\pm i$ eigenspaces of J. Let $W_o \in \mathcal{G}_c$ be $W_o = A + iZ_o$. Let $\mathcal{Q}$ be the subalgebra of $\mathcal{G}_c$ spanned over $\mathbf{C}$ by W_o and $\mathcal{P}$. It is easily verified that $\mathcal{Q}$ projects to the $+i$ eigenspace of J in $\mathcal{Y}$. Furthermore, this image is ad R-invariant and defines the bundle of vector fields of type $(0,1)$.

Next, we describe the Koszul form. Let $\beta \in \mathcal{G}'$ be the functional which equals λ on $\mathcal{P}$ and is 0 on A. Let B_β be the bi-linear form on $\mathcal{G}$ defined by $B_\beta(X,Y) = \beta([X,Y])/2$. This form projects to a non-degenerate, ad R-invariant, J-invariant form ϕ on $\mathcal{Y}$. This form defines the Koszul symplectic form on Y. (See Proposition 2 of [P2].) We let $g(X,Y) = \phi(JX,Y)$. Then g defines the pseudo-Riemannian structure on Y. We let g_c be the extension of g to $\mathcal{Y}_c$. The Hermitian form is then defined by

$$H = (g + i\phi)/2.$$

These forms, too, project to $T(Y)$.

The form ϕ has an important invariance principal. Let $\gamma(t) = \exp tA$ in G. Then, ad $\gamma(t)$ is the extension of $\delta(t)$ to $\mathcal{G}$ which is trivial on A. Clearly ad ${}^*\gamma(t)\beta = t\beta$.

This results in the equality

$$\text{ad }^*\gamma(t)B_\beta = tB_\beta.$$

Since ϕ is non-degenerate on $\mathcal{G}/\mathcal{R}$, it follows that

$$\det_{\mathcal{G}/\mathcal{R}} \text{ ad } \gamma(t) = t^n. \tag{11}$$

(Note that n is one half the dimension of $\mathcal{G}/\mathcal{R}$.)

The subbundle $T_{\mathcal{P}}(Y)$ of $T^{0,1}(Y)$ defined by the image of $\mathcal{P}$ will play a special role in our discussions. It is the bundle of vector fields of type $(0,1)$ which are tangent to the leaves of the N-foliation. Let $\mathcal{W}$ be the subspace of $\mathcal{Q}$ spanned by W_o. Since W_o normalizes $\mathcal{R}$, this subspace projects to a R invariant subspace of $\mathcal{Y}_c$ which defines a bundle $T_{\mathcal{W}}(Y)$. Then we have the direct sum decomposition

$$T^{0,1}(Y) = T_{\mathcal{W}}(Y) \oplus T_{\mathcal{N}}(Y).$$

There are analogous decompositions for froms. We write

$$T^{0,1}(Y)^* = T_{\mathcal{W}}(Y)^* \oplus T_{\mathcal{N}}(Y)^*.$$

From the description of the Koszul structure given above, this decomposition is orthogonal with respect to H^*. Thus, we may write $\overline{\partial} = \overline{\partial}_{\mathcal{W}} + \overline{\partial}_{\mathcal{N}}$, where each part has values in the appropriate space of sections of the appropriate bundle. From formula10n and the orthogonality, it follows that

$$\Delta = \Delta_{\mathcal{W}} + \Delta_{\mathcal{N}}$$

where

$$\begin{aligned} \Delta_{\mathcal{W}} &= -\,\overline{\partial}^*_{\mathcal{W}}\overline{\partial}_{\mathcal{W}} \text{ and} \\ \Delta_{\mathcal{N}} &= -\,\overline{\partial}^*_{\mathcal{N}}\overline{\partial}_{\mathcal{N}} \end{aligned}$$

To explicitly describe these operators, we choose a subset $\{Z_1, \ldots, Z_{n-1}\} \subset \mathcal{P}$ consisting of joint eigenvectors of $\delta(t)$, which projects to a basis of $\mathcal{P}/\mathcal{R}_c$. These elements, together with W_o, project to a basis for $\mathcal{Q}/\mathcal{R}_c$. There is a subset $\{W_1, \ldots, W_{n-1}\} \subset \mathcal{P}$ which is dual to the Z_i in the sense that

$$H(Z_i, W_j) = \delta_{i,j}.$$

Note that $H(W_o, W_o) = 1$ and that W_o is orthogonal to the Z_i as well as the W_i. The following simple lemma is left to the reader. The idea of the proof is to prove it at the identity coset, and then to use G-invariance to conclude it in general.

Lemma 12. *Let f and g be C^∞, right R-invariant functions. Then for all $x \in G$,*

$$H^*(\overline{\partial}_{\mathcal{W}} f(x), \overline{\partial}_{\mathcal{W}} g(x)) = R(W_o)f(x)R(\overline{W}_o)\overline{g}(x)$$

$$H^*(\overline{\partial}_{\mathcal{N}} f(x), \overline{\partial}_{\mathcal{N}} g(x)) = \sum_1^n R(Z_i)f(x)R(\overline{W}_i)\overline{g}(x). \tag{12}$$

As a manifold, $G = N \times \mathbf{R}^+$. We shall denote the $\mathbf{R}^+$ coordinate by 'y' and the N coordinate by 'u'. In these coordinates,

$$R(A) = y\frac{\partial}{\partial y}.$$

For $X \in \mathcal{N}$, we shall let $R_b(X)$ be its action on $C^\infty(N)$ as a left invariant vector field. We shall use the same symbol to denote the corresponding action on $C^\infty(G)$ which is independent of the y-variable. Then

$$R(X)f(u,y) = R_b(\delta(y)(X))f(u,y).$$

In formula12n above, the Z_i are all eigenvectors of $\delta(y)$. In fact, the W_i may be chosen to be eigenvectors as well. If Z_i corresponds to the eigenvalue y^α, then W_i corresponds to $y^{1-\alpha}$. Thus, formula12n above may be expressed as

$$H^*(\overline{\partial}_{\mathcal{N}} f, \overline{\partial}_{\mathcal{N}} g)(u,y) = \sum yR_b(Z_i)f(u,y)R_b(\overline{W}_i)\overline{g}(u,y). \tag{13}$$

It follows that there is an operator Δ_b on $C^\infty(N/R)$ such that

$$\Delta_{\mathcal{N}} = y\Delta_b.$$

To compute $\Delta_{\mathcal{W}}$, note that since Z_o is central in $\mathcal{N}$, $R_b(Z_o) = L(Z_o)$. Thus, $R(W_o) = y(\dfrac{\partial}{\partial y} + iL(Z_o))$. Lemma12n then says that:

$$\Delta_{\mathcal{W}} = -R(W_o)^* R(W_o). \tag{14}$$

The adjoint of $R(W_o)$ is easily computed from Lemma13n below.

Lemma 13. *The invariant measure on Ω is $y^{-(n+1)}dydu$ where du is the Haar measure on N/R.*

Proof It suffices to prove that the stated measure is invariant under both the N and the $\mathbf{R}^+$ actions. The N invariance is clear. The action of s in $\mathbf{R}^+$ on the general point (uR, y) in $N/R \times \mathbf{R}^+$ is given by

$$(uR, y) \to (\delta(s)(u)R, ys).$$

We may identify N/R with $\mathcal{N}/\mathcal{R}$ in such a way that the $\delta(s)$ action on N/R is the natural $\delta(s)$ (linear) action on $\mathcal{N}/\mathcal{R}$. This identification preserves the Haar measures. It follows that the Jacobian of the transformation defined by the above formula is $J(s) = s \det_{\mathcal{N}/\mathcal{R}} \delta(s)$. Our lemma will follow if we show that $J(s) = s^{n+1}$. This follows easily from formula11n.

It will be seen in section 3 below that our current realization of Δ is not the most convenient one. Consider the unitary isomorphism U of $L^2(N/R \times \mathbf{R}^+, y^{-(n+1)}dydu)$ into $L^2(N/R \times \mathbf{R}^+, y^{-2}dydu)$ given by

$$Uf(u, y) = y^{-(n-1)/2} f(u, y). \tag{15}$$

For any operator B on $L^2(N/R \times \mathbf{R}^+, y^{-(n+1)}dydu)$, we define $\tilde{B} = UBU^{-1}$. Note that $R(A)\tilde{} = y\dfrac{\partial}{\partial y} + (n-1)/2$. Thus, an easy computation using formula14n shows that

$$\tilde{\Delta} = \tilde{\Delta}_{\mathcal{W}} + \Delta_{\mathcal{N}}$$

where

$$\tilde{\Delta}_{\mathcal{W}} = y^2(\frac{\partial^2}{\partial y^2} + L(Z_o)^2) + \frac{1-n^2}{4} - i(n-1)yL(Z_o). \tag{16}$$

Now, we shall define the space of functions on which we shall study the eigenvalue problem. The pair of operators Δ_b and $L(Z_o)$ commute on $C^\infty(N/R)$. Let $\mathcal{V}$ be the space of joint C^∞ vectors in $L^2(N/R)$ for this pair. This is the largest subspace of $L^2(N/R)$ which contains $C_c^\infty(N/R)$ and is invariant under the closure of both $L(Z_o)$ and Δ_b. We topologize this space in the usual manner. We define $\mathcal{C}$ to be the space of all C^∞ mappings $f : \mathbf{R}^+ \to \mathcal{V}$ such that there is an $\alpha \in \mathbf{R}$ such that $y^\alpha f(y)$ is bounded in $\mathcal{V}$ as $y \to 0$ (the elements of moderate growth). Elements of $\mathcal{C}$ may, of course, be identified with complex valued functions on $\Omega = N/R \times \mathbf{R}^+$. Conversely, Δ may be considered as a $\mathcal{V}$-valued differential operator. It is clear that Δ has regular singularities at $y = 0$ as a $\mathcal{V}$-valued differential operator. Thus, the boundary value of any eigenfunction for Δ in $\mathcal{C}$ will be in $\mathcal{V}$. (See §1 above.)

