Pre-Calculus Students' Interactions with Mathematical Symbols

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Introduction

A large percentage of students come to college unprepared to take required mathematics courses. These students often begin their math requirements with a pre-calculus course for which they might not receive credit (McGowen, 2000; Schattschneider, 2002). In many cases, these pre-calculus classes are not working to prepare students for higher mathematics courses (Schattschneider, 2002); students are spending time and money retaking classes to a point where they may become blocked from reaching their educational goals (McGowen, 2000).

Teachers and researchers have observed that many difficulties in mathematics can be attributed to students' problems with understanding and manipulating algebraic symbols (e.g. Driscoll, 1999; Gray & Tall, 1994; Keiran, 2007; Kinzel, 1999; Stacey & Macgregor, 1999). Students bring their own interpretations of symbols to college classrooms based on interactions with symbols in previous classes (Kinzel, 1999; Stacey & MacGregor, 1999). Misunderstandings related to symbols and syntax in mathematics may lead some students to actually develop their own techniques in problem solving due to their personal interpretations of the symbols involved (Gray & Tall, 1994). The purpose of this study was to investigate college pre-calculus students' techniques for solving mathematical problems. The research question is: In what ways do the symbols in a mathematical problem influence pre-calculus students' initial goals in problem solving?

Background

Arcavi (1994) and Fey (1990) label the underlying understanding of algebraic symbols

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and their uses as *symbol sense*. Neither Arcavi nor Fey attempt to formally define symbol sense, claiming that to do so is difficult because it interacts with other senses such as number sense or function sense. Instead, they provide examples of what it would mean for a student to have symbol sense. Fey (1990) describes symbol sense as an "informal skill required to deal effectively with symbolic expressions and algebraic operations" (p. 80), while Arcavi (1994) explains it as "a quick or accurate appreciation, understanding, or instinct regarding symbols" (p. 31) that is involved at all stages of mathematical problem solving.

Working fluently with symbols in mathematics requires developing strong symbol sense. Arcavi (1994) states that many students fail to see algebra and its symbols as tools for understanding, communicating, and making connections, even after several years of study. He sees development of symbol sense as a necessary component of sense-making in mathematics. Table 1 shows several examples from Arcavi of behaviors that illustrate symbol sense.

Table 1

Behaviors that Illustrate Symbol Sense (Arcavi, 1994)

Elements of Symbol Sense
1. Understanding when to call on symbols for problem solving and when to abandon them for better tools
2. Having a feeling for an optimal choice of symbols
3. Ability to choose possible symbolic representations and to replace them if the first choice proves useless problem solving
4. Ability to extricate oneself from confusion by using other available tools to help regain symbol meaning
5. Realizing that a symbolic expression is needed and the ability to engineer it
6. Understanding different roles played by symbols
7. Understanding the need to continuously check symbols meaning and compare with one's own expectations and intuitions

Working from Arcavi's (1994) descriptions, Pierce and Stacy (2001) isolated key elements of symbol sense needed at the solving/manipulating stage of problem solving in order to take a closer look at the stage that is most affected when using computer-algebra system (CAS) technology. Pierce and Stacey call this collection of key elements *algebraic insight* and define it as " the algebraic knowledge and understanding which allows a student to correctly enter expressions into a CAS, efficiently scan the work and results for possible errors, and interpret the output as conventional mathematics" (Pierce & Stacey, 2001, p. 419). Figure 1 shows Pierce and Stacey's depiction of problem solving involving stages of formulation, solving, interpreting, and checking. As indicated in the figure, symbol sense is important in at least three of these stages, but they define algebraic insight as the specific symbol sense needed at the solving stage.

