Links in learning logarithms

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The beautiful history of the development of logarithms (Smith & Confrey, 1994), coupled with the power of the logarithmic function to model various situations and solve practical problems, makes the continued effort to support students' understanding of logarithms as critical today as it was when slide rules and logarithmic tables were commonly used for computation. They continue to play an important role despite the fact that calculators are now used for many computations involving logarithms: logarithmic scales can increase the range over which numbers can be viewed in a meaningful way. As described in the senior secondary curriculum, logarithmic scales are used regularly in astronomy, chemistry, acoustics, seismology, and engineering and students should be able to "identify contexts suitable for modelling by logarithmic functions" and be able to "use logarithmic functions to solve practical problems (Australian Curriculum and Assessment Reporting Authority [ACARA], 2012)

By the time students enter a mathematics course at University, they have likely been exposed to logarithms in Year 10 as an enrichment topic and logarithmic functions in Senior Secondary Mathematics. While the preparation for University study is slightly different in the United States, concerns regarding student struggle with understanding is consistent. We have repeatedly witnessed students' difficulties with logarithms while teaching university mathematics courses. Our observations mirror those of other teachers. For example, Hurwitz (1999) notes that, "Students often have difficulty thinking of a logarithm as the output of a function because the notation used for logarithms does not look like the familiar $f(x)$ notation" (p. 334). Gramble (2005) relays that teachers often tell students that logarithms are exponents, but, "for some reason students hear the terms exponents and logarithms but often do not understand the relationship between them" (p. 66). Hurwitz's statement can be applied to other functions such as $f(x) = \sin(x)$, and $f(x) = a^{(x)}$ that are included in the Mathematical Methods curriculum and which differ from earlier functions encountered; however, students' struggles with logarithms and logarithmic functions are the focus here. Exploration of challenges in understanding logarithms as real numbers and logarithmic functions as well as their graphs provide insight that can be used as the basis for instruction. In this paper, we discuss and share evidence of students' difficulties collected from various courses over time. We share concepts related to logarithms that could help students build an understanding of these functions, and we present some ways that misconceptions related to these concepts are manifested to suggest what teachers can listen for as they explore logarithms with students.

What is a logarithm?

Teachers all over the globe undoubtedly hear the question, "What is a logarithm?" repeatedly. Typically it is defined as an exponent: The logarithm of a positive number x is the power to which a given number b (called the base) must be raised in order to produce the number x (ACARA, 2011). For example, because we know that $5^2 = 25$ is true, we can also say that the logarithm with base 5 of 25 is 2 (that is, the equation $\log_5 25 = 2$ has identical meaning to the first equation). We also define a logarithm as a function—specifically an inverse function to raising a number to a power where the exponent is the output of the function. Formally, a logarithmic function with base a is defined as $y = \log_b(x)$ (or similarly $f(x) = \log_b(x)$) for $x > 0$ and $b > 0$, $b \neq 1$, if and only if $x = b^y$ (e.g., see Larson, Edwards & Hostetler, 2006). It is clear that a rich understanding of logarithms relies on students' understanding of both of these definitions, which involve many different mathematical concepts. In this paper we focus on the importance of understanding the notion of function, mathematical notation, and properties used in relation to functions in order to work with logarithms.

Concepts and ideas that build towards understanding

Developing an understanding of functions includes being able to classify a relationship as a function. For example, understanding that $y = \log_b(x)$, where $x > 0$, is not just an exponent (or number) but is a function and, as such, has a domain and range, its input values produce unique output values, and it has an inverse function $f^{-1}(x) = b^x$. These ideas can be used to build understanding of logarithms. For example, students can understand why $\log_b(1) = 0$ for all bases by recognising that the inverse relationship $b^0 = 1$. As a function, $\log_b(x)$ can also be composed with other functions, including its inverse relationship b^x. Recognising the relationship that results from the composition of inverse functions leads to understanding the logarithmic function as a key tool for solving exponential equations.

Understanding logarithmic functions relies in part on being able to interpret the notation and symbols involved. A critical component of this is being able to interpret the logarithmic notation as representing both an object and a process (Liang & Wood, 2005; Tall & Razali, 1993; Weber, 2002b). For instance, $\log_a$ is used as a referent to a specific logarithmic function and as an indicator of the value used in an exponentiation process. In all logarithmic functions of the form $\log_a(x) = y$, the input value $x > 0$ can be thought of as both a domain value for the logarithmic function and the product of y factors of a (Weber, 2002b). Trying to make sense of this dual role played by the notation can cause great confusion for many students (Tall & Razali, 1993).