Similar comments hold for $\tilde{\Delta}$. Clearly, $\sigma(\tilde{\Delta})(s) = \sigma(\Delta)(s + (n-1)/2)$. Thus, there is an obvious correspondence between the characteristic exponents. The operator U of course gives rise to a one-to-one correspondence between the relevant eigenfunctions for Δ and those for $\tilde{\Delta}$. It is trivial to see that the boundary values for corresponding characteristic exponents also correspond under U. Thus, for our purposes, it is entirely equivalent to consider $\tilde{\Delta}$ in place of Δ. Actually, it is also convenient to 'shift' the eigenvalue slightly. To explain this, we note the following:

Proposition 14. *As a $\mathcal{V}$-valued differential operator, the operator $\tilde{\Delta}$ has regular singularities at $y = 0$. The indicial polynomial is $\sigma(\tilde{\Delta})(s) = s(s-1) + (1-n^2)/4$.*

In studying the eigenvalue problem, we introduce the operator

$$\Delta_\lambda = \tilde{\Delta} - (1-n^2)/4 + (1-\lambda^2)/4$$

This operator has indicial polynomial

$$p(s) = (s - (1-\lambda)/2)(s - (1+\lambda)/2)$$

This meets the requirements for defining the boundary values provided that $\lambda \notin \mathbf{Z}$. In this case the boundary value of any eigenfunction from $\mathcal{C}$ is a well defined element of $\mathcal{V}$.

Section 3. *The N-transform*

In this section, we introduce a transformation on $L^2(N/R)$ which yields a transformation of $\tilde{\Delta}$ into the image of the Casimir operator in some representation space of the covering group of $\mathrm{Sl}(2,\mathbf{R})$. Let $f \in L^2(N/R)$. We let the partial Fourier transformation of f in the Z_o direction at λ be denoted by f_λ. Thus

$$f_\lambda(u) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} f((\exp tZ_o)u) e^{-i\lambda t} dt.$$

Then f_λ transforms covariantly under the one parameter group $L = (\exp \mathbf{R} Z_o)R$ with respect to the character χ defined by $\chi(\exp tZ_o u) = \exp -i\lambda t$. For a.e. λ, the function $|f_\lambda|$ is in $L^2(N/L)$ and

$$||f||^2 = \int_{-\infty}^{\infty} ||f_\lambda||^2 d\lambda. \tag{17}$$

We denote the space of all such covariant, $L^2(N/L)$, functions by $\mathcal{H}_\lambda$. We consider this space as a Hilbert space under the obvious norm.

We define, for $\lambda \neq 0$,

$$\nu(f)_\lambda = |\lambda|^{(1-n)/2} f_\lambda \circ \delta(|\lambda|^{-1}).$$

Then, for $\lambda > 0$, $\nu(f)_\lambda$ belongs to $\mathcal{H}_1$ while for $\lambda < 0$, $\nu(f)_\lambda$ belongs to $\mathcal{H}_{-1}$. The mapping of $f_\lambda \to \nu(f)_\lambda$ defines an isometry of $\mathcal{H}_\lambda$ onto $\mathcal{H}_{\pm 1}$. Finally, we define

$$T_N^{\pm} f(u,x) = (2\pi)^{-1/2} \int_0^{\infty} e^{ix\lambda} \nu(f)_{\pm\lambda}(u) d\lambda$$

for $u \in N/R$ and $x \in \mathbf{R}$. This integral exists for a.e. u since, from formula17n and the Fubini theorem, the integrand is in $L^2(\mathbf{R}^+)$ for a.e. u.

The image of $T_N^{\pm}$ is easily described. Let $L^2_+(\mathbf{R})$ be the subspace of $L^2(\mathbf{R})$ consisting of those functions whose Fourier transform is supported in $\mathbf{R}^+$. We identify the Hilbert space tensor product $L^2_+(\mathbf{R}) \otimes \mathcal{H}_{\pm 1}$ with a space of functions on $N/R \times \mathbf{R}$. It is clear that $T_N^{\pm}$ defines a mapping of $L^2(N/R)$ onto the corresponding space. Furthermore, the map

$$T_N : f \to (T_N^+ f, T_N^- f)$$

defines a unitary isomorphism of $L^2(N/R)$ onto $L^2_+(\mathbf{R}) \otimes \mathcal{H}$ where $\mathcal{H} = \mathcal{H}_1 \times \mathcal{H}_{-1}$.

The significance of the operator T_N is that it 'decouples' the operators $L(Z_o)$ and $\Delta_{\mathcal{N}}$. To explain this, note that $\mathcal{H}_\lambda$ may be interpreted as the space of square integrable sections over N/L of the Hermitian line bundle $E = N \times_\chi \mathbf{C}$. The manifold N/L has a complex structure where the space of tangent vectors of type $(0,1)$ is defined by the projection of $\mathcal{P}$ to $\mathcal{N}_c/\mathcal{L}$. This structure is pseudo-Kahlerian, relative to the Hermitian form defined from the projection of the restriction of H to $\mathcal{N}$. In this complex structure, E is a holomorphic line bundle. As was done above, we define the Laplace-Beltrami operator $\Box^{(\lambda)}$ which acts on C^∞ sections of E. We shall denote the operators $\Box^{(\pm 1)}$ by $\Box^{\pm}$, respectively. Let $\Box$ be the unbounded operator on $\mathcal{H}$ defined by $\Box = \Box^+ \times \Box^-$. Then, we have the following:

Proposition 15. *The operator T_N intertwines Δ_b and $i\dfrac{\partial}{\partial x}\otimes\Box$. It also intertwines $L(Z_o)$ and $\dfrac{\partial}{\partial x}\otimes I$.*

Proof The statement about $L(Z_o)$ is clear from the fact that $(L(Z_o)f)_\lambda = i\lambda f_\lambda$.

To prove the other statement, we interpret elements of $L^2(N/R)$ as functions on N as usual. For f and g in $L^2(N/R)$, we have (from formula12n and the Plancherel formula for $\mathbf{R}$),

$$\begin{aligned}-(\Delta_b f, g) &= \int_{N/R} R_b(Z_i)f(u)R_b(\overline{W}_i)\overline{g}(u)du\\ &= \int_{-\infty}^{\infty}\int_{N/L} R_b(Z_i)f_\lambda(u)R_b(\overline{W}_i)\overline{g}_\lambda(u)dud\lambda.\end{aligned}$$

We write this integral as a sum of an integral over $\lambda > 0$ and one over $\lambda < 0$. We then change λ to $-\lambda$ in the second integral. We then make a changes of variables in each integral, replacing u by $\delta(\lambda^{-1})(u)$. Arguing as in the proof of formula13n, we see that

$$-(\Delta_b f, g) = \sum_{\pm}\int_0^{\infty}\int_{N/L}\lambda R_b(Z_i)\nu(f)_{\pm\lambda}(u)R_b(\overline{W}_i)\overline{\nu}(g)_{\pm\lambda}(u)dud\lambda.$$

It follows easily that

$$-(\Delta_b f, g) = \sum_{\pm}(-i\frac{\partial}{\partial x}\otimes\Box^{(\pm 1)}T_N^{\pm}f, T_N^{\pm}g).$$

The rest of the proposition is clear.

To summarize the conclusions of this section, $I\otimes T_N$ maps $L^2(\mathbf{R}^+, dy/y^2)\otimes L^2(N/R)$ onto $L^2(\mathbf{R}^+, dy/y^2)\otimes L^2_+(\mathbf{R})\otimes\mathcal{H}$. We let H^+ be the upper half-plane $y > 0$ in $\mathbf{C}$ and identify $L^2(\mathbf{R}^+, dy/y^2)\otimes L^2_+(\mathbf{R})$ with a space $L^2_+(H^+, dxdy/y^2)$ of square integrable functions on H^+ in the obvious manner. (This space consists of functions whose Fourier transform in x is supported in $\mathbf{R}^+$.) Then $I\otimes T_N$ intertwines $\tilde{\Delta}_\lambda$ with the following operator:

$$\Delta_\lambda^S = y^2(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2})\otimes I + y\frac{\partial}{\partial x}\otimes(i(\Box - n + 1)) + \frac{1-\lambda^2}{4}. \tag{18}$$

where $\Box = (\Box^+, \Box^-)$ on $\mathcal{H}^+\times\mathcal{H}^-$.