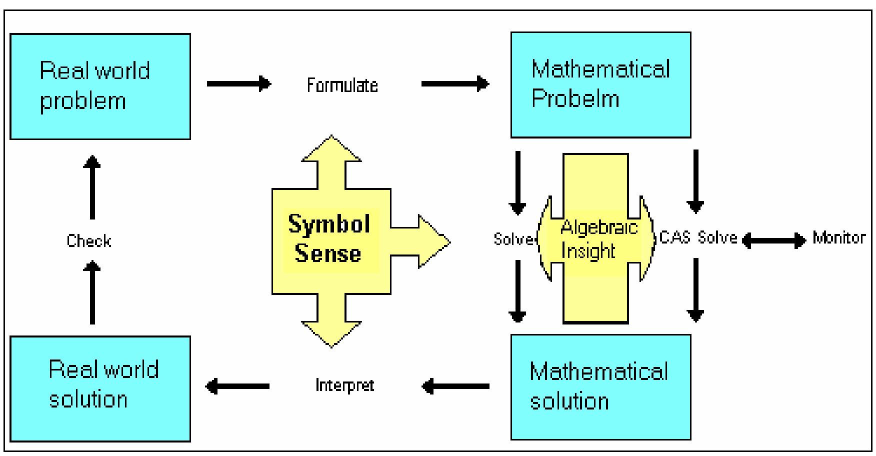

Figure 1. Algebraic Insight (Pierce & Stacey, 2001)

Simon, Tzur, Heinz, and Kinzel (2004) contend that, as a person works on a mathematical task, they engage in reflection on the activity-effect relationships in their work. Learners begin by setting a goal that is based on their current conceptions and is directed toward a particular problem situation. Learners then select (mental) activities with which they are already familiar. While carrying out this activity, learners continually monitor the effects and distinguish between results of the activity that made positive progress and those that made negative progress towards their goal. They create mental records of the relationships between each execution of the activity and the effect that it produced. By reflecting on these records and looking for patterns between the activities and the effects of the activities, the learner abstracts a new activity-effect relationship, which is the basis for a more advanced conception (Simon et al., 2004; Tzur & Simon, 2004). Thus, at any phase of the problem-solving diagram in Figure 1, reflections on the activities chosen and the effects of those activities in relation to how they help make progress toward a goal may be taking place. In this study, I look at just the initial goals and activities, and explore they ways in which they may be influenced by mathematical symbols.

Framework

Pierce and Stacey (2001) divide algebraic insight into two components. *Algebraic expectation* is the aspect of algebraic insight needed for working within a symbolic expression. It is the thinking that takes place as one considers possible results or outcomes of an algebraic activity. Three key elements of algebraic expectation include: (a) recognition of basic properties; (b) identification of structure, and (c) identification of key features of an expression that determine features expected in the solution. The second component, *linking representations*, is the insight needed to make connections between symbolic and graphic forms or symbolic and numeric forms. A mathematical idea can be represented symbolically, graphically, numerically, or in other ways. Having algebraic insight involves being able to anticipate what the graphical or numerical representation looks like given a symbolic representation, or vice versa. Pierce and Stacey see a need for students to recognize the meanings of both letter and operator symbols in order to inform their understanding of transitions between symbols and graphs or tables.

Pierce and Stacey (2001) assert that the nature of algebraic insight is the same for work done by-hand or with a CAS. Thus, I contend that the framework for assessing algebraic insight is also appropriate for examining students' problem solving both with non-CAS graphing calculators and without a calculator. The two components that make up algebraic insight, algebraic expectation and the ability to link representations, should be equally identifiable when analyzing students' work done by-hand or with graphing calculators. Thus, I adapted their framework to evaluate students' symbol sense when solving problems with or without a calculator, and added elements from Table 1 using Arcavi's (1994) description of symbol sense to better address the tasks on which pre-calculus students worked in the study. [1] This symbol sense framework provided observable instances to which I could attend when analyzing students' mathematical thinking.

Table 2 shows the main elements of the resulting symbol sense framework along with common instances and examples of each. Much of Pierce and Stacey's (2001) original framework 1 remains intact because it is a powerful framework for reflecting on symbol sense at the solving stage. I made only four alterations: (a) element 1.1 and its common instances were included to address work in the formulation stage of problem solving, (b) Pierce and Stacey's instance "recognize simple factors" was incorporated as an example of "identify strategic groups of components", (b) two instances for linking symbolic and numeric representations were removed because they did not address the types of problems solved in pre-calculus, and (d) element 4.1 and common instances were added to address work in the solving stage. Most of the examples shown in the table come directly from Pierce and Stacey.