The development of the idea that logarithmic functions are characterized by their own unique set of properties that distinguish this function from polynomial and other functions is a significant characteristic. For example, the functional relationship $f(mn)$ is applied differently for different function families: For $f(x) = x^2 + 2x + 1$, we can replace the input x with mn to produce $(mn)^2 + 2(mn) + 1$ and then simplify using algebraic properties. However, for $f(x) = \log_b(x)$, when we replace the input we get $\log_b(mn)$ which, due to properties of exponentials, can be written as $\log_b(m) + \log_b(n)$. We want students to understand and be able to use these properties to simplify expressions, but also to recognise problem situations where logarithms can be applied (for example, in situations where adding two numbers might be far easier and more accurate than multiplying them). Properties and connections between the logarithmic and exponential forms are needed to solve exponential and logarithmic equations (Weber, 2002a). Because students have more experience with algebraic functions and their properties, building an understanding of the unique characteristics of the logarithmic and exponential functions can be challenging.

Student difficulties

Logarithms as inverse functions

Many students understand logarithms as something you primarily *do* or *convert*. While interviewing two students, the first author found that both could easily solve problems like $\log_2(x + 7) = 3$ by converting the expression to $2^3 = x + 7$. However, when given $\log_4(-16)$, both students tried to complete the same type of conversion by using guess and check to find a value to which 4 could be raised to produce -16, believing that the answer should either be a negative number or a fraction (Kenney, 2006). Both suggested that there would be an answer if they kept looking, but they were not able to find one. This suggests a misunderstanding of the concept of the inverse relationship of exponentiation. Teachers hope that students will be able to use their understanding of exponents and inverse functions to build an understanding of logarithmic functions (Weber, 2000a; 2000b). Yet if logarithms represent only an action such as converting logarithmic to exponential form as in the example above, rather than an inverse function object, certain understandings, such as restrictions on the domain, may not develop.

Teachers often use graphs with logarithms to support the development of a relationship between the function and its inverse. In Figure 1, we see that Nora, a college algebra student, has created an image similar to those seen in many textbooks. The image intends to convey the notion that the two curves are inverse functions (Kastberg, 2002).

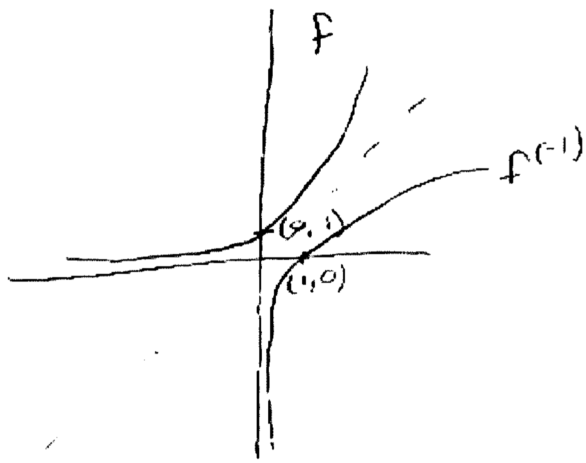

Figure 1. Nora's drawing of a logarithmic function.

Nora's discussion of her drawing reveals that she can regenerate the image the teacher shared in class.

N: I don't remember too much about the graph... Just the basic log graph is going to look something like... It is going to have an asymptote at [y = 0]. And that is going to go through (0,1) [draws exponential function]. And f inverse is going to go through (1,0) and it is going to have an asymptote like [the y-axis] and then it will have symmetry [draws $y = x$].

S: So which one of those is the log graph? Or is it all the log graph?

N: It is all log, but this is just the f and this is the f inverse [labels the exponential function f and the logarithmic function $f^{(-1)}$].

For Nora, the image created was "all log." When asked to graph a logarithm function, she always drew both an exponential and a logarithm. While her ability to generate the image is a good starting point, building an understanding of the image would allow her to use it to check or explore questions involving logarithms. For example, if asked to use this graph to find log(−10) Nora might initially incorrectly assign a value. This would prompt the teacher to focus efforts on differentiating between the two functions in the image.