The observant reader will note that the first term on the right in the above formula is just the Laplace-Beltrami operator for the upper half-plane H^+. There is, in fact, a deep relationship between Δ_λ^S and the Casimir operator for the group $S = \tilde{\mathrm{Sl}}(2,\mathbf{R})$ (' $\tilde{}$ ' refers to the simply-connected covering space).

We choose as a basis for the Lie algebra $\mathcal{S} = sl(2,\mathbf{R})$, the elements defined below.

$$H_0 = \begin{bmatrix} 0 & 1\\ -1 & 0\end{bmatrix}, H_1 = \begin{bmatrix} 0 & 1\\ 1 & 0\end{bmatrix}, H_2 = \begin{bmatrix} 1 & 0\\ 0 & -1\end{bmatrix}. \tag{19}$$

We define subgroups $\tilde{K}$ and A of S by exponentiating $\mathbf{R}H_0$ and $\mathbf{R}H_2$ respectively. The subgroup Z of $\tilde{K}$ defined by

$$Z = \exp \pi \mathbf{Z} H_0$$

is the center of S.

We also define

$$N_0 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

As a manifold, $S = H^+ \times \mathbf{R}$, the explicit isomorphism being defined by the mapping κ where

$$\kappa(x, y, \theta) = \exp xN_0 \exp(\frac{1}{2} \log y) H_2 \exp \theta H_0. \tag{20}$$

We use the inverse of this mapping to define coordinates (x, y, θ) for S.

We let C represent the Casimir element in the enveloping algebra. This is the element is the element in the universal enveloping algebra of $\mathcal{S}$ defined by

$$C = \frac{1}{4}(H_0^2 - H_1^2 - H_2^2).$$

Then, according to [G], p.25, as an operator acting on $C^\infty(S)$, C is given by

$$C = y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}) + y \frac{\partial^2}{\partial x \partial \theta}. \tag{21}$$

Next, we make a fundamental assumption: *We shall assume that the operators* $\square^{\pm}$ *have essentially self adjoint closures on* $\mathcal{H}_{\pm}$. In [P1], we gave general conditions under which this assumption is valid. For example, if $\mathcal{P}$ is abelian or if ad A has only positive eigenvalues (so that δ is essentially a dilation), then this condition is satisfied. Note that this condition is not automatic due to the non-definite nature of H. This assumption allows us to define a unitary representation σ of $\tilde{K}$ in $\mathcal{H}$ by

$$\sigma(\exp \theta H_0) = \exp(-i\theta[\square - n + 1]). \tag{22}$$

We then let

$$\pi = \mathrm{ind}_L(\tilde{K}, S, \sigma).$$

This is the representation of S defined by left translation in the space of square integrable sections of the Hilbert bundle $S \times_\sigma \mathcal{H}$. These sections may be thought of as $\mathcal{H}$ valued functions u on S which satisfy

$$u(gk) = \sigma(k^{-1})u(g) \tag{23}$$

for all $g \in S$ and all $k \in K$. We may then conclude from formula21n that

$$\pi(C) = y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}) y \frac{\partial}{\partial x}[i(\square - n + 1)].$$

As a Hilbert space, the representation space of π is $L^2(H^+, dxdy/y^2) \otimes \mathcal{H}$. It follows that in this realization

$$\pi(C) = y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}) \otimes I - y\frac{\partial}{\partial x} \otimes (i(\Box - n + 1)). \tag{24}$$

Formally, this operator is identical with Δ_1^S (formula18n). However, we must pay close attention to the spaces on which these operators act. Recall that that the space of eigenfunctions for $\tilde{\Delta}$ under consideration consists of C^∞ mappings into $\mathcal{V}$ where $\mathcal{V}$ is the space of joint C^∞ vectors for Δ_b and $L(Z_o)$. (See the discussion following formula16n.) Let $\mathcal{V}_S$ be the subspace of $L^2(\mathbf{R}, \mathcal{H}(\sigma))$ of joint C^∞ vectors for $\dfrac{\partial}{\partial x}$ and $\dfrac{\partial}{\partial x}\sigma(H_0)$. Since the N-transform is unitary, $T_N(\mathcal{V})$ is the subspace $\mathcal{V}_S^+$ of $\mathcal{V}_S$ consisting of those functions whose Fourier transformation in x is supported in $\mathbf{R}^+$. (Proposition15n.)

Let $\mathcal{C}_S$ (respectively $\mathcal{V}_S^+$) be the space of C^∞ mappings of $\mathbf{R}^+$ into $\mathcal{V}_S$ (respectively $\mathcal{V}_S^+$) of moderate growth as $y \to 0$. The N-transformation of the space $\mathcal{C}$ is $\mathcal{C}_S^+$. The boundary values of eigen-functions in $\mathcal{C}_S^+$ will belong to $\mathcal{V}_S^+$. Clearly, T_N will commute with the corresponding boundary maps. Thus, it will suffice to invert the boundary map on $\mathcal{C}_S^+$. This will be done in the next section.

Section 4. $\tilde{Sl}(2, \mathbf{R})$

We continue with the notation established in the last section. We shall also let Λ be the Cartan involution. This is the automorphism of S whose differential is the negative of the transpose mapping on the Lie algebra.

As discussed above, our goal in this section is to study the boundary values associated with the eigenvalue problem

$$Cu = \frac{(\lambda^2 - 1)}{4}u$$

where C is the Casimir element and u satisfies formula23n.

We shall need to avoid certain 'singular' values of λ. We shall assume, to begin with, that $\lambda \notin \mathbf{Z}$. Additionally, we shall assume that the set

$$(\lambda + 1 - 2\mathbf{N}) \cup -(\lambda + 1 - 2\mathbf{N}) \tag{25}$$

does not intersect the 'singular support' of σ. To define 'singular support', let σ be decomposed as a direct integral as

$$\sigma = \oplus \int_{\mathbf{R}} \sigma_\alpha \otimes I_\alpha d\mu(\alpha)$$

where $\sigma_\alpha(\exp tH_0) = e^{i\alpha t}$ and I_α is the identity representation in a Hilbert space $\mathcal{H}_\alpha$. We write $\mu = \mu_1 + \mu_2$ where μ_1 is absolutely continuous with respect to Lebesgue measure and μ_2 is singular with respect to Lebesgue measure. The support of μ_2 is referred to as the singular support of σ.

As explained in the previous section, we are mainly interested in studying the boundary values in the $NA\tilde{K}$ decomposition of S. However, it seems to be necessary to first consider the 'compact picture' and then to relate this to the non-compact situation. Thus, we consider the decomposition $S = \tilde{K}A\tilde{K}$.

Let $\mathcal{H}_\infty(\sigma)$ denote the space of C^∞-vectors for σ. Let $\mathcal{V}_K$ be the space of C^∞ maps f of $\tilde{K}$ into $\mathcal{H}_\infty(\sigma)$ which satisfy

$$f(kz) = \sigma(z^{-1})f(k) \tag{26}$$

for all $k \in \tilde{K}$ and $z \in Z$. C^∞ functions f satisfying formula23n are in one to one correspondence with C^∞ maps of A into $\mathcal{V}_K$. Specifically, such f correspond with the map $a \to R(a)f|\tilde{K}$.

We would like to explicitly describe the C action on such functions. We define two operators on $\mathcal{V}_K$ which will be important. The first operator is simply pointwise multiplication by $\sigma(H_0)$. We denote this by '$\sigma(H_0)$'. The second is differentiation in the H_0 direction in the $\tilde{K}$ variable. We denote this by $R(H_0)$. Finally, we coordinatize A by the map $t \to \exp\frac{1}{2}(\log t)H_2$. Actually, we shall only consider that portion of A coordinatized by $t \in (0,1)$. (This does not cause trouble since it turns out that we will only be interested in values of t near zero.)

We claim that on functions satisfying formula23n, in these coordinates, C is described by:

$$\Omega = \Theta^2 + \frac{t^2+1}{t^2-1}\Theta + \frac{t}{(t+1)^2}[\frac{t}{(t-1)^2}(R(H_0) - \sigma(H_0))^2 - R(H_0)\sigma(H_0)]. \tag{27}$$

where $\Theta = t\dfrac{\partial}{\partial t}$.

To see this, let $M = \mathbf{R}^3$. We denote the typical point of M by (μ, ζ, ν). Let $\eta : M \to G$ be the mapping defined by

$$\eta(x) = (\exp\mu H_0)(\exp\zeta H_2)(\exp\nu H_0).$$

This map is a local diffeomorphism on the set M_o in M defined by $\zeta < 0$. Note, however, that η is not injective on M_o since, for example, η is constant on the set $\{(n\pi, \zeta, -n\pi) | n \in \mathbf{Z}\}$. The Casimir operator lifts to an operator Ω on M_o. The explicit form of this operator may be found in [B], p.596, formula (4.19). After some simplification, one sees that

$$4\Omega = \frac{\partial^2}{\partial\zeta^2} + 2\coth 2\zeta\frac{\partial}{\partial\zeta} + (\sinh 2\zeta)^{-2}((\frac{\partial}{\partial\mu} - \frac{\partial}{\partial\nu})^2 - 4\sinh^2\zeta\frac{\partial^2}{\partial\mu\partial\nu}).$$

We introduce the change of variables

$$\zeta = (\log t)/2.$$

In these variables, M_o is described by $0 < t < 1$.