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[1] Pierce and Stacey (2001) maintain that the common instances of each element of algebraic insight may need to change to relate to the particular level of student and content of study (their instances were chosen for work with a beginning calculus course).

Table 2

Symbol Sense Framework

Elements	Common Instances	Examples
1.1 Link verbal and algebraic representations	1.1.1 Know how and when to use symbols	choose appropriate symbols in word problems
	1.1.2 Knowing when to abandon symbols	Use graph or table to explore features of the problem
	1.1.3 Ability to select possible symbolic reps	Creating an equation from a word problem
	1.1.4 Know that chosen rep can be abandoned	Changing names when 2 letters represent same thing
2.1 Recognize basic properties	2.1.1 Know meaning of symbols	In $f(x) = ax^2 + bx + c$, *letters* are parameters, names and variables
	2.1.2 Know order of operations	$a+b/c$ or $(a+b)/c$
	2.1.3 Know properties of operations	each op has inverse operation: $(a+b)^2 \neq a^2 + b^2$
2.2 Identify structure	2.2.1 Identify objects	See fn as an object; Identifying equivalent expressions
	2.2.2 Identify Strategic groups of components	Decompose fractions; See domain; Recognize simple factors
2.3 Identify key features	2.3.1 Identify form	$2 + e^x$ is exponential
	2.3.2 Identify dominant term	Degree of polynomial; multiplicity
	2.3.3 Link form to solution type	$x^2+3x+12$ has imaginary zeros
3.1 Link sym and graph reps	3.1.1 Link form to shape	$\frac{x^2-1}{x+1}$ graphs as a line
	3.1.2 Link key features to likely position	Noticing intercepts, maximum points, etc.
	3.1.3 Link key features to intercepts or asymptotes	Intercepts = zeros of function; Zeros of denominator indicate a hole or vertical asymptote
3.2 Link sym and num reps	3.2.1 Link key features to critical intervals for table	Knowing that the zero is not an integer
4.1 Recognize meaning	4.1.1 Link sym meaning to problem	Providing answer to the question being asked
	4.1.2 Use symbols to personal expectations	Correctly using the equal sign

Methodology

Site and Participants

I conducted a qualitative multi-case study of six pre-calculus college students. A mathematics department at a public state university in the southern United States served as the site for this study. Participants were selected from class sections taught by mathematics graduate students in the department. To assist in the purposeful sampling of participants, a survey was issued to all pre-calculus classes taught during the semester. The survey assessed two main criteria: level of ability in mathematics, and familiarity and use of the graphing calculator.[2] For the first criterion, two A-level students, two B or C level students, and two students who were retaking the course were selected in the sampling process. Choosing students with different levels of success in mathematics helped produce variation in the small group of participants.

The six invited participants consist of five female students, Nina, Jill, Beth, Molly, and Elyse, and one male student, Shawn (all pseudonyms). Each student represents one case in the qualitative case study. Jill and Nina were enrolled in a different section with a different instructor than the other four students, but both instructors taught in a similar manner and covered the same material in the five-week semester. Nina and Beth were retaking the course during the summer term after failing to receive a satisfactory grade in the course in the spring semester. Shawn and Molly both had not taken a mathematics course in three years, while Jill and Elyse had taken at least one college-level course in the past year.

Data Collection

To help overcome difficulties in interpreting students' mental processes, I conducted personal interviews that contained carefully constructed tasks and questions to identify students'

[2] This was to address a second research question related to students' uses of graphing calculators. This issue is not being addressed in this paper.

symbol sense (Goldin, 2000). An interview guide for each interview was prepared ahead of time with questions and tasks to present to the participants. Different questions were used with different participants, depending on the detail of responses and on the types of follow-up questions needed for a particular response.

Students worked on mathematical tasks (shown in Table 3) in an interview conducted near the beginning of the summer term. As they worked on the tasks, students talked aloud about their thoughts and actions. Because these students were at the beginning of a pre-calculus course, the tasks involved concepts learned in algebra and focused on functions and equation solving. I expected that students would be familiar with the types of problems in the tasks. The students who were retaking the course would have reviewed these types of problems as part of the course in the previous semester.