Logarithmic notation

Students begin to build their understanding of function notation as they make sense of how to evaluate functions using the notation (Hurwitz 1999). For example, $f(x) = 2x + 3$ tells the learner to double an input and add three. This notation also allows students to work with functions as objects or processes by ignoring the "$f(x)$" symbols on the left and dealing only with the right hand side. Hurwitz (1999) finds that logarithmic notation, however, leaves students "bereft of a succinct way to verbalize the operation performed on the input" (p. 344) since no clear operation is signified in $f(x) = \log_2(x)$. This is not unlike the challenge students face when they begin to interpret $y = 2x$, where x is multiplied by 2 is signified by the juxtaposition of the x and the 2 rather than a multiplication symbol. It is also similar to their interaction with other transcendental functions such as $f(x) = \sin(x)$. Additional complications come into play with logarithmic functions because we have special unique notation for certain bases, namely $\log(x)$ for $\log_{10}(x)$ and $\ln(x)$ for $\log_e(x)$.

Students struggle to interpret logarithmic notation and often reach for their existing understandings drawn from polynomials. For example, consider the following unequal expressions: $\log_3(x) + \log_3(x + 1)$, and $\log_5(x) + \log_5(x + 1)$. When a class of 59 college algebra students were asked to identify whether the two expressions were equal or unequal, 16 students claimed that the expressions were equal, reasoning that log was irrelevant because it could be "cancelled out" (Kenney, 2006). A follow-up interview with one student, Lynn, further demonstrated how students might eliminate the log notation from the expressions in order to act on the remaining terms. Given $\ln(x) - \ln(x + 3)$, Lynn shared the following:

L: The ln-s are… they cancel out to give you x minus $x + 3$, so x minus x would be 0 so you'd just have the plus 3, well, –3.
I: What happens to the ln-s again?
L: They cancel out.
I: Where do they go?
L: I guess they just disappear.

This experience is similar to a finding from Yen (1999) who analysed types of errors made by Australian students on a mathematics examination and saw that student often divided an equation like $\ln(7x - 3) = \ln(2x)$ by ln as though it was a variable in the equation. In the absence of clear understanding of the role of the symbols involved in logarithmic notation, students fall back on their understanding of polynomials to reason about logarithms.

In the same algebra survey (Kenney, 2006), 35 out of 59 students answered that $\ln(x)$ was equal to $\log(x)$ although these special notations for the natural log ($\log_e$) and common log ($\log_{10}$) were distinguished in class discussions and activities. Students were also asked to identify the base of each logarithm in all class discussions. However the students predominantly interpreted ln and log as interchangeable symbols. Teachers need to help students recognize that mathematics has a specialized language and uses conventions to represent different ideas (Stacey & MacGregor, 1997). In addition, gathering student insights about the meaning of the symbols allows time for the development of personal meaning for them.

Unique properties of logarithms

The unique properties of logarithms are powerful in their ability to transform expressions and support the solving of equations. Yet without connections between ideas and development of these properties, there is room for overgeneralizations from instruction and from prior knowledge (Liang & Wood, 2005). One such example is the overgeneralisation of the idea that logs of the same quantity using different bases yield different results. In the college algebra survey, 57 out of 59 students correctly answered that $\log_3(2)$ was not equal to $\log_4(2)$ because the bases were different. Yet, 26 students used this same reasoning to say that $\log_a(1)$ was different from $\log_b(1)$ because the bases were different (Kenney, 2006).

Properties are powerful tools students use to simplify expressions and solve equations, however overgeneralisation of polynomial ideas applied to notation coupled with new properties can cause confusion. During class, Jamie was confused about expanding the expression $\log_a 6xy^5z^4$ and asked: "Are you supposed to work it out?" The teacher responded to Jamie's question by doing the problem on the board. Her answer was $\log_a 6 + \log_a x + 5\log_a y + 4\log_a z$. After the teacher finished, Jamie asked: "So if it says write as a sum, you don't work it out? Just go that one step? I worked it out and found a number." For Jamie the idea of an expression and an equation were conflated and so the properties were used to "convert" the expression into an equation and obtain a numeric answer (Kastberg, 2002).