A straight forward computation yields the formula

$$\Omega = \Theta^2 + \frac{t^2+1}{t^2-1}\Theta + \frac{t}{(t+1)^2}[\frac{t}{(t-1)^2}(\frac{\partial}{\partial\mu} - \frac{\partial}{\partial\nu})^2 - \frac{\partial^2}{\partial\mu\partial\nu}].$$

This easily implies formula27n.

From formula27n, we may consider Ω as a $\mathcal{V}_K$ valued differential operator on $C^\infty((-1,1),\mathcal{V}_K)$. It is clear that Ω has regular singularities at 0 and, for moderately growing solutions on $(0,1)$, the boundary values will belong to $\mathcal{V}_K$. We shall study the boundary value problem on this space.

The group $\tilde{K}$ acts on $\mathcal{V}_K$ both by left translation and by multiplication by $\sigma(k^{-1})$. The boundary map commutes with these actions. Clearly, for all $\lambda \in \mathbf{C}$, the operator $P_\lambda = \Omega + (1-\lambda^2)/4$ is C^∞ on $C^\infty((-1,1),\mathcal{V}_K)$ and has regular singularities at 0 in the sense defined in §1. Then $\sigma(P_\lambda)(s) = (s-(1+\lambda)/2)(s-(1-\lambda)/2)$. The characteristic exponents are

$$s_\pm = (1 \pm \lambda)/2.$$

Now, let A^λ_σ be the space of all solutions to $P_\lambda u = 0$ in $C^\infty((0,1),\mathcal{V}_K)$ which have moderate growth as $t \to 0$. For each $u \in A^\lambda_\sigma$ there is a pair of boundary values $B(u)^\pm$ corresponding to $s_\pm$. These boundary values are uniquely determined elements of $\mathcal{V}_K$. We set $B(u) = B^+(u)$. (Recall that we have normalized λ by requiring $\operatorname{re} \lambda \leq 0$.)

We may identify $B(u)$ with an element of $\mathcal{H}_\infty(\pi_{\lambda,\sigma})$ for a certain representation $\pi_{\lambda,\sigma}$ of $S \times \tilde{K}$. We let $\mathcal{P}$ be the span of N_0 and H_2. Let P_o be the connected subgroup of S corresponding to $\mathcal{P}_o$ and $P = ZP_o$. We define a character τ on P_o by requiring its differential to be zero at N_0 and $1+\lambda$ at H_2.

There is a unique representation $\tau_{\lambda,\sigma}$ of $P \times \tilde{K}$, valued in $\mathcal{H}(\sigma)$, which equals τI on P_o, σ on $Z \subset P_o$ and σ on $\{e\} \times \tilde{K}$. Let $\mathcal{H}(\pi_{\lambda,\sigma})$ to be the space of all $\mathcal{H}(\sigma)$ valued functions f on $S \times \tilde{K}$ such that

(1)
$$f(gp) = \tau_{\lambda,\sigma}(p^{-1})f(g)$$
for all $p \in P \times \tilde{K}$ and $g \in S \times \tilde{K}$.

(2) $||f||$ defines, upon restriction to $\tilde{K}$, an element of $L^2(\tilde{K}/Z)$. (We use Haar measure on $\tilde{K}/Z$, normalized to give the quotient unit measure).

We let $\pi_{\lambda,\sigma}(g)$ be the representation of $S \times \tilde{K}$ defined by left translation by g^{-1} in this space. This representation is unitary on $\tilde{K} \times \tilde{K}$. We will usually realize this representation in the obvious space of $\mathcal{H}(\sigma)$ valued functions on $\tilde{K}$, obtained by restricting elements of $\mathcal{H}(\pi_{\lambda,\sigma})$ to $\tilde{K} \times \{e\}$. On $\{e\} \times \tilde{K}$, this representation is just the pointwise action of σ. The reason we have included the extra factor of $\tilde{K}$ in our definition of $\pi_{\lambda,\sigma}$ is to make the space of C^∞ vectors be C^∞ maps in the $\mathcal{H}_\infty(\sigma)$ topology. Explicitly, we have the following lemma.

Lemma 16. *The space $\mathcal{H}_\infty(\pi_{\lambda,\sigma})$ is $\mathcal{V}_K$.*

Proof The fact that $\mathcal{H}_\infty(\pi_{\lambda,\sigma}) \subset \mathcal{V}_K$ is easily seen. To prove the equality, we must show that the $\pi_{\lambda,\sigma}$ action is C^∞ on $\mathcal{V}_K$. Explicitly, we need to show that for all f in $\mathcal{V}_K$ and for all $g \in \mathcal{H}(\pi_{\lambda,\sigma})^*$, the mapping of S into $\mathbf{C}$ given by

$$s \to < \pi_{\lambda,\sigma}(s)f, g >$$

is C^∞.

The action of $\pi_{\lambda,\sigma}(s)$ is given by a formula of the form

$$\pi_{\lambda,\sigma}(g)f(k) = \sigma(a(g,k))f(b(g,k))$$

where a and b are C^∞ maps of $S \times \tilde{K}$ into $\tilde{K}$. The above scalar product may be defined by an integral over a compact set. Since f is C^∞, it is clear that this defines a C^∞ function, as desired.

There is a mapping $T_{\lambda,\sigma}$ from $\mathcal{H}_\infty(\pi_{\lambda,\sigma})$ into A^λ_σ. To define this, note first that there is an obvious pairing between $\pi_{\lambda,\sigma}$ and $\pi_{-\overline{\lambda},\sigma}$. In fact, if $f \in \mathcal{H}(\pi_{\lambda,\sigma})$ and $g \in \mathcal{H}(\pi_{-\overline{\lambda},\sigma})$, then we define

$$< f, g >= \int_{\tilde{K}/Z} (f(x), g(x))dx$$

where the scalar product is the $\mathcal{H}(\sigma)$ scalar product. It is well known that $\pi_{\lambda,\sigma}$ and $\pi_{-\overline{\lambda},\sigma}$ are contragredient representations under this pairing.

Now let n be an *even* integer. For each $w \in \mathcal{H}(\sigma)$ let $v_n(w)$ be the function on $\tilde{K}$ defined by

$$v_n(w)(\exp tH_0) = \sigma(\exp -tH_0)we^{-int} \tag{28}$$

The function $v_n(w)$ may be considered either as belonging to $\mathcal{H}(\pi_{\lambda,\sigma})$ or $\mathcal{H}(\pi_{-\overline{\lambda},\sigma})$ as the need arises. Given $F \in \mathcal{H}_\infty(\pi_{\lambda,\sigma})$, we define a C^∞ function $T^w_{\lambda,\sigma}(F)$ on S by

$$T^w_{\lambda,\sigma}(F)(s) =< \pi_{\lambda,\sigma}((\Lambda(s^{-1}), e))F, v_0(w) > . \tag{29}$$

where $v_0(w)$ is considered as an element of $\mathcal{H}(\pi_{-\overline{\lambda},\sigma})$. ($\Lambda$ is the Cartan involution.) The mapping $w \to T^w_{\lambda,\sigma}(F)(s)$ defines, by duality, an element $T_{\lambda,\sigma}(F)(s)$ of $\mathcal{H}(\sigma)$. Explicitly,

$$T_{\lambda,\sigma}(F)(s) = \int_{\tilde{K}/Z} \sigma(k)F(\Lambda(s)k, e)dk$$

It is easily seen that this element, in fact, belongs to $\mathcal{H}_\infty(\sigma)$. Furthermore, $T_{\lambda,\sigma}(F)$ satisfies formula23n. We say that $T_{\lambda,\sigma}(F)$ has moderate growth if it has moderate growth when considered as a $\mathcal{V}_K$ mapping as explained below formula26n.

Proposition 17. *For all $F \in \mathcal{H}_\infty(\pi_{\lambda,\sigma})$, $T_{\lambda,\sigma}(F)$ is an eigenvector with eigenvalue $q = (1-\lambda^2)/4$ for $\pi(C)$. Furthermore, $T_{\lambda,\sigma}(F)$ has moderate growth.*

Proof For the first statement, it suffices to show that $\pi_{\lambda,\sigma}(C) = qI$ on $\mathcal{H}(\pi_{\lambda,\sigma})$. Let $A = -i\sigma(H_0)$. We note that the function $v_n(w)$ is uniquely determined by the facts that $v_n(w)(e) = w$ and

$$\pi_{\lambda,\sigma}(-iH_0)v_n(w) = (A+n)v_n(w).$$

The infinitesimal action of $\mathcal{S}$ on the v_n is easily described. The Let X and Y in $\mathcal{S}_c$ be defined by

$$X = (H_2 - iH_1)/2 \text{ and } Y = (H_2 + iH_1)/2.$$

Let x, y and h be respectively, the images of X, Y and $-iH_0$ under $\pi_{\lambda,\sigma}$. Then $[h, x] = -2x$, $[h, y] = 2y$ and $[y, x] = h$.