Table 3

Analyzed Tasks

#	**Task**
1	Solve for x: $\frac{x+16}{x^2-3x-12}=0$
2	Solve for x: $x^3+2x-4=8$
3	Solve the linear inequality: $\left|3x-\frac{2}{7}\right|+1.2>5$

Data Analysis

Students' work on each task was analyzed to examine their symbol sense by looking at their initial reactions to the symbols and structure of the problem. I used instances of symbol sense that were observable in written work or inferred from discussions to investigate how symbols and symbolic structures influenced students' ideas for working with the problem. I assigned codes from the symbol sense framework to lines of the transcript associated with students' initial reactions to a given problem. Using a spreadsheet, I recorded the structure of the problem, symbol sense codes, student's goals, activities used to try to reach the goal, the perceived effects of that activity, and the written work. Table 4 is an example of the spreadsheets created for each student for each task. These tables allowed me to look for patterns both within and across cases.

Table 4

Example Analysis Spreadsheet: Nina's work in Task 1

LINE	STRUCTURE	CODE	GOAL	ACTIVITY-EFFECT	WORK
1	Rational Equation	**2.2.2** **2.3.1**	Simplify	A1: Set numerator equal to 0. Tries to factor denominator. E1: Gets frustrated and nervous because she can't factor	$\frac{x-16=0}{(\quad)(\quad)}$
	Rational Equation	*2.2.1* **2.3.3**	Find *x*	A1: Decides that *x*=0 if the whole equation equals 0 E1: Realizes that 0-16 is -16, not 0. Decides she will need different *x*-values for the numerator and denominator.	
	Rational Equation	*2.1.3*	Divide to get 0	A1: Decides she just needs to find where denominator is zero because dividing by 0 gives 0. E1: Cannot find a number that works. Gives up.	

It is important to note that my overall analysis involved looking at students symbol sense throughout the entire problem. Often the structure of the problem changed as students began manipulating the symbols (e.g. from a rational equation to a polynomial equation if students multiplied both sides by the denominator), which may have caused students to alter their focus. However, I have chosen to address the research question in this paper by looking only at the initial effect that symbols had on students' problem solving. The transcripts shown below include only students' discussions related to the first row of each of the tables used in analysis.

Findings

The following sections contain data and analyses for each of the three tasks. Each section includes a brief analysis of each student (case) and a cross-case analysis for the specific task.

Task 1

The first task that students attempted was a rational equation. Table 5 contains verbatim transcripts of each student's initial reaction to the task, as well as the symbol sense codes attributed to the statements and the initial goal demonstrated for the problem. Codes in bold font indicate an instance of symbol sense was present; codes in italics indicate evidence of a lack of a particular instance of symbol sense. Note that it was possible to code a student as both having and lacking the same symbol sense within a given discussion.

Table 5

Initial Reactions to Task 1: Solve $\frac{x+16}{x^2-3x-12}=0$

Student	Transcript	Codes	Goal
Jill	"Okay the first thing I see it's that it's a perfect square, so I would probably do *x* +4 and *x*-4. Then I'd do the foil thing (pointing to denominator). I'd do *x, x*…gosh I haven't done these in so long. Let's see, that has to be a 4 and a 3…so I guess…I'm just	**2.2.2** *2.3.1* *2.3.3*	Factor and Cancel