Building deeper understanding

These examples illustrate common misconceptions that emerge as teachers support students to understand the logarithmic function. Misconceptions based on prior knowledge are a part of developing understanding. When mathematics learners of any age develop new ideas, they use old ideas to make sense of the new. So, for example, a relationship such as $\log_b(M + N) = \log_b(M) + \log_b(N)$ can appear reasonable when one's thinking about numbers and polynomials has been based on the distributive property. But now learners must differentiate that knowledge and reason in new ways by considering the meaning of the new notations they are using.

Building connections to prior experiences

It is important for students to recognise the procedures signified by the function notation. Conceptual understanding of logarithmic functions, however, includes viewing logarithms as objects that can be decomposed and recomposed into new objects, following the properties of logarithms, while also anticipating the exponentiation process needed to solve for x (Weber, 2002b). When students are first introduced to the idea of a variable, we often hear teachers make connections to everyday objects (e.g., "pretend I have one apple, and two more apples, how many apples would I have?") to help students build the idea of an 'x' as an object that can combined with other like objects. A similar exercise could be effective with logarithms. Consider asking students to simplify $x + x + x$, and then $\log_2(x) + \log_2(x) + \log_2(x)$. This is a simple exercise, but it builds on students' existing understanding of how to notate three objects and develops an understanding of $\log_2(x)$ as an object itself. Asking students to evaluate both answers for $x = 2$ can also build an understanding of the role played by $\log_2$ and that it cannot be 'cancelled' away. This may also allow students to learn to interpret the notation. Drawing parallels or comparing to students existing understanding and new ideas builds meaning for the new ideas.

Teachers can also connect to students' experiences with functions such as square root. Like logarithms, square roots represent an inverse relationship, have restrictions on the input value's domain, and have unique notation that indicate the process to take (e.g., find the number that when multiplied by itself results in the input value). Students have 'taken' the square root (similar to the request to take the log of both sides of an equation), and used the square root button on a calculator to perform the process suggested by the symbol. Linking the square root notation with ln and log could improve students' understanding of the dual role that these symbols play and recognition that they cannot be removed from equations without some process acting upon them.

Attending to discourse

When teaching logarithms, it is important to pay attention both to the words we use and the ideas that students share about their understanding. For example, we often hear teachers refer to the fact that $\log_b(b^x) = x$ because the inverse relationships 'cancel' each other out. For students, the term cancel becomes a reason to incorrectly remove the log term from a log equation. It is important to question students when they use this term, to challenge their ideas about what it means to 'cancel' and when it can and cannot be used.

Teachers' use of representations can sometimes lead to limitations in student thinking. Showing students how to construct the graph of $\log_b(x)$ from the exponential graph as in Figure 1 is important, but without further discussion this may lead to incorrect conceptions like Nora's. Similarly, students are often told that they can solve an equation like $3^x = 4^{x-2}$ by taking either the *log* or *ln* of both sides. These two choices are usually offered because they are available on a calculator and because the answer is the same regardless of the base used. However, statements like these could lead to the conception portrayed by the students who saw $\log_{10}(x)$ and $\ln(x)$ as the exact same functions.

Follow-up questions during class discussions and open-ended assessment items can help a teacher know what understandings students have developed about logarithms and logarithmic functions. Consider how the students whose thinking we have described might answer some of the following questions:

- What does $\log_3$ mean in the expressions $\log_3(x)$ and $\log_3(x + 1)$?
- Describe what you think your calculator is doing when you use it to find $\ln(3)$.
- Give an example of an exponential equation whose solution is a negative number.
- How could we plot the graph of $\log_2(x)$ without starting with the exponential graph?
- Produce an argument that could convince a friend that $\log_b(M + N) \neq \log_b M + \log_b N$.

Summary

Logarithms continue to play an important role in mathematics (most significantly in calculus), science, and engineering. It is therefore important for students to understand logarithms as real numbers as well as the characteristics of logarithmic functions. As teachers, we want students to be able to develop a differentiated understanding of logarithmic functions rather than a set of discrete skills, and to be able to regenerate and self-check ideas by relating to prior knowledge. Students moving from Year 10 to senior secondary mathematics need to build from an understanding of logarithms as real numbers toward an understanding of logarithms as the range of a logarithmic function. It is important that students take their time to build understandings of these new objects using processes that are more familiar, to learn to notate their reasoning rather than copying images provided. To achieve this, teachers are encouraged to explore results of students' efforts through informal questioning can result in more flexible student understanding of this unique and powerful function.

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