Lemma 18. $2xv_n(w) = (\lambda+1-n-A)v_{n-2}(w)$, $2yv_n(w) = (\lambda+1+n+A)v_{n+2}(w)$, and $hv_n(w) = (n+A)v_n(w)$.

Proof Note that

$$hxv_n(w) = xhv_n(w) - 2xv_n(w) = (A+n-2)xv_n(w).$$

Thus, $xv_n(w) = v_{n-2}(w_o)$ where $w_o = xv_n(w)(e)$. This is easily evaluated using the covariance of $v_n(w)$ as an element of $\mathcal{H}(\pi_{\lambda,\sigma})$ and the fact that $H_1 = 2N_0 - H_0$. We find that $2w_o = (\lambda+1-A-n)w$. This proves the first equality. The rest of the lemma follows similarly.

In these terms, it is easily seen (using the above lemma) that on the span of the $v_n(w)$,

$$\pi_{\lambda,\sigma}(C) = -\frac{1}{4}(h^2 + 2xy + 2yx) = \frac{1}{4}(1-\lambda^2).$$

Since the $v_n(w)$ are clearly dense in $\mathcal{H}_\infty(\pi_{\lambda,\sigma})$, this proves the first statement.

Next we prove the moderate growth. For this, let $a = \exp\frac{1}{2}(\log t)H_2$ and let $G = T_{\lambda,\sigma}(F)$. We shall show that $a \to t^{\text{ re }\lambda/2}R(a)G|\tilde{K}$ defines a bounded mapping of $(0,1)$ into $\mathcal{V}_K$. This is equivalent to saying that for all n amd m, the function

$$a \to t^{\text{ re }\lambda/2}R(a)\sigma(H_0)^n R(H_0)^m G|\tilde{K} \tag{30}$$

is bounded as a map into the space of continuous, $\mathcal{H}(\sigma)$ valued functions.

Of course, it suffices to prove that the functions in formula30n have uniformly bounded scalar product with any $w \in \mathcal{H}(\sigma)$. For this, we note that for k in $\tilde{K}$, and $w \in \mathcal{H}(\sigma)$,

$$(G(ka), w) = T^w_{\lambda,\sigma}(F)(ka) =< \pi_{\lambda,\sigma}(k^{-1})F, \pi_{-\overline{\lambda},\sigma}(a^{-1})v_0(w) > .$$

A similar equality is true for $\sigma(H_0)^n R(H_0)^m G$. The moderate growth follows from the following lemma. We shall postpone the proof of this lemma until after Proposition26n. .

Lemma 19. *For a as above, $t^{\text{ re }\lambda/2}||\pi_{-\overline{\lambda},\sigma}(a^{-1})v_0(w)||$ is bounded as $t \to 0$.*

Let $F \in \mathcal{H}_\infty(\pi_{\lambda,\sigma})$. From Proposition17n, $T_{\lambda,\sigma}(F)$ defines an $\mathcal{H}_\infty(\sigma)$ valued distribution on $(0,\infty)$ which extends (due to the moderate growth) to a distribution on $(-\infty,\infty)$ which is supported in $[0,\infty)$. It follows that the boundary value of this element is a well defined element of $\mathcal{V}_K$. Thus, we obtain a mapping $T: \mathcal{H}_\infty(\pi_{\lambda,\sigma}) \to \mathcal{V}_K$. We identify $\mathcal{V}_K$ with $\mathcal{H}_\infty(\pi_{\lambda,\sigma})$ by means of Lemma16n. This operator may be explicitly computed.

We define an operator D^λ on $\mathcal{H}(\sigma)$ by

$$D^\lambda = \pi^{-1}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos t)^{\lambda-1}\sigma(\exp(tH_0))dt.$$

Initially this is defined only for re $\lambda > 0$. However, the following lemma allows us to define this operator (as a continuous operator on $\mathcal{H}_\infty(\sigma)$) for all $\lambda \notin 2\mathbf{N}^-$.

Lemma 20. *For each $w \in \mathcal{H}_\infty(\sigma)$, $\lambda \to D^\lambda(w)$ extends holomorphically to $\mathbf{C}\setminus\mathbf{Z}^-$. The operator D^λ so defined is bounded in $\mathcal{H}(\sigma)$ for all $\lambda \in \mathbf{C}\setminus\mathbf{R}^-$.*

Proof Let σ be decomposed as a direct integral as

$$\sigma = \oplus\int_{\mathbf{R}} \sigma_\alpha \otimes I_\alpha d\mu(\alpha) \tag{31}$$

where $\sigma_\alpha(\exp tH_0) = e^{i\alpha t}$ and I_α is the identity representation in a Hilbert space $\mathcal{H}_\alpha$. In this notation, the derivative of σ is multiplication by $i\alpha$. It follows that the elements of $\mathcal{H}_\infty(\sigma)$ have rapidly vanishing norms at $\alpha = \pm\infty$. (This in fact characterizes $\mathcal{H}_\infty(\sigma)$.)

From the last formula on p. 8 of [M], we obtain the identity (for $\operatorname{re}\lambda > 0$)

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos t)^{\lambda-1} e^{iat}dt = \frac{\pi,(\lambda)2^{-\lambda+1}}{,((1+\lambda+\alpha)/2),((1+\lambda-\alpha)/2)}. \tag{32}$$

Using the identity

$$,(\frac{1}{2}+z),(\frac{1}{2}-z) = \frac{\pi}{\cos(\pi z)}, \tag{33}$$

([M], p. 2), it is easily seen that this expression is bounded (as a function of α) except for $\lambda < 0$. Furthermore, for $\lambda < 0$, this expression grows at most like $|\alpha|^{-\lambda}$ at $\pm\infty$. Our lemma follows easily.

The operator D^λ acts in a pointwize fashion on $\mathcal{H}_\infty(\pi_{\lambda,\sigma})$. We may write, with the obvious abuse of notation:

$$T_{\lambda,\sigma}D^\lambda = D^\lambda T_{\lambda,\sigma} \text{ and}$$
$$BD^\lambda = D^\lambda B.$$

The following explains the significance of D^λ.

Proposition 21. $T = \lambda D^{-\lambda}$.

Proof We decompose σ according to formula31n. If w belongs to $\mathcal{H}(\sigma)$, we shall denote its α-component in $\mathcal{H}_\alpha$ by w^α.

For each α, we have operators and spaces defined by σ_a in the same manner as the spaces defined by σ. We shall use α, rather σ_α, as a parameter in denoting theses spaces. The space $\mathcal{V}_K$ is the direct integral of the spaces $\mathcal{V}_{K,\alpha}\otimes\mathcal{H}_\alpha$ in the sense that elements F of $\mathcal{V}_K$ are defined by a measurable field $F_\alpha \in \mathcal{V}_{K,\alpha}\otimes\mathcal{H}_\alpha$ where, for all $n \in \mathbf{Z}^+$, $R(H_0)^n F_\alpha$ has rapidly decaying sup norm at $\alpha = \pm\infty$. (See [G].)

Now, let $u \in A^\lambda_\sigma$ be such that u_α has moderate growth for a.e. α. Then $u_\alpha \in A^\lambda_\alpha \otimes \mathcal{H}_\alpha$. It is easily shown that $B(u)_\alpha = B_\alpha \otimes I(u_\alpha)$ for a.e. α. From this, we see that for $F \in \mathcal{H}_\infty(\pi_{\lambda,\sigma})$, $T(F)_\alpha = T_\alpha \otimes I(F_\alpha)$. It suffices to compute the T_α.

Let v^α_n be the elements of $\mathcal{H}_\infty(\pi_{\lambda,\alpha})$ defined by

$$v^\alpha_n(\exp -tH_0) = e^{-i(n+\alpha)t}$$

Let

$$u_n^\alpha = T_{\lambda,\alpha}(v_n^\alpha).$$

Then u_n^α is a C^∞ eigenvector for C corresponding to the eigenvalue $q = (1-\lambda^2)/4$. Furthermore,

$$u_n^\alpha((\exp \mu H_0)g(\exp \nu H_0)) = e^{-i(\beta\mu+\alpha\nu)}u_n^\alpha(g).$$

where $\beta = n + \alpha$. In particular, u_n^α is determined by the function $\tilde{u}_n^\alpha(t) = u_n^\alpha(\exp \zeta H_2)$ where $\zeta = \log t/2$ as before. From formula27n, we see that $\tilde{u}_n^\alpha$ satisfies the ordinary differential equation $P_o\tilde{u}_n^\alpha = q\tilde{u}_n^\alpha$ where

$$P_o = \Theta^2 + \frac{t^2+1}{t^2-1}\Theta + \frac{t}{(1+t)^2}[\frac{t}{(t-1)^2}(\beta-\alpha)^2 - \beta\alpha]. \tag{34}$$

The indicial polynomial for $P_o - q$ is the same as that for P_λ. Since we have assumed that the roots do not differ by an integer, it follows from the theory of ordinary differential equations with regular singularities ([C], Theorem 3, p.158) that $\tilde{u}_n^\alpha$ has a unique expansion of the form

$$\tilde{u}_n^\alpha(t) = u_{n,+}^\alpha(t)t^{s_+} + u_{n,-}^\alpha(t)t^{s_-} \tag{35}$$

where $u_{n,\pm}^\alpha$ are C^∞ on some neighborhood of $t = 0$. From Theorem6n, the boundary value of the solution u_n^α is the function

$$\lambda e^{-i(\beta\mu-\alpha\nu)}u_{n,+}^\alpha(0).$$

More generally, we say that an element v of $\mathcal{H}_\infty(\pi_{\lambda,\alpha})$ is $\tilde{K}$-finite if it is a finite linear combination of such v_n^α. We denote the space of all such elements by $\mathcal{H}_f(\pi_{\lambda,\alpha})$. It is well known that this space is invariant under the infinitesimal action of $\mathcal{S}$. We define the linear functional ζ on $\mathcal{H}_f(\pi_{\lambda,\alpha})$ by

$$< v, \xi >= T^\alpha(v)(e).$$

Note that from the above comments $< v_n^\alpha, \xi >= u_{n,+}^\alpha(0)$.