	guessing and checking right now.”		
Molly	“I normally don’t know what to do with the *x* squared on the bottom. And sometimes I’m confused with whether or not I should multiply this (denominator) on both sides, or set it up on both sides to get it on top and then solve it…So I could technically put that (left hand side) into the calculator and see what happens but I don’t think anything will…See if it gives me a break down of it.”	*1.1.2* *2.2.2* *2.3.1* *3.1.3*	Identify procedure
Nina	“I see a fraction and it’s all equal to zero, so we’ll set the top equal to zero (writes x-16=0), and then I’m going to undo the bottom…I feel like I should simplify, well, it is simplified”	**2.2.2** **2.3.1** *2.2.2* *2.3.3*	Simplify
Beth	“First I would have to factor this out. So I would do this (writes T chart). Then this is my *a* (pointing to the coefficients), this is my *b*, this is my *c*. And I can take out (writes (x)(x)) then…well x times x, that's true. And then what numbers would I have to multiply this by to get 12...yeah. No - I'd have to multiply this (first term) times this (12) to get 3.”	**2.1.1** **2.3.1** *2.2.2* *2.3.3*	Factor
Shawn	“I'd say you'd have to factor out the bottom (long pause)… I'm trying to think about how I would factor that, and if it would make sense, cause I would think that if you were trying…oh you’re trying to solve. I was thinking simplify before.”	**2.3.1** **4.1.1** *2.3.3*	Simplify/ Solve
Elyse	“The first thing I’m thinking is I need to somehow get x by itself…I know that you can probably factor the bottom, but I don’t want to do that right now… I think I’m going to start by adding 16 over here (indicates numerator) to cancel it out (writes +16 in the numerator) and add 16 to the bottom (writes + 16 below the denominator. (writes $x/(x^2-2x)$). Ok, at least one of the x’s is isolated.	*2.1.2* *2.1.3* *2.2.2* *2.3.3*	Get rid of fraction

In this first task, Jill’s initial reaction to the problem was to mistakenly identify the numerator as a difference of squares and to factor it as such. Her initial goal was not related to solving for x or to understanding the meaning of the problem. She was determined to factor the numerator and denominator so that two of the factors cancel. Her goal was influenced by both her symbol sense for identifying structure and her lack of symbol sense for correctly identifying the form of the numerator.

For Molly, the rational structure and quadratic denominator brought to mind several possible goals, but she was not certain which may be appropriate. She struggled with the idea of trying to use a graphing calculator for this problem, but lacked the symbol sense for understanding how the symbolic expression may link to a graphical representation. She also struggled with the symbol sense needed to select an appropriate activity for working with the rational structure.

Nina's first reaction to the problem was to set the numerator equal to zero, but she then quickly turned her attention to the denominator. Although she wrote an equation that she could easily have solved, her overall goal was to simplify. This came from her symbol sense for identifying the form of the problem as a rational expression, but she failed to link that form to an appropriate solution type.

Beth focused her attention on the quadratic expression in the denominator and seems to have ignored the rest of the terms completely. She drew a T-chart shown in line 1 (which is a tool for organizing the coefficients, *a, b*, and *c*, in a quadratic function ax^2+bx+c and find factors.) She did not fill in the chart with values, but mentally checked computations for factors of 12 instead. Her goal was simply to factor. She had symbol sense for understanding the meaning of the symbols in the quadratic term, but lacked the symbol sense for linking the larger structure to a solution type.

Shawn started the first task by focusing on the denominator of the rational expression. He said that he needed to factor the quadratic denominator, but he seemed concerned about the effect of this activity. After a moment of silent reflection, he explained that he had thought that he was supposed to simplify the expression, and he did not see how factoring would help. He did not clarify, but it is possible that he realized that he would not find the numerator (x-16) as a factor of the denominator. Once he realized that the question was asking him to solve for x and not simplify, he felt more comfortable with his chosen activity.

Elyse interpreted the need to solve for x as a need to isolate x on one side. However, her actions and discussions suggest that her initial goal was actually to get rid of the fraction by any means possible. She chose to isolate the x in the numerator by adding 16 to the numerator and denominator. Her lack of symbol sense for knowing both order and properties of operations here caused her to create her own methods for reaching her goal.

All six students struggled with the symbol sense to identify the numerator as a strategic term on which to focus for solving the rational equation and to link the rational form to an appropriate solution type. The influence in this problem seems to have come from the combination of two structures: (a) The rational structure was something with which several students were uncomfortable and wanted to eliminate; and (b) the quadratic structure provided students with a manipulation (factoring) to perform when they did not know what else to do. This combination created a desire to manipulate that dominated students' thinking and kept them from attending to what it might mean for this type of expression to be equal to zero.

Task 2

The second task was a cubic polynomial equation that could not be easily solved algebraically. The purpose for giving this problem was to observe what students paid attention to when the structure was not as familiar to them (as, for example, quadratic structures might be). Table 6 shows transcripts of students' initial reactions to this problem.