Lemma 22. *For all $X \in \mathcal{P}$, and all $\tilde{K}$-finite vectors v,*

$$< \pi_{\lambda,\sigma}(X)v, \xi >=< \tau_{\lambda,\alpha}(X)v, \xi > .$$

Proof It suffices to consider $v = v_n^\alpha$. Consider $T^\alpha(\pi_{\lambda,\sigma}(N_0)v_n^\alpha)$. Since we are evaluating at e, this is uniquely determined by the function

$$\eta(t) =< \pi_{\lambda,\alpha}(\exp \zeta H_2)\pi_{\lambda,\alpha}(N_0)v_n^\alpha, v_0^\alpha >$$

where $\zeta = (\log t)/2$. Using the identity $[H_2, N_0] = 2N_0$, we see that

$$\eta(t) = t < \pi_{\lambda,\alpha}(N_0)\pi_{\lambda,\sigma}(\exp \zeta H_2)v_n^\alpha, v_0^\alpha > .$$

We bring $\pi_{\lambda,\alpha}(N_0)$ across the bi-linear pairing as $-\pi_{-\overline{\lambda},\alpha}(N_0)$. Of course, $\pi_{-\overline{\lambda},\alpha}(N_0)v_0^\alpha$ is $\tilde{K}$-finite. Thus we may expand η as a finite sum of terms of the form of t times an expression similar to that in formula35n. The presence of the t term shows that the boundary value is zero at e, in agreement with the claim of the lemma.

To consider H_2, we now define

$$\eta(t) =< \pi_{\lambda,\alpha}(\exp \zeta H_2)\pi_{\lambda,\alpha}(H_2)v_n^\alpha, v_0^\alpha > .$$

Then, $\eta = \frac{1}{2}t\frac{d}{dt}\tilde{u}_n^\alpha$ where $\tilde{u}_n^\alpha$ is as above. Our formula follows easily by differentiation in formula35n.

Now, let us compute T^α. It is obvious that T^α intertwines the $\tilde{K}$ actions. In particular, it follows that there is a constant c_n^α such that $T^\alpha(v_n^\alpha) = c_n^\alpha v_n^\alpha$. Proposition21n will follow from formula32n, once we have proven the following:

Lemma 23.

$$c_n^\alpha = \frac{\lambda,\ (-\lambda)2^{\lambda+1}}{,\ ((1-\lambda+\alpha)/2),\ ((1-\lambda-\alpha)/2)}.$$

Proof We note that

$$c_n^\alpha =< v_n^\alpha, \xi >$$

where ξ is as above.

Let X and Y be the elements of $\mathcal{S}_c$ defined above. Writing $2X = (H_2 - 2iN_0) + iH_0$ and using Lemma22n, we see that

$$2 < \pi_{\lambda,\alpha}(X,0)v_n, \xi >= (1 + \lambda - n - \alpha) < v_n, \xi > .$$

From Lemma18n, we get:

$$(1 + \lambda - n - \alpha)c_{n-2}^\alpha = (1 + \lambda - n - \alpha)c_n^\alpha.$$

If $\lambda - \alpha$ is not of the form $2k + 1$ for some integer k, the coefficients are never zero and we conclude that $c_{n+2}^\alpha = c_n^\alpha$. (Note that, by definition, n is even.) If $\lambda - \alpha$ is of the form $2k+1$, then $\lambda + \alpha$ is not of this form and the same conclusion follows from similar arguments applied to Y.

To evaluate c_o^α, we we note that the boundary value corresponding to v_0^α may be explicitly computed. This involves explicitly computing the expression in formula35n. The computation is essentially the same as that of formula (11.3) in [B], so we shall be brief. Since we want to let $t \to 0$, we shall limit ourselves to $\zeta < 0$. In formula34n, we let $\beta = \alpha$, and make the substitution $y = (\sinh \zeta)^2 = t^{-1}(t-1)^2/4$. This results in equation (10.16), p. 626 of [B] where $n = m = 2\alpha$. We continue as in [B], *loc. cit.*, concluding that the function $Y_{n,m}$ defined in [B] satisfies equation (10.18) of [B] where $\sigma = \lambda/2$. Since u_o is C^∞ at $\zeta = 0$ and assumes the value one at this point, we can immediately conclude that formula (10.20) of [B] is valid, with $\Theta(q) = 1$. (See formula (10.11) of [B].) This eventually yields formula (11.3) of [B]. For small t, y is approximately $t^{-1}/4$. It follows (using Theorem6n) that, in the notation of [B],

$$c_0^\alpha = \lambda\beta_{\alpha,\alpha}(-\lambda/2, 0) = \frac{\lambda,\ (-\lambda)2^{\lambda+1}}{,\ ((1-\lambda+\alpha)/2),\ ((1-\lambda-\alpha)/2)}. \quad (36)$$

This finishes the proof of the lemma and of Proposition21n.

From Proposition21n, the Poisson kernel is, formally, $(\lambda D^{-\lambda})^{-1}T_{\lambda,\sigma}$. It is clear from formula32n and the 'non-singularity' assumption on σ (formula25n), that $D^{-\lambda}$ is injective. We may explicitly compute its inverse. Let us first assume that $\operatorname{re}\lambda < 0$.

Using formula33n, we write

$$(c_0^\alpha)^{-1} = \frac{\pi}{2}\left(\frac{-\sin\pi\lambda}{\cos\lambda\pi+\cos\alpha\pi}\right)\frac{,\,(\lambda)2^{-\lambda}}{,\,((1+\lambda+\alpha)/2),\,((1+\lambda-\alpha)/2)}. \tag{37}$$

This formula shows that $(\lambda D^{-\lambda})^{-1} = -\frac{\pi}{4}E^\lambda D^\lambda$ where E^λ is the operator on $\mathcal{H}(\pi_{\lambda,\sigma})$ defined by

$$\frac{\sin\pi\lambda}{\cos\lambda\pi+\cos\alpha\pi}. \tag{38}$$

(Note that this computation shows that for $\lambda \in \mathbf{C}\setminus\mathbf{R}$, D^λ has a bounded inverse.)

To explicitly express E^λ in terms of σ, let $Z = e^{i\pi\alpha}$ and $W = e^{i\pi\lambda}$. If $\operatorname{im}\lambda < 0$, we have the identity

$$\begin{aligned}\frac{\sin\pi\lambda}{\cos\lambda\pi+\cos\alpha\pi} =& iZ((Z+W)^{-1}-(Z+W^{-1})^{-1})\\ =& -i\sum(-1)^n W^{-|n|}Z^n = -i\sum(-1)^n W^{-|n|}e^{in\pi\alpha}.\end{aligned}$$

Then, we see that

$$E^\lambda = -i\sum(-1)^n e^{-i\pi\lambda|n|}\sigma(\exp n\pi H_0). \tag{39}$$

If λ is real, the computation of the inverse is much subtlier. Formula37n is valid only if $\cos\lambda\pi+\cos\alpha\pi \neq 0$. We write $\sigma = \sigma_3 \oplus \sigma_4$ where σ_4 has discrete spectrum, supported in $\{\alpha \mid \cos\lambda\pi+\cos\alpha\pi = 0\}$ and σ_3 is supported off of this set. On $\mathcal{H}(\sigma_3)$, the inverse is described by formula37n. This will, typically, be an unbounded operator, even on $\mathcal{H}_\infty(\sigma_3)$.

On $\mathcal{H}(\sigma_4)$, the inverse will be obtained by inverting formula36n. Explicitly, $\alpha = \pm(\lambda-1+2k)$ for some non-negative integer k. (k is non-negative due to the non-singularity assumption on λ.) In the case $\alpha = \lambda+1+2k$, we have:

$$\begin{aligned}(c_0^\alpha)^{-1} =& \frac{,\,(k+1),\,(-\lambda-k)}{\lambda,\,(-\lambda)2^{\lambda+1}}\\ =& \frac{(-1)^k,\,(k+1)}{\lambda,\,(1+\lambda+k),\,(-\lambda)2^{\lambda+1}\sin\pi\lambda}\end{aligned}$$

Recalling that by assumption $\lambda < 0$, we see that the corresponding operator is unbounded (if the support of σ_4 is infinite), although it is continuous on $\mathcal{H}_\infty(\sigma_4)$. In the case $\alpha = -\lambda-1-2k$, we obtain the identical expression for $(c_0^\alpha)^{-1}$.

We now define the Poisson kernel. Explicitly, we define

$$P_{\lambda,\sigma} = (\lambda D^{-\lambda})^{-1}T_{\lambda,\sigma}.$$

Here, $(D^{-\lambda})^{-1})$ is considered as a partially defined operator on $\mathcal{H}_\infty(\pi_{\lambda,\sigma})$. Our main result of this section is the following.

Theorem 24. *Let λ and σ be non-singular. Then on A_σ^λ, the range of B is contained in the domain of $P_{\lambda,\sigma}$ and on this set $P_{\lambda,\sigma}B$ is the identity map.*

Proof We begin with the following lemma. The proof is essentially the same as in [KKMOOT].

Lemma 25. *B is injective on A_σ^λ.*

Proof Suppose $u \in A_\sigma^\lambda$ is such that $Bu = 0$. From direct integral theory, it follows that for a.e. α, $Bu^\alpha = 0$. Let

$$\tilde{u}^\alpha = \int_{\tilde{K}/Z} L(k)\sigma_\alpha(k^{-1})u^\alpha dk$$

where $L(k)$ is the left regular representation of $\tilde{K}$. Then $\tilde{u}^\alpha$ is an eigenfunction for the Casimir operator which satisfies

$$\tilde{u}^\alpha(k_1 x k_2) = \sigma_\alpha((k_1 k_1)^{-1})\tilde{u}^\alpha(x)$$

for all $x \in S$ and $k_i \in \tilde{K}$. Thus, $\tilde{u}^\alpha$ is a σ_α-spherical function for S. The function $u_0^\alpha = T_{\lambda,\alpha}(v_0^\alpha)$ is also such a spherical function corresponding to the same eigenvalue of the Casimir operator. It follows from Corollary 3.3, Chapter X of [He], suitably generalized, that

$$\tilde{u}^\alpha = \tilde{u}^\alpha(e)u_0^\alpha.$$

Since $B(u_0^\alpha) \neq 0$, we must have $0 = \tilde{u}^\alpha(e) = u^\alpha(e)$. Applying the same argument to translates of u^α shows that $u^\alpha = 0$ and hence that $u = 0$, proving the lemma.

Now, let $u \in A_\sigma^\lambda$. By construction,

$$BT_{\lambda,\sigma}(Bu) = \lambda D^\lambda Bu = B\lambda D^\lambda u.$$

It follows that $T_{\lambda,\sigma}(Bu) = \lambda D^\lambda u$, which is equivalent with $u = P_{\lambda,\sigma}Bu$. This finishes the theorem.

The boundary theory described above is adapted to the $\tilde{K}A\tilde{K}$ decomposition of S. As commented above, we require a theory adapted to the $\tilde{K}AN$ decomposition. The difference is somewhat analogous to doing harmonic analysis on the upper half plane instead of the unit disc.

We use the mapping κ of formula20n to define coordinates (x, y, θ) on S where $(x, y, \theta) \in \mathbf{R} \times \mathbf{R}^+ \times \mathbf{R}$. In these coordinates, the Laplace-Beltrami operator on A_σ is given by formula21n.

Let $\mathcal{V}_S$ and $\mathcal{C}_S$ be as defined following formula24n. Now, we define $\mathcal{C}_S^\lambda$ to be the space of all eigenvectors for C in $\mathcal{C}_S$ with eigenvalue $q = (1 - \lambda^2)/4$. Our most immediate problem is to prove the non-triviality of this space. For this, we shall use the operator $T_{\lambda,\sigma}$ above, but described in the 'non-compact' picture. We define $N' = \exp \mathbf{R}N_0$ and $\overline{N}' = \exp \mathbf{R}N_0^t$. Then $P\overline{N}$ is an open dense subset of S and $P \cap \overline{N}' = e$. It follows that the image of $\overline{N}'$ in $\tilde{K}$ under the projection mapping has complement of measure zero. Thus, the restriction mapping from $\mathcal{H}(\pi_{\lambda,\sigma})$ to functions of $\overline{N}'$ is one-to-one. This allows us to realize $\pi_{\lambda,\sigma}$ in a space of $\mathcal{H}(\sigma)$ valued functions on $\overline{N}'$. The content of the following proposition is well known (at least for the case that σ is one dimensional) and easily proved. (See, for example [W].)

Proposition 26. *Let $F \in \mathcal{H}(\pi_{\lambda,\sigma})$. Then*

$$||F||^2 = \int_{\overline{N}'} ||F(x)||^2 (x^2+1)^{\text{ re } \lambda} dx.$$

Also

$$\int_{\overline{N}'} F(x)dx = \int_{-\pi/2}^{\pi/2} (\cos\theta)^{\lambda-1} F(\theta) d\theta.$$

Let $F \in \mathcal{H}(\pi_{\lambda,\sigma})$ and $G \in \mathcal{H}(\pi_{-\overline{\lambda},\sigma})$. Then

$$< F, G >= \int_{\overline{N}'} (F(x), G(x))dx$$

where dx is Lebesgue measure on $\overline{\mathbf{R}}$.

The intertwining operator between the two realizations is fairly easy to describe. Let $g = \exp xN_0^t$. Then

$$g = \exp\theta H_0 \exp(\log s)H_2 \exp nN_0$$

where

$$n = x(x^2+1)^{-1}, \quad s = (x^2+1)^{1/2}, \quad \theta = -\tan^{-1} x.$$

Thus, the mapping from the $\tilde{K}$ realization to the $\overline{N}'$ realization is defined by sending F into the function $\tilde{F}$ defined by

$$\tilde{F}(x) = (x^2+1)^{-(\lambda+1)/2} F(\theta). \tag{40}$$

In particular, the function of formula28n, considered as an element of $\mathcal{H}_\infty(\pi_{-\overline{\lambda},\sigma})$, corresponds to the function $\tilde{v}_0(w)$ defined by

$$\tilde{v}_0(w)(x) = (x^2+1)^{(\overline{\lambda}-1)/2} \sigma(\exp((\tan^{-1} x)H_0)(w)). \tag{41}$$

We may use this information to prove Lemma19n, section one. (Recall that we deferred the proof of this lemma.) It is easily seen that

$$\pi_{-\overline{\lambda},\sigma}(a^{-1})v_o(w)(x) = t^{(\overline{\lambda}-1)/2} v_o(w)(t^{-1}x).$$

Thus, from Proposition26n, $||\pi_{-\overline{\lambda},\sigma}(a^{-1})v_o(w)||^2$ is given by

$$t^{\text{ re } \lambda - 1} \int (\frac{t^{-2}x^2+1}{x^2+1})^{\text{ re } \lambda} (t^{-2}x^2+1)^{-1} dx$$

Lemma19n follows easily from this.

Let $\mathcal{L}$ denote the Schwartz space of rapidly vanishing, C^∞ mappings $\tilde{F}$ from $\mathbf{R}$ into $\mathcal{H}_\infty(\sigma)$ which have all their derivatives rapidly vanishing. We let $\mathcal{L}_+$ be the set of such functions which have Fourier transform supported in $\mathbf{R}^+$. These spaces are non-trivial. For example, the inverse Fourier transform of any element of $\mathcal{D}(\mathbf{R}^+, \mathcal{H}_\infty(\sigma))$ belongs to $\mathcal{L}_+$. The elements of $\mathcal{L}$ are bounded, L^1 maps into $\mathcal{H}(\sigma)$. It follows from Proposition26n that we may in fact interpret $\mathcal{L}$ as a subspace of $\mathcal{H}_\infty(\pi_{\lambda,\sigma})$. More over, it is clear from formula40n that the elements of $\mathcal{L}$ define C^∞ mappings in the $\tilde{K}$ realization. Thus, $\mathcal{L} \subset \mathcal{H}_\infty(\pi_{\lambda,\sigma})$. When discussing elements of $\mathcal{L}$, we shall use a tilde when they are thought of as defined on $\mathbf{R}$ and will omit the tilde when they are thought of as elements of $\mathcal{H}_\infty(\pi_{\lambda,\sigma})$. This is consistent with the notation defined in formula40n. The following lemma establishes the non-triviality of $\mathcal{C}_S^\lambda$.

Corollary 27. *Let $\tilde{F} \in \mathcal{L}$. Then $T_{\lambda,\sigma}(F) \in \mathcal{C}_S^\lambda$. If $\tilde{F} \in \mathcal{L}_+$ then $T_{\lambda,\sigma}(F) \in \mathcal{C}_S^+$ as well.*

Proof For $G \in \mathcal{H}(\pi_{-\overline{\lambda},\sigma})$ let

$$u_G(g) = < \pi_{\lambda,\sigma}(\Lambda(g^{-1}))F, G > .$$

Clearly, if $g = \exp nN_0$,

$$u_G(g) = \int_{\mathbf{R}} (\tilde{F}(x-n), \tilde{G}(x))dx$$

This is essentially a convolution on $\mathbf{R}$. It follows that the Fourier transform of u_G in n will be supported in $\mathbf{R}^+$ for $F \in \mathcal{L}_+$. From Proposition26n, the function $||\tilde{G}||$ is square integrable with respect to dx since, by our normalizations, re $\lambda \leq 0$. It follows that $M : G \to u_G|N'$ is a continuous mapping of $\mathcal{H}(\pi_{-\overline{\lambda},\sigma})$ into $L^2(N')$.