Table 6

Initial Reactions to Task 2: Solve $x^3 + 2x - 4 = 8$

Student	Transcript	Codes	Goal
Jill	"Okay the first thing I'd do is move the 4 over. Hmm…then I would move the 12 over and equals…wait. Why'd I do that?... I guess I did that so that way it would maybe it would be easier	**1.1.3** **2.1.3** **2.3.1**	Manipulate

	and that way I could just plug into these and it would all equal zero. But then again that would just be guess and check– wait I guess…no you can't do quadratic…because it's *x* cubed.		
Molly	"Well, um, I'm thinking that I need to substitute 8 in, but I might be wrong…I'm used to seeing it set up like $f(x)$ equal…so it' s a little different. But I could try that and then I could graph it to see. Does that sound okay? Or I could graph it and then try to figure out where x=8 is."	**3.1.2** *2.1.1* *2.2.1* *4.1.1*	Evaluate
Nina	"Okay I would set everything equal to zero. (writes) Cause if I set everything to zero then I can simplify it and figure out what the *x*'s are. (combines like terms). Um, it's a cubic. Those are always so much fun (sarcastic laugh). (writes 2 sets of parenthesis). I feel like I shouldn't factor because it's cubic…I feel like you have to keep everything on one side. I feel like I wouldn't be able to just factor out that *x* just to factor it out. Um…I could try it anyways.	**1.1.3** **2.2.2** **2.3.1** **2.1.3**	Find a value for *x*
Beth	"What I would do is get all of my *x*'s on one side and all my numbers on the other. So add 4 to both sides (writes +4) (writes x^3 +2x=12) to get 12, and then typically I would just plug in numbers here. I can use my calculator for this."	**1.1.2** **1.1.3** **4.1.2** *2.1.1*	Find a value for *x*
Shawn	"Oh this cannot…I can't…with the cube. There was a book problem like this, and I asked and [the teacher] said we're going to do this next week. So, if we did this next week I could probably… Wait, let me just work it out (Moves constants to right side and factors *x* out on left. Sets factors to 12 and solves for *x*)."	**2.3.1** *2.3.3* *4.1.1*	Find a value for *x*
Elyse	"Okay I feel better about this one…there's no fraction. Okay. Here I know that you want to isolate the *x*'s from the non-*x* numbers, so I'll add 4 to both sides (writes +4 on both sides and computes by hand and writes result). Okay, and then…you can divide that by 2 to get that *x* alone (divides 2*x* on left and 12 on right by 2)	*2.1.2* *2.2.1*	Isolate an *x*

In this task, Jill's initial reaction was to combine the constant terms in the equation. Her reaction to the problem was guided by a procedure inherent in the symbols. Initially, she just moved the constants around, adding 4 to the right side to get x^3+2x =12, and then subtracting the 12 to put it back on the left side of the equation. It was only after these manipulations that she stopped and questioned why she might have chosen this activity. Her initial goal was to manipulate the terms. She demonstrates the symbol sense for selecting an appropriate representation of the symbolic structure with which to work.

Molly's first reaction was to identify the value 8 on the right side of the equation as a value to substitute in for x. She interpreted 8 as being equivalent to the value $f(8)$ for this function. This interpretation indicates that Molly did not identify the equal sign as a symbol for equivalence, but instead as a symbol indicating some action to take. Her activities on this problem were driven by this interpretation. The equal sign and the 8 were symbols for which she had created her own meaning in order to reach her goal and complete the task.

Nina immediately rewrote the problem to set it equal to zero, claiming that this would allow her to simplify the problem and determine the values for x. She then identified that it was a cubic function, drew two sets of parentheses, and then questioned whether factoring was a reasonable activity for this type of problem. She possessed the symbol sense for identifying both the form of the problem and for identifying the structure needed to work with the problem. However, even with a sense that factoring to produce $x(x^2+2)=12$ was not a legitimate or useful activity, she decided to try this anyway to meet her goal of finding x.

Beth's goal was also to find a value for x. She chose to rearrange the given equation to separate the terms involving x from constant terms on opposite sides of the equal sign. Once she obtained the equation $x^3+2x=12$, she used a guess and check strategy to meet her goal. Beth possessed the symbol sense for choosing a useful form of the symbolic representation and for linking the meaning of the symbols to her own expectations for the problem. However, she seems to have used the equal sign as a separation tool, isolating the variables from the constants, which is why I assigned a code of lacking knowledge of the meaning of symbols (=) here.