Now, let $w \in \mathcal{H}(\sigma)$. The mapping $a \to \pi_{-\overline{\lambda},\sigma}(a^{-1})v_o(w)$ is C^∞ as a mapping of $\exp \mathbf{R}H_2$ into $\mathcal{H}(\pi_{-\overline{\lambda},\sigma})$. Furthermore, for all $a \in \exp \mathbf{R}H_2$,

$$\begin{aligned} T_{\lambda,\sigma}(F)^w(ga) &= < \pi_{\lambda,\sigma}(a\Lambda(g^{-1}))F, v_o(w) > \\ &= < \pi_{\lambda,\sigma}(\Lambda(g^{-1}))F, \pi_{-\overline{\lambda},\sigma}(a^{-1})v_o(w) > . \end{aligned} \tag{42}$$

Again, the result is square integrable. The $L^2(\mathbf{R})$ rate of growth as $t \to 0$, is determined by the norm of $\pi_{-\overline{\lambda},\sigma}(a^{-1})v_o(w)$ in $\mathcal{H}(\pi_{-\overline{\lambda},\sigma})$ which is computed in Lemma19n. We see that $T_{\lambda,\sigma}(F)$ has moderate growth as an $L^2(\mathbf{R})$ valued map. It follows (by applying the same arguments to the derivatives of F and to powers of $\sigma(H_0)$ on F) that $T_{\lambda,\sigma}(F)$ in fact has its image in $\mathcal{V}_S$ and has moderate growth as a $\mathcal{V}_S$ valued map. growth as a map into $\mathcal{C}^+$. This proves the corollary.

Now, it is easily seen that in the coordinates defined by κ, the Laplace-Beltrami operator has regular singularities. Thus, any solution $u \in \mathcal{C}_S^\lambda$ has a boundary value $B_N(u)$ which is an element of $\mathcal{V}_S$. The description of the boundary map is essentially the same here as given was above Theorem24n.

Proposition 28. *Let F be as in Corollary27n. Then $B_N(T_{\lambda,\sigma}(F)) = \lambda D^{-\lambda}F \circ \Lambda|N'$.*

Proof To begin the proof, first, note that since the $\tilde{K}AN'$ decomposition is unique, we may define an C^∞ mapping $\tau : (\mu, t, \nu) \to (x, y, \theta)$ by the requirement

$$(\exp \mu H_0)(\exp(\frac{1}{2}\log t)H_2)(\exp \nu H_0) = (\exp xN_0)(\exp(\frac{1}{2}\log y)H_2)(\exp \theta H_0) \tag{43}$$

This mapping may be easily computed using, for example, the matrix realization of $\mathrm{Sl}(2, \mathbf{R})$. One finds that

$$\begin{aligned} x &= b(1-t^2)\cos\mu \sin\mu \\ y &= bt \\ \theta &= \tan^{-1}(t\tan\mu) + \nu \end{aligned} \tag{44}$$

where $b = (t^2 \sin^2 \mu + \cos^2 \mu)^{-1}$. The branch of the inverse tangent is determined by the requirement that at $t = 1$, $\theta = \mu + \nu$.

We would like to apply the change of coordinates theorem (Theorem10n) above. If $\cos\mu \neq 0$, then τ extends to $-\infty < t < \infty$ as a C^∞ function. Let $\pi/4 < \delta < \pi/2$ and let

$$R(\delta) = \{(\mu,\nu) \mid |\mu| < \delta\}.$$

Let $\epsilon = \epsilon(\delta) = \cot\delta$. Then, for all $(\mu,\nu) \in R(\delta)$, the function

$$t \to tb(t,\mu,\nu)$$

is strictly increasing on $(-\epsilon,\epsilon)$ and contains $(-\epsilon,\epsilon)$ in its image. Since b depends only on μ and t, it is easily seen that there is an interval $(-a,a)$ in $(-\epsilon,\epsilon)$ for which τ satisfies conditions (1)-(4) of §1.

We define $\mathcal{V}_K(\delta)$ to be the subspace of elements f of $C^\infty(R(\delta),\mathcal{H}_\infty(\sigma))$ which satisfy

$$f(\mu,\nu) = \sigma(\exp(-\nu H_0))f(\mu,0).$$

It is clear that for all δ as above, P_λ restricts to an operator (which we call $P(\delta)$) on $\mathcal{D}((-\epsilon,\epsilon),\mathcal{V}_K(\delta))$ which has regular singularities. In fact, $P(\delta)$ and P_λ have the same characteristic polynomial and the restriction mapping commutes with the boundary mappings on moderately growing solutions, as the reader may readily verify for himself.

Next, consider the space $\mathcal{V}_S$. This space injects into $C^\infty(\mathbf{R},\mathcal{H}_\infty(\sigma))$. We extend elements of $C^\infty(\mathbf{R},\mathcal{H}_\infty(\sigma))$ to functions on $\mathbf{R}^2$ which satisfy

$$g(x,\theta) = \sigma(\exp(-\theta H_o))g(x,0).$$

We let $\mathcal{W}$ denote the space of all elements of $C^\infty(\mathbf{R}^2,\mathcal{H}_\infty(\sigma))$ which satisfy this condition. The operator $C_\lambda = C - (1-\lambda^2)/2$ has regular singularities on functions valued in $\mathcal{W}$ and the boundary map commutes with the injection of $\mathcal{V}_S$ into $\mathcal{W}$.

Let

$$T : C^\infty((-a,a),\mathcal{W}) \to C^\infty((-\epsilon,\epsilon),\mathcal{V}_K(\delta)))$$

be the mapping defined from τ as in §1. Then T intertwines C_λ and $P(\delta)$. Thus, the boundary values of corresponding eigenfunctions must be related by Theorem10n. Combining this with Proposition21n, we conclude that for $-\delta < \mu < \delta$,

$$(\cos\mu)^{-2s_+}BV_N(T_{\lambda,\sigma}(F))(\tan\mu) = \lambda D^{-\lambda}F(\exp(\mu H_0)).$$

Since δ is arbitrary, this in fact holds for $-\pi/2 < \mu < \pi/2$. From formula40n, this is equivalent with

$$BV_N(T_{\lambda,\sigma}(F))(x) = \lambda D^{-\lambda}F(\Lambda(\exp xN_0)).$$

Our result follows from this.

Finally, let us prove formula2n. Let $L_s = T_N^{-1}(\mathcal{L}^+)$ where $\mathcal{L}^+$ is as in Corollary27n and T_N is as in §3. Then L_s is a dense subset of $L^2(N/R)$. Let $f_o \in L_s$ and set $\tilde{F}_o = T_N(f_o)$. Let F be the element of $\mathcal{H}_\infty(\pi_{\lambda,\sigma})$ defined by

$$\tilde{F}(\exp xN_0^t) = F_o(-x).$$

Then $T_{\lambda,\sigma}(F)$ is an eigenfunction for the Casimir operator and, according to Proposition28n, has boundary value equal to $\lambda D^{-\lambda}F_o$. Explicitly, from formula29n, formula41n and formula42n, in $N'A\tilde{K}$ coordinates,

$$T_{\lambda,\sigma}(F)(x',y) = y^{(\lambda-1)}/2\int_{\mathbf{R}}\sigma(k)\tilde{F}_o(x'-x)((xy^{-1})^2+1)^{(\lambda-1)/2}dx$$

where $k = \exp(-\tan^{-1}(xy^{-1})H_0)$.

Now, let $\tilde{\Box}$ be $T_N^{-1}(\Box+n-1)T_N$. From formula22n, T_N intertwines $\sigma(\exp tH_0)$ and $\exp -it\tilde{\Box})$. Proposition15n implies that T_N intertwines translation in the Z_o direction with translation in x. It follows that the above formula may be written

$$T_{\lambda,\sigma}(F) = I\otimes T_N(f)$$

where

$$f(n,y) = y^{(\lambda-1)/2}\int_{-\infty}^{\infty}e^{i\theta\tilde{\Box}}f_o(n(\exp(-xZ_o))((xy^{-1})^2+1)^{(\lambda-1)/2}dx$$

where $\theta = \tan^{-1}(xy^{-1})$ and $n\in N/R$. The integral will converge because $T_N^{-1}\tilde{L}(x)\tilde{F}_o$ will decay rapidly as $x\to\infty$ in $L^2(N/R)$ norm. From the comments above formula18n, f is in the kernel of $\tilde{\Delta}_\lambda$. Finally, then, applying U^{-1} where U is as in formula15n, we conclude that $y^{(n-1)/2}f$ is in the kernel of $\Delta_\lambda = \Delta+(n^2-\lambda^2)/2$. The boundary value of f is $\lambda\tilde{D}^\lambda f$ where $\tilde{D}^\lambda = T_N^{-1}D^\lambda T_N$. This finishes the proof of formula2n.

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