Given the cubic form, Shawn decided that he needed to factor, but identified that this would not factor in the same way that a quadratic function would. He chose to move the constant terms to the right side of the equation, and to factor x out of the remaining x-terms on the left side in the same way that Nina had. However, unlike Nina, who engaged in this activity even though she felt strongly that it was not appropriate, Shawn lacked the symbol sense for linking this form to an appropriate solution type or for linking the meaning of the structure he had created to the original problem.

Elyse decided that her goal on this problem was to get x by itself. This idea was similar to her ultimate goal from Task 1, but she chose different activities to accomplish the goal. This time she chose to start by combining the constant terms, and then, noticing the coefficient in front of one of the x terms, divided only the $2x$ and 12 by 2 to get $x^3+x=6$. She again demonstrated a lack of symbol sense for knowing order of operations and did not link the form of the problem to the solution type. Her goal, influenced by this lack of symbol sense, was to isolate only one x in the problem (i.e., she would have felt she'd reached her goal even if there were additional x's on the other side of the equation).

The main influence on all of the participants' work on this problem seems to have been the cubic polynomial. They all referred to the fact that there was an x^3 term in the problem. Some compared it to a quadratic to identify things that they could *not* do (e.g., they could not use the quadratic formula). Without a definite, memorized rule for dealing with cubic equations, students were more creative in their activity choices. Several of the students set a similar goal for finding or isolating x, but chose different activities for reaching that goal based on their interpretations of the structure.

Task 3

The last task in this interview was a linear inequality that also involved an absolute value expression. Table 3 shows students reactions to this problem.[3]

Table 6

Initial Reactions to Task 3: Solve $\left|3x - \frac{2}{7}\right| + 1.2 > 5$

Student	Transcript	Codes	Goal
Jill	"Okay, first thing I would do is move the 1.2 over, so I would have 3*x* –2/7, absolute value, then I would do (uses calculator) I don't feel like doing that in my head. So 3.8. Then I would add 2/7 on this side…to get a decimal – I like decimals better than fractions – And then I would just divide by 3	**2.1.2** *2.2.1* *2.3.3*	Isolate x
Nina	"Okay, so it's an absolute value. Oh my gosh I forget how to do these. Okay. The first thing I would be to…add…oh okay. 3*x*…keep everything in the absolute value, subtract the 1.2. I can't remember if I'm supposed to change the sign. I think that's when I divide I change the sign, so I'm gonna leave it."	**2.1.3** *2.1.3* *2.3.3*	Manipulate
Beth	"I have to find *x* again. So (writes find *x*). Again, I would probably, and that's absolute value. I would probably go and punch in numbers on the calculator. And let's see 3…I'll just pick a number. Let's say 2 …(uses calculator) Okay. Well I know that 6.9 is greater than 5. So that could work!	*2.1.1* *2.2.1* *2.3.3*	Find a value for *x*
Elyse	"Um, I know…that means absolute value between those lines…Absolute value would be the distance from zero on the number line, in either direction, and you would not have a negative. I want to take care of what's in that absolute value section first. And then turn that into a regular number, and then add it to the rest of the equation after I figure out whatever is going on in there. So I think…you need to…to get rid of that fraction, times it by its reciprocal."	**2.1.1** *2.1.2* *2.1.3* *2.3.3*	Simplify inside the absolute value

Just as in Task 2, Jill's initial goal on this task was to manipulate the terms. The manipulation activities that she used to meet this goal do not seem to have been greatly influenced by the inequality sign; she treated this inequality problem in the same way that she might treat a linear equation. Her activities do seem to have been influenced by the absolute

[3] Due to time constraints, Molly and Shawn were not able to work on this problem in the initial interview. However, the other students' work on this problem was interesting and important for addressing the research question, and was therefore not excluded from this report.

value sign, although not in a way that indicates a conceptual understanding of this symbol. The order of operations that she followed indicate that she treated the absolute value sign like parentheses in an equation. Thus, although she knows order of operations, she did not link the overall form of this problem to a correct solution type or identify the absolute value expression as an object that needed to be treated differently.

The absolute value sign initially had an influence on Nina's work, as indicated by her decision to keep the terms inside the absolute value sign together. She also it like parenthesis. She subtracted 1.2 to the right side of the inequality. She then paid attention to the inequality symbol by trying to recall rules for working with this symbol and recalled that the sign changed direction when dividing. Nina lacked the symbol sense needed to link the form of this problem with its correct solution type, and to understand how the absolute value and inequality symbols affected the problem.

Beth related this problem to her work in Task 2 and chose the same guess and check activity that she had used in the previous problem. She chose x=2 as a guess, used the calculator for computations, found a value that was greater than five, and concluded that she had her answer. She did not link this symbolic structure to the correct solution type, but instead interpreted the problem in the same way she had interpreted a polynomial equation. She did mention the absolute value sign, but it is not clear if she knew its meaning or used it in her calculations, since 3*2 – 2/7 produced a positive number on her calculator (She did not use the ABS function or mention that she needed a positive value).

Elyse's first reaction to the problem was to focus on the absolute value symbol. She clearly defined what absolute value meant to her, bringing to mind the fact that applying the absolute value to a number always produced a positive result. She decided that to solve the problem, she needed to work inside the absolute value sign. She treated the absolute value sign as parentheses that dictated an order of operations to follow. She was concerned again by the presence of a fraction in the problem, claiming that she needed to get rid of it. She demonstrates a lack of symbol sense for knowing properties of operations for accomplishing this, however, believing that just introducing the reciprocal will remove the fraction.

Three of the students treated this task in the similar way to Task 2, indicating that the inequality symbol did not change their goal for solving a polynomial problem. None of the students linked the inequality and absolute value forms to an appropriate solution type. The absolute value symbol did influence students' goals and activities in this problem, although it was mainly interpreted as having a similar meaning to parentheses that dictated order of operations.

Discussion

Overall, it seems clear that students paid attention to different parts of a given problem and that their initial goals were based on the symbols on which the focused their attention. On these three tasks, students primarily demonstrated symbol sense for selecting appropriate forms of the symbolic representation on which to work, but lacked the symbol sense for linking the forms to a correct anticipated solution type. Only one of the students attempted to link the symbolic representation to a graphical or numeric one using the graphing calculator, even on the polynomial task (Task 2) where symbolic manipulation was not useful.

Tzur and Lambert (in review) suggest that certain questions, observations, or objects can serve as a prompt for orienting a students' goals and activities as they engage in reflection on activity-effect relationships (AER) as identified by Simon et al. (2004). Tzur and Lambert assert that what prompts an activity and triggers the AER mechanism can take various forms. I contend that, in this study, students' goals and activities were often prompted by the presence of particular mathematical symbols or symbolic structures such as inequality signs, equal signs, fractions, and absolute value symbols. Students often had trouble recalling rules for working with these items. They anticipated the presence of particular symbols or symbolic forms in their work, and sometimes create new ways of accomplishing their goals that did not follow mathematical rules.

Implications

It is important for teachers to be aware of the inconsistencies between teachers' and students' goals in problem solving. It was clear that students lacked many of the desired instances of symbol sense that Arcavi (1994) suggests are so important to the broader theme of sense making in mathematics in general. However, it is not too late, even at the college level, to engage students in conversations about the meaning of symbols and how meanings link to students' prior experiences and to specific problems.

Building symbol sense can help build students fluency with the complicated language of mathematics. One way to build symbol sense in a pre-calculus class would be to include more symbols in discussions and in examples. Students are uncomfortable with $<$, $|x|$, a/b, $\sqrt{\ }$, x^3, because they are often not required to work with them as much in problem solving. The students in this study were surprised when more than one of these symbols was used in the same problem because this was not the way that most of their teachers had presented them in the past.

This detailed look into the ways in which students interpret mathematical symbols can be useful in identifying ways to strengthen students' understanding of symbols and to improve their mathematical capabilities. It may also improve college teachers' awareness of the networks of understandings that students have developed about mathematical symbols and they ways in which they learn to interpret the mathematics.